ETL, DEL, and Past Operators

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Abstract

[8] merges the semantic frameworks of Dynamic Epistemic Logic DEL *([1, 3]) and* Epistemic Temporal Logic ETL *([2, 6]). We consider the logic* TDEL *on the merged semantic framework and its extension with the labeled past-operator "P_ϵ" ("The event ϵ has happened before which..."). To axiomatize the extension, we introduce a method to transform a given model into a* normal *form in a suitable sense. These logics suggest further applications of* DEL *in the theory of agency, the theory of learning, etc.*

1. Introduction

[8] provides a framework for generating the models of *Epistemic Temporal Logic* (ETL, [2, 6]) from the models of *Dynamic Epistemic Logic* (DEL, [1, 3]). In the framework, the temporal transitions in DEL are captured by sequences of event models, called *DEL-protocols*, and each transition made by a product update is encoded into the tree structures of ETL. This allows us to say that DEL-models *generate* ETL-models. The framework allows for a systematic comparison between the two major trends, DEL and ETL, in describing agents' intelligent interactions, and suggests a direction for the studies of new logics that are hybrids of the two.

The main objective of the present paper is to push that investigation further. [8] studies the logic TPAL of ETL-models generated by protocols consisting of public announcements. However, public announcements are just one kind of event model. Thus we might ask what the logic would be like if we extend the setting of TPAL to the full class of event models. In Section 2, we apply the basic methods in TPAL and obtain an axiomatization of the class of the ETL-models generated from the class of all DEL-protocols. We call this extended system TDEL.

After axiomatizing TDEL, in Section 4 we will study the extension of TDEL with the labeled past-operator P_ϵ, where P_ϵ reads as "the event ϵ has occurred before which φ." We call the resulting system TDEL+P. This is a very natural operator to add to the context of TDEL, where all successive updates by event models are encoded as tree structures. A similar operator has been investigated in [12] in the original DEL-context; our objective in the present paper can be characterized as investigating that operator in the TDEL-context.

The axiomatization of TDEL+P will be based on one distinctive feature of the DEL-generated ETL-models. Given a set X of event models, DEL-generated ETL-models can be transformed into the models that consist only of the event models in X or event models with trivial preconditions, and this transformation preserves the truth of formulas whose only event models are those in X. We call this model transformation *normalization*. In Section 3, we will show that DEL-generated ETL-models can be normalized in this sense, and will apply this fact to the axiomatization of TDEL+P.

TDEL and its extension TDEL+P suggest further interesting applications in the theory of agency and the theory of learning. In modeling agency, some systems model intentionality in terms of agents' goals to bring about certain states. And, for instance in [7], for an agent to intend to bring about a state at which φ holds, it is not sufficient for her just to bring about that state. In the history leading up to that state, she must also have believed that her actions would lead to a φ state (so she does not bring it about by accident). This seems exactly to call for a way to express what an agent used to believe, about what was then her future. Also, when expressing that an agent learned something from an event, we want to be able to say something like, "After ϵ took place, i knew that φ. But before ϵ, i did not know φ." Expressing this sentence requires both a future and a past modality. We will discuss these issues further in Section 5.

2. TDEL

We start by generating ETL-models from DEL-models, though a detailed exposition for ETL and DEL is omitted. Readers who are not familiar with the systems are invited to refer to e.g. [2, 6] for ETL and to e.g. [10] for DEL. Below, we fix a finite set $\mathcal{A}$ of agents and a countable set At of propositional letters.

2.1. DEL-Generated ETL-Models

Definition 2.1 An *epistemic model* $\mathcal{M}$ is a tuple $\langle W, \sim, V\rangle$, where W is a nonempty set, $\sim: \mathcal{A} \to W \times W$, and $V : At \to 2^W$. The set W represents the set of possible situations, $\sim$, the indistinguishability relation over the possible situations for an agent i, and V, the valuation function. We denote W, $\sim$ and V by $Dom(\mathcal{M})$, $Rel(\mathcal{M})$, and $Val(\mathcal{M})$ respectively. Also, we write $\sim_i$ for $\sim(i)$ by convention. ◁

Definition 2.2 An *event model* $\mathcal{E}$ is a tuple $\langle E, \to, \mathsf{pre}\rangle$, where E is a nonempty set, $\to: \mathcal{A} \to E \times E$, and $\mathsf{pre} : E \to \mathcal{L}_{EL}$, where $\mathcal{L}_{EL}$ is the set of epistemic formulas. E represents the set of possible events, $\to_i$, the indistinguishability relation over the possible events for an agent i, and pre assigns the preconditions for the possible events. We denote the domain E of $\mathcal{E}$ by $Dom(\mathcal{E})$, and write $\to_i$ for $\to(i)$ by convention. ◁

Let $\mathbb{E}$ be the class of pointed event models $(\mathbb{E}, e)$. Let $\mathbb{E}^*$ be the class of finite sequences of pointed event models.

Definition 2.3 A *DEL-protocol* is a set $\mathsf{P} \subseteq \mathbb{E}^*$, which is closed under finite prefix. Let $ptcl(\mathbb{E})$ be the class of DEL-protocols. Given an epistemic model $\mathcal{M}$, a *state-dependent DEL-protocol* is a function $\mathsf{p} : Dom(\mathcal{M}) \to ptcl(\mathbb{E})$. ◁

Given a sequence $\sigma = \epsilon_1 \dots \epsilon_n \in \mathbb{E}^*$, we write $\sigma_{(n)}$ for the initial segment of σ of length n ($n \leq \mathsf{len}(\sigma)$), and σ_n for the nth component of σ. When $n > \mathsf{len}(\sigma)$ or $n = 0$, σ_n and $\sigma_{(n)}$ are empty. If $\sigma = (\mathcal{E}_1, e_1)(\mathcal{E}_2, e_2) \dots (\mathcal{E}_n, e_n) \in \mathbb{E}^*$, we write σ^L and σ^R for $\mathcal{E}_1 \cdots \mathcal{E}_n$ and $e_1 \cdots e_n$ respectively. Thus, for example, if $\sigma = (\mathcal{E}_1, e_1) \dots (\mathcal{E}_n, e_n)$, then $(\sigma^L)_{(3)} = \mathcal{E}_1\mathcal{E}_2\mathcal{E}_3$ and $(\sigma^R)_3 = e_3$. Clearly, $(\cdot)^L, (\cdot)^R$ on the one hand and $(\cdot)_n, (\cdot)_{(n)}$ on the other commute. Thus, we omit parentheses when there is no danger of ambiguity.

Definition 2.4 (σ^L-Generated Model) Let $\mathcal{M} = \langle W, \sim, V\rangle$ be an epistemic model and p, a state-dependant DEL-protocol on $\mathcal{M}$. Given a sequence $\sigma \in \mathbb{E}^*$, the *σ^L-generated model*, $\mathcal{M}^{\sigma^L,\mathsf{p}} = \langle W^{\sigma^L,\mathsf{p}}, \sim_i^{\sigma^L,\mathsf{p}}, V^{\sigma^L,\mathsf{p}}\rangle$, is defined by induction on the initial segment of σ^L:

- $W^{\sigma^L_{(0)},\mathsf{p}} := W$, for each $i \in \mathcal{A}$, $\sim_i^{\sigma^L_{(0)},\mathsf{p}} := \sim_i$ and $V^{\sigma^L_{(0)},\mathsf{p}} := V$.
- $w\tau \in W^{\sigma^L_n,\mathsf{p}}$ iff
  1. $w \in W$,
  2. $\sigma^L_{(n)} = \tau^L$,
  3. $w\tau_{(n-1)} \in W^{\sigma^L_{(n-1)},\mathsf{p}}$,
  4. $\tau \in \mathsf{p}(w)$, and
  5. $\mathcal{M}^{\sigma^L_{(n-1)},\mathsf{p}}, w\tau_{(n-1)} \models \mathsf{pre}(\tau^R_n)$
- For each $w\tau, v\tau' \in H_n$ ($0 < n < \mathsf{len}(\sigma^L)$), $w\tau \sim^{\sigma^L_{(n)}} v\tau'$ iff $w\tau_{(n-1)} \sim_i^{\sigma^L_{(n-1)},\mathsf{p}} v\tau'_{(n-1)}$ and $\tau^R_n \to_i (\tau')^R_n$ in τ^L_n.
- For each $p \in \mathsf{At}$, $V^{n+1,\mathsf{p}}(p) = \{w\sigma \in W^{n+1,\mathsf{p}} \mid w \in V(p)\}$.

Note that, in the definition of $\sim_i$, $\tau^L = (\tau')^L = \sigma^L_n$, and thus $\sigma^L = (\sigma')^L$. ◁

Definition 2.5 (DEL-Generated ETL-Model) Let $\mathcal{M} = \langle W, \sim, V\rangle$ be an epistemic model and p a state-dependent DEL-protocol on $\mathcal{M}$. *An* ETL*-model* $\mathsf{Forest}(\mathcal{M}, \mathsf{p}) = \langle \mathsf{H}, \sim, U\rangle$ *generated from* $\mathcal{M}$ *by* p is defined as follows:

- $\mathsf{H} := \{h \mid \exists w \in W, \sigma \in \bigcup_{w \in W} p(w)$ such that $h = w\sigma \in W^{\sigma^L,\mathsf{p}}\}$.
- For all $h, h' \in \mathsf{H}$ with $h = w\sigma$ and $h' = v\sigma'$, $h \sim_i h'$ iff $w\sigma \sim_i^{\sigma^L,\mathsf{p}} v\sigma'$.
- For each $p \in \mathsf{At}$ and $h = w\sigma \in \mathsf{H}$, $h \in V'(p)$ iff $h \in V^{\sigma^L,\mathsf{p}}(p)$.

We define the class $\mathbb{F}_{st}(\mathbb{E})$ to be the class of all ETL-models of the form $\mathsf{Forest}(\mathcal{M}, \mathsf{p})$. ◁

Given $X \subseteq \mathbb{E}$, we denote by $\mathbb{F}_{sd}(X)$ the class of ETL-models generated from epistemic models $\mathcal{M}$ by state-dependent protocols p consisting only of elements in X, i.e., for every w in $\mathcal{M}$, if $\sigma \in \mathsf{p}(w)$, $\sigma \subseteq X^*$.

Example 2.6 (Public Announcements) We illustrate the above construction in public announcement logic with each event model denoting an announcement or observation of some true formula. Let $\mathcal{M}$ be a model that consists of w, v, u, each of which are indistinguishable (the $\sim$ relation in $\mathcal{M}$ is an equivalence relation on w, v, u), where $V(p) = \{w, v\}$ and $V(q) = \{v\}$. This model is represented by the three points labeled with w, v, u, respectively at the bottom of Figure 1. Consider the protocol p where $\mathsf{p}(w) = \{p, pq, \neg q\}$, $\mathsf{p} = \{p, pq, \neg q\}$ and $\mathsf{p} = \{\neg q, \neg q\top, p\}$. The DEL-generated ETL-model $\mathsf{Forest}(\mathcal{M}, \mathsf{p})$ can be visualized as follows:

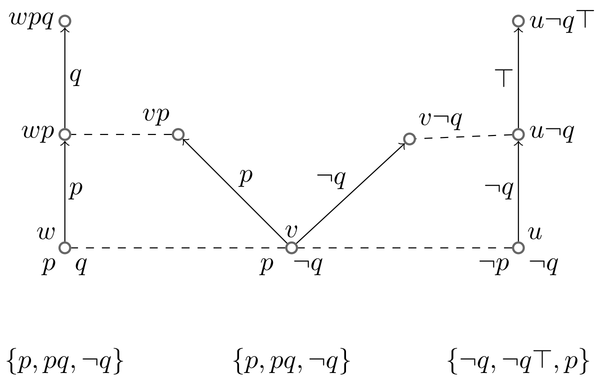

Figure 1. A DEL-generated ETL model.

2.2. Axiomatization of TDEL

The language $\mathcal{L}_{TDEL}$ of TDEL extends the language $\mathcal{L}_{EL}$ of epistemic logic by the operator $\langle\epsilon\rangle$, where $\epsilon \in \mathbb{E}$. The dual of $\langle\epsilon\rangle$ is $[\epsilon]$ defined by $\neg\langle\epsilon\rangle\neg$ as usual.

Let $\mathcal{H} \in \mathbb{F}_{sd}(\mathbf{E})$ with

$$\mathcal{H} = \mathsf{Forest}(\mathcal{M}, \mathsf{p}) = \langle \mathsf{H}, \{\sim_i\}_{i\in\mathcal{A}}, V\rangle.$$

The semantics of the knowledge operator and the event model operator are defined by:

- $\mathcal{H}, h \models K\varphi$ iff for all h' such that $h \sim_i h'$, $\mathcal{H}, h' \models \varphi$.
- $\mathcal{H}, h \models \langle\epsilon\rangle\varphi$ iff $h\epsilon \in \mathsf{H}$ and $\mathcal{H}, h\epsilon \models \varphi$.

The boolean cases are defined in the standard way.

Example 2.7 (Semantics in TDEL) Let $\mathcal{H}$ be the model $\mathsf{Forest}(\mathcal{M}, \mathsf{p})$ in Figure 1. For instance, we have $\mathcal{H}, w \models \langle p\rangle\langle q\rangle K(p\wedge q)$ but $\mathcal{H}, w \not\models \langle p\wedge q\rangle K(p\wedge q)$. This illustrates the fact that in TDEL we cannot treat sequences of events as single events, while in DEL we can. Also the fact that we have $\mathcal{H}, w \models (p \wedge q) \wedge \neg\langle p \wedge q\rangle\top$ violates the schema $\langle\epsilon\rangle\top \leftrightarrow \mathsf{pre}(\epsilon)$, which is valid in DEL. In TDEL, we only have $\langle\epsilon\rangle\top \rightarrow \mathsf{pre}(\epsilon)$.

Definition 2.8 The axiomatization TDEL of $\mathbb{F}_{sd}(\mathbb{E})$ is given by the following axiom schemes and inference rules.

Axioms

PC Propositional validities

K_i $K_i(\varphi \rightarrow \psi) \rightarrow (K_i\varphi \rightarrow K_i\psi)$

F1 $\langle\epsilon\rangle p \leftrightarrow \langle\epsilon\rangle\top \wedge p$

F2 $\langle\epsilon\rangle\neg\varphi \leftrightarrow \langle\epsilon\rangle\top \wedge \neg\langle\epsilon\rangle\varphi$

F3 $\langle\epsilon\rangle(\varphi \wedge \psi) \leftrightarrow \langle\epsilon\rangle\varphi \wedge \langle\epsilon\rangle\psi$

F4 $\langle\epsilon\rangle K_i\varphi \leftrightarrow \langle\epsilon\rangle\top \wedge \bigwedge_{\{(\epsilon')^R \in Dom(\epsilon^L) \mid \epsilon^R \rightarrow_i (\epsilon')^R\}} K_i(\langle\epsilon'\rangle\top \rightarrow \langle\epsilon'\rangle\varphi)$

A1 $\langle\epsilon\rangle(\varphi \rightarrow \psi) \rightarrow (\langle\epsilon\rangle\varphi \rightarrow \langle\epsilon\rangle\psi)$

A2 $\langle\epsilon\rangle\top \rightarrow \mathsf{pre}(\epsilon^R)$

Inference Rules

MP If $\vdash \varphi \rightarrow \psi$ and $\vdash \varphi$, then $\vdash \psi$.

k-Nec If $\vdash \varphi$, then $\vdash K_i\varphi$.

e-Nec If $\vdash \varphi$, then $\vdash [\epsilon]\varphi$.

◁

Readers are invited to verify that these are sound with respect to $\mathbb{F}_{sd}(\mathbb{E})$.

2.3. Completeness Proof

The proof is given by a variant of the Henkin-style construction. The basic construction is the same as the one in [8] with minor modifications.

Definition 2.9 (Legal Histories) Let W_0 be the set of all TDEL-maximal consistent sets. We define λ_n and H_n ($0 \leq n \leq d(\Sigma)$) as follows:

- Define $H_0 = W_0$ and for each $w \in H_0$, $\lambda_0(w) = w$.
- Let $H_{n+1} = \{h\epsilon \mid h \in H_n$ and $\langle\epsilon\rangle\top \in \lambda_n(h)\}$. For each $h = h'\epsilon \in H_{n+1}$, define $\lambda_{n+1}(h) = \{\varphi \mid \langle\epsilon\rangle\varphi \in \lambda_n(h')\}$.

Given $h \in H_n$, we write $\lambda(h)$ for $\lambda_n(h)$. ◁

The following can be straightforwardly verified by appealing to the construction and F2.

Lemma 2.10 *For each $n \geq 0$, for each $\sigma \in H_n$, $\lambda_n(\sigma)$ is a maximally consistent set.*

Let $\mathcal{H}^{can}_0 = (H_0, \sim^0, V^0)$, where $\sim^0$ and V^0 are defined by

- $w \sim^0_i v$ iff $\{\varphi \mid K_i\varphi \in w\} \subseteq v$.
- For each $p \in \mathsf{At}$ and $w \in H_0$, $p \in V(w)$ iff $p \in w$.

Definition 2.11 (Canonical Model) The canonical model $\mathcal{H}^{can}$ is a triple $\langle \mathsf{H}^{can}, \{\sim^{can}_i\}_{i\in\mathcal{A}}, V^{can}\rangle$, where each item is defined as follows:

- $\mathsf{H}^{can} =_{\text{def}} \bigcup_{i=0}^{\infty} H_i$.
- For each $w\sigma, w'\sigma' \in \mathsf{H}^{can}$, $w\sigma \sim^{can}_i w'\sigma'$ iff$_{\text{def}}$ $w\sigma \sim^{\sigma^L}_i w'\sigma'$, where $\sim^{\sigma^L}$ is defined by induction in the following way:

- $\sim_i^{\sigma^L_{(0)}} = \sim_i^0$
- For each $w\tau, v\tau' \in H_n$ $(0 < n < \mathsf{len}(\sigma^L))$, $w\tau \sim^{\sigma^L_{(n)}} v\tau'$ iff $w\tau_{(n-1)} \sim_i^{\sigma^L_{(n-1)}} v\tau'_{(n-1)}$ and $\tau^R_n \rightarrow_i (\tau')^R_n$ in τ^L_n.

- For every $P \in \mathsf{At}$ and $h = w\sigma \in \mathsf{H}^{can}$, $w\sigma \in V^{can}(P)$ iff $w \in V^0(P)$.

◁

Proposition 2.12 *Let $w\sigma \sim_i^{can} v\tau$ with $w, v \in W^0$, $\sigma = \sigma_1 \dots \sigma_n$ and $\tau = \tau_1 \dots \tau_n$. If $K_i\varphi \in \lambda(w\sigma)$, then* $K_i(\langle\tau_1\rangle\top \rightarrow \langle\tau_1\rangle(\langle\tau_2\rangle\top \rightarrow \langle\tau_2\rangle(\dots(\langle\tau_n\rangle\top \rightarrow \langle\tau_n\rangle\varphi)\dots) \in \lambda(w)$.

Proof. By induction on n. When $n = 0$, σ, τ are empty and thus the claim clearly holds. For the inductive step, assume that $K_i\varphi \in \lambda(\sigma)$. Then, by the construction of $\mathcal{H}^{can}$, $\langle\sigma_n\rangle K_i\varphi \in \lambda(w\sigma_{n-1})$. By F4, for all events e in σ^L_n such that $\sigma^R_n \rightarrow_i e$:

$$K_i(\langle\sigma^L_n, e\rangle\top \rightarrow \langle\sigma^L_n, e\rangle\varphi) \in \lambda(w\sigma_{(n-1)}).$$

Here, by the construction of $\mathcal{H}^{can}$, $\sigma_n \rightarrow_i \tau_n$. By applying the IH, we are done. QED

Lemma 2.13 (Truth Lemma) *For every $\varphi \in \mathcal{L}_{\mathsf{TDEL}}$ and $h \in \mathsf{H}^{can}$,*

$$\varphi \in \lambda(h) \quad \textit{iff} \quad \mathcal{H}^{can}, h \models \varphi.$$

Proof. We show by induction on the structure of $\varphi \in \mathcal{L}_{\mathsf{TDEL}}$ that for each $h \in \mathsf{H}^{can}$, $\varphi \in \lambda(h)$ iff $\mathcal{H}^{can}, h \models \varphi$. The base and the boolean cases are straightforward.

For the knowledge modality, let $h \in \mathsf{H}^{can}$ with $h = w\sigma_1 \cdots \sigma_n$ $(w \in W_0)$ and assume $K_i\psi \in \lambda(h)$. Suppose $h' \in \mathsf{H}^{can}$ with $h \sim_i^{can} h'$. By construction of the canonical model, we know that $h' = v\tau_1 \cdots \tau_n$ for some $v \in H_0$ and $\tau_1 \dots \tau_n \in \mathbb{E}^*$ with $w \sim_i^0 v$. By Proposition 2.12, we have $K_i(\langle\tau_1\rangle\top \rightarrow \langle\tau_1\rangle(\langle\tau_2\rangle\top \rightarrow \langle\tau_2\rangle(\cdots\langle\tau_{n-1}\rangle(\langle\tau_n\rangle\top \rightarrow \langle\tau_n\rangle\psi)\cdots)) \in \lambda(w)$.
Since $w \sim_i^0 v$, we have by the construction of $\mathcal{H}^{can}$,
$\langle\tau_1\rangle\top \rightarrow \langle\tau_1\rangle(\langle\tau_2\rangle\top \rightarrow \langle\tau_2\rangle(\cdots\langle\tau_{n-1}\rangle(\langle\tau_n\rangle\top \rightarrow \langle\tau_n\rangle\psi)\cdots) \in \lambda(v)$.
Now note that
$\langle\tau_1\rangle\top \in \lambda(v), \langle\tau_2\rangle\top \in \lambda(v\tau_1), \dots, \langle\tau_n\rangle\top \in \lambda(v\tau_1 \dots \tau_{n-1})$.
Thus, we have
$\langle\tau_2\rangle\top \rightarrow \langle\tau_2\rangle(\cdots\langle\tau_{n-1}\rangle(\langle\tau_n\rangle\top \rightarrow \langle\tau_n\rangle\psi)\cdots) \in \lambda(v\tau_1)$
$\langle\tau_3\rangle\top \rightarrow \langle\tau_3\rangle(\cdots\langle\tau_{n-1}\rangle(\langle\tau_n\rangle\top \rightarrow \langle\tau_n\rangle\psi)\cdots) \in \lambda(v\tau_1\tau_2)$
$\vdots$
$\langle\tau_n\rangle\psi \in \lambda(v\tau_1 \cdots \tau_{n-1})$

Therefore, $\psi \in \lambda(v\tau_1 \cdots \tau_n) = \lambda(h')$. By the induction hypothesis, $\mathcal{H}^{can}, h' \models \psi$. Therefore, $\mathcal{H}^{can}, h \models K_i\psi$, as desired.

For the other direction, let $h \in \mathsf{H}^{can}$ and assume $K_i\psi \notin \lambda(h)$. For simplicity, let $h = w\sigma_1$ with $w \in W_0$ and $\sigma_1 \in \mathbb{E}$. The argument can easily be generalized to deal with the general case along the lines of the argument above. Since $\lambda(h)$ is a maximally consistent set, we have $\neg K_i\psi \in \lambda(h)$. Thus, by Definition 2.9, $\langle\sigma_1\rangle\neg K_i\psi \in \lambda(w)$. Using axiom $F2$, $\neg\langle\sigma_1\rangle K_i\psi \in \lambda(w)$; and so, by Axiom $F4$, $\neg\langle\sigma_1\rangle\top \vee \neg\bigwedge_{\{\tau \mid \sigma_1 \rightarrow_i \tau \text{ in } \sigma^L_1\}} K_i(\langle\tau\rangle\top \rightarrow \langle\tau\rangle\psi) \in \lambda(w)$. Since $\langle\sigma_1\rangle\top \in \lambda(w)$ by construction, it follows that $\neg\bigwedge_{\{\tau \mid \sigma_1 \rightarrow_i \tau \text{ in } \sigma^L_1\}} K_i(\langle\tau\rangle\top \rightarrow \langle\tau\rangle\psi) \in \lambda(w)$.

Now consider the set $v_0 = \{\theta \mid K_i\theta \in \lambda(w)\} \cup \{\neg\bigwedge_{\{\tau \mid \sigma_1 \rightarrow_i \tau \text{ in } \sigma^L_1\}}(\langle\tau\rangle\top \rightarrow \langle\tau\rangle\psi)\}$. We claim that this set is consistent. Suppose not. Then, there are formulas $\theta_1, \dots, \theta_m$ such that $\vdash \bigwedge_{j=1}^m \theta_j \rightarrow \bigwedge_{\{\tau \mid \sigma_1 \rightarrow_i \tau \text{ in } \sigma^L_1\}}(\langle\tau\rangle\top \rightarrow \langle\tau\rangle\psi)$ and for $j = 1, \dots, m$, $K_i\theta_j \in \lambda(w)$.

By standard modal reasoning, $\vdash \bigwedge_{j=1}^m K_i\theta_j \rightarrow \bigwedge_{\{\tau \mid \sigma_1 \rightarrow_i \tau \text{ in } \sigma^L_1\}} K_i(\langle\tau\rangle\top \rightarrow \langle\tau\rangle\psi)$. This implies that $\bigwedge_{\{\tau \mid \sigma_1 \rightarrow_i \tau \text{ in } \sigma^L_1\}} K_i(\langle\tau\rangle\top \rightarrow \langle\tau\rangle\psi) \in \lambda(w)$. However, this contradicts the fact that $\neg\bigwedge_{\{\tau \mid \sigma_1 \rightarrow_i \tau \text{ in } \sigma^L_1\}} K_i(\langle A\rangle\top \rightarrow \langle A\rangle\psi) \in \lambda(w)$, since $\lambda(w)$ is a maximally consistent set.

Now using standard arguments (Lindenbaum's lemma), there exists a maximally consistent set v with $v_0 \subseteq v$. By the construction of v, we must have $w \sim_i^0 v$. Also, since v is an mcs such that $\neg\bigwedge_{\{\tau \mid \sigma_1 \rightarrow_i \tau \text{ in } \sigma^L_1\}}(\langle A\rangle\top \rightarrow \langle A\rangle\psi) \in \lambda(v)$, there is some τ_1 such that $\neg(\langle\tau_1\rangle\top \rightarrow \langle\tau_1\rangle\psi \in \lambda(v)$. Otherwise, v is inconsistent. Therefore, for such τ_1, we have $\langle\tau_1\rangle\top \in \lambda(v), \neg\langle\tau_1\rangle\psi \in \lambda(v)$. Here, by axiom $F2$, $\langle\tau_1\rangle\neg\psi \in \lambda(v)$. Hence $\neg\psi \in \lambda(v\tau_1)$ and therefore $\psi \notin \lambda(v\tau)$. By the induction hypothesis, $\mathcal{H}^{can}, v\tau_1 \not\models \psi$. This implies $\mathcal{H}^{can}, w\tau_1 \not\models K_i\psi$, as desired.

For the event model operator, assume that $\langle\epsilon\rangle\psi \in \lambda(h)$. Since $\langle\epsilon\rangle\top \in \lambda(h)$ (for $\neg\langle\epsilon\rangle\top \in \lambda(h)$ makes $\lambda(h)$ inconsistent), $\psi \in \lambda(h\epsilon)$. By the induction hypothesis, we have $\mathcal{H}^{can}, h\epsilon \models \psi$, which implies $\mathcal{H}^{can}, h \models \langle\epsilon\rangle\psi$.

For the other direction, assume $\mathcal{H}^{can}, h \models \langle\epsilon\rangle\psi$. Then, $\mathcal{H}^{can}, h\epsilon \models \psi$. By the inductive hypothesis, we have $\psi \in \lambda(h\epsilon)$ and thus $\langle\epsilon\rangle\psi \in \lambda(h)$. QED

All that remains is to show is that $\mathcal{H}^{can}$ is in the class of intended models (i.e., is an element of $\mathbb{F}_{sd}(\mathbb{E})$).

Lemma 2.14 *The canonical model $\mathcal{H}^{can}$ is in $\mathbb{F}_{sd}(\mathbb{E})$. That is, there is an epistemic model $\mathcal{M}$ and local protocol p on $\mathcal{M}$ such that $\mathcal{H}^{can} = \mathsf{Forest}(\mathcal{M}, \mathsf{p})$.*

Proof. Let $\mathcal{M}_{can} = (W_0, \{\sim_i^0\}_{i \in \mathcal{A}}, V^0)$ and define $\mathsf{p}_{can} : W_0 \rightarrow \mathbb{E}^*$ so that $\mathsf{p}_{can}(w) = \{\sigma \mid w\sigma \in \mathsf{H}^{can}\}$. Suppose that $\mathcal{H}^{p_{can}} = \mathsf{Forest}(\mathcal{M}_{can}, \mathsf{p}_{can})$. We claim that $\mathcal{H}^{can}$ and $\mathcal{H}^{p_{can}}$ are the same model. For this, it suffices to show that for all $w \in W_0$ and $\sigma \in \mathbb{E}^*$ we have $w\sigma \in \mathsf{H}^{can}$ iff $w\sigma \in W^{\sigma,p_{can}}$. For this implies $\mathsf{H}^{can} = \mathsf{H}^{p_{can}}$, where $\mathsf{H}^{p_{can}}$ is the domain of $\mathcal{H}^{p_{can}}$. Then, by inspecting the construction of Forest and Definition 2.11, we see that $\mathcal{H}^{can}$ and $\mathcal{H}^{p_{can}}$ are the same model .

We will show by induction on the length of $\sigma \in \mathcal{E}^*$ that for any $w \in W_0$, $w\sigma \in \mathsf{H}^{can}$ iff $w\sigma \in W^{\sigma,p_{can}}$. The base case ($\mathsf{len}(\sigma) = 0$) is clear. Assume that the claim holds for all σ with $\mathsf{len}(\sigma) = n$.

Given any $\sigma \in \mathbb{E}^*$ with $\mathsf{len}(\sigma) = n$, we first show by subinduction (on the structure of A) that, for all $A \in \mathcal{L}_{EL}$, $\mathcal{H}^{can}, w\sigma \models A$ iff $\mathcal{M}^{\sigma,p_{can}}, w\sigma \models A$. The base and boolean cases are straightforward. Suppose that $\mathcal{H}^{can}, w\sigma \models K_iB$. We must show $\mathcal{M}^{\sigma,p_{can}}, w\sigma \models K_iB$. Let $v\sigma \in W^{\sigma,p_{can}}$ with $w\sigma \sim_i^{\sigma,p} v\sigma$. By the main induction hypothesis, we have both $v\sigma \in \mathsf{H}^{can}$ and $w\sigma \in W^{\sigma,p_{can}}$. By construction, since $w\sigma \sim_i^{\sigma,p_{can}} v\tau$, we have $w \sim_i^0 v$. Furthermore, $w\sigma \sim_i^{can} v\tau$. Hence, $\mathcal{H}^{can}, v\sigma \models B$. By the subinduction hypothesis, $\mathcal{M}^{\sigma,p_{can}}, v\sigma \models B$. Therefore, $\mathcal{M}^{\sigma,p_{can}}, w\sigma \models K_iB$.

Coming back to the main induction, assume $w\sigma_{(n)}\sigma_{n+1} \in \mathsf{H}_{can}$. This implies that $\langle\sigma_{n+1}\rangle\top \in \lambda(w\sigma_{(n)})$. By truth lemma, we have $\mathcal{H}^{can}, w\sigma_{(n)} \models \langle\sigma_{n+1}\rangle\top$. This, together with axiom $A2$, implies $\mathcal{H}^{can}, w\sigma \models \mathsf{pre}(\sigma^R_{n+1})$. From the above subinduction, it follows that $\mathcal{M}^{\sigma_{(n)},p_{can}}, w\sigma_{(n)} \models \mathsf{pre}(\sigma^R_{n+1})$ (recall that $\mathsf{pre}(e) \in \mathcal{L}_{EL}$ for all events e by definition). Thus, by the construction of p_{can}, we have $w\sigma_{(n)}\sigma_{n+1} \in W^{\sigma_{(n)},p_{can}}$. This shows that if $w\sigma_{(n)}\sigma_{n+1} \in \mathsf{H}^{can}$ then $w\sigma_{(n)}\sigma_{n+1} \in W^{\sigma_{(n)}\sigma_{n+1},p_{can}}$. The other direction is similar. QED

The proof of the completeness theorem follows from Lemma 2.13 and Lemma 2.14 using a standard argument.

Theorem 2.15 TDEL *is sound and complete with respect to* $\mathbb{F}_{sd}(\mathbb{E})$.

2.4. TDEL Restricted to Some Class of Protocols

TDEL axiomatizes the class $\mathbb{F}_{sd}(\mathbb{E})$. However, note that the completeness proof above does not depend on the fact that $\mathbb{E}$ is the set of *all* pointed event models, but only the fact that $\mathbb{F}_{sd}(\mathbb{E})$ contains the ETL-models generated from epistemic models $\mathcal{M}$ by the protocol p that allows *all possible finite sequences of* $\mathbb{E}$ at each w in $\mathcal{M}$, i.e $\mathsf{p}(w) = \mathbb{E}^*$.

Thus, even if we restrict our attention to some $X \subseteq \mathbb{E}$, the proof should work as well for the class $\mathbb{F}_{sd}(X)$. However, here we have to be careful that such an X must at least contain all the "relevant" pointed event models: if $(\mathcal{E}, e) \in X$, then $(\mathcal{E}, f) \in X$ for all f such that $e \to f$ in $\mathcal{E}$. Otherwise the knowledge modality case of Lemma 2.13 since we need all the "relevant" histories in the present sense must be included in the canonical model.

Let $X \subseteq \mathbb{E}$. Call X *e-closed* if, for all $\mathcal{E}$, if there is $\epsilon \in X$ such that $\epsilon^L = \mathcal{E}$, then for every event e in $\mathcal{E}$, (ϵ^L, e) is in X. Denote by $\mathcal{L}_{\mathsf{TDEL}}(X)$ the fragment of $\mathcal{L}_{\mathsf{TDEL}}$ that only allows the event model operators $\langle\epsilon\rangle$ such that $\epsilon \in X$. Also, let $\mathsf{TDEL}(X)$ be the axiomatization as above except that the axiom schema and the $[\epsilon]$-necessitation rule can be instantiated by the event models in X. The following is a corollary of our completeness proof.

Corollary 2.16 *For all e-closed subsets X of* $\mathbb{E}$, $\mathsf{TDEL}(X)$ *is complete with respect to* $\mathbb{F}_{sd}(X)$.

Thus, by changing the parameter X, we could have axiomatizations for various kinds of logic of protocols. In fact, the logic of public announcement protocols, as is presented in [8] is a particular version of $\mathsf{TDEL}(X)$. We could also consider the logics of secret message protocols, etc.

3. Normalization of DEL-Generated ETL-Models

Before we study the proposed extension, we need to turn our attention to a distinctive property of DEL-generated ETL-models. The rough idea is that, given a set X of event models, DEL-generated ETL-models can be transformed into the models that consist of the event models in X and the event models with trivial preconditions in such a way that the truth of the formulas expressed with event models in X is preserved. We call this model transformation *normalization*. To formulate this notion here, we need some definitions.

Definition 3.1 We say that two event models $(E, \to, \mathsf{pre})$ and $(E', \to', \mathsf{pre}')$ are *isomorphic*, if $(E, \to)$ and $(E', \to')$ are isomorphic. Clearly, such an isomorphic relation partitions the set of event models. Given an event model $\mathcal{E}$, let $[\mathcal{E}]$ be the class of event models isomorphic to $\mathcal{E}$. We call $[\mathcal{E}]$ *the type of* $\mathcal{E}$. Also given a finite e-closed subset X of $\mathbb{E}$, we denote by PRE_X the conjunction of the preconditions of the events that occur in X. ◁

Definition 3.2 (Normalization Function) Let X be a finite e-closed subset of $\mathbb{E}$. The *normalization function with respect to* X is a function $f_X : \mathbb{E} \to \mathbb{E}$ such that, for every pointed event model $(\mathcal{E}, e)$ with $\mathcal{E} = (E, \to, \mathsf{pre})$, $f_X((\mathcal{E}, e)) = (\mathcal{E}', e)$, where $\mathcal{E}' = (E', \to', \mathsf{pre}')$ is defined by:

- $E' = E$
- $\to' (i) = \to (i)$

- $\mathsf{pre}'(e) = \mathsf{pre}(e) \vee \neg\mathsf{pre}(e) \vee PRE_X$.

$\triangleleft$

The purpose of having this function is to replace certain pointed event models ϵ with isomorphic pointed models with tautologous preconditions. Therefore, this role of the normalization function does not turn on the particular form ($\mathsf{pre}(e) \vee \neg\mathsf{pre}(e)$) of the tautology, as given in the third clause of the definition. However, having the tautology of such a form, we can guarantee that, if $\epsilon \neq \epsilon'$, then $f_X(\epsilon) \neq f_X(\epsilon')$. Also the third disjunct in the third clause guarantees that, for all $\epsilon \in \mathbb{E}$, $f_X(\epsilon) \notin X$.

Definition 3.3 Given a finite e-closed subset X of $\mathbb{E}$, a *substitution function for X* is a function $\sigma_X : \mathbb{E} \to \mathbb{E}$ such that, for all $\epsilon \in \mathbb{E}$,

$$\sigma_X(\epsilon) = \begin{cases} \epsilon & \text{if } \epsilon \in X \\ f_X(\epsilon) & \text{otherwise} \end{cases}$$

Given a DEL-generated ETL-model $\mathcal{H}$ and a history $h = w\epsilon_1 \dots \epsilon_n$ in $\mathcal{H}$, we denote $w\sigma_X(\epsilon_1) \dots \sigma_X(\epsilon_n)$ by $\sigma_X(h)$. $\triangleleft$

Definition 3.4 (Normalization) Let X be an e-closed subset of $\mathbb{E}$. The *normalization* $\mathcal{H}\sigma_X$ of a DEL-generated ETL-model $\mathcal{H} = (H, \sim, V)$ with respect to X is a tuple $(H', \sim', V')$. σ_X that satisfies the following conditions:

$\mathsf{H}' := \{\sigma(h) \mid h \in \mathsf{H}\}$

$\sigma(h) \sim_i' \sigma(g)$ iff $h \sim_i g$.

$V'(p) := \{\sigma(h) \mid h \in V(p)\}$

$\triangleleft$

Example 3.5 (Normalization) We can now illustrate the manner in which a model can be normalized, and how that process depends on the set of event models we are interested in. The process uniformly replaces any event not in the set with an event that has tautological preconditions. Let our initial model be the one from Figure 1. If we normalized this model with respect to the set $\{p, q, \neg q, \top\}$, the model would not change, since this is the set of all events in the model. For the other extreme case, if we normalized with respect to the set $\emptyset$, indicating tautologous preconditions by indexed $\top$'s, we would obtain the following:

On the other hand, if we normalized with respect to some subset of the expressions in the model, we would replace some events and keep others.

Proposition 3.6 *Let $\mathcal{H}$ be a* DEL*-generated* ETL*-model. Then $\mathcal{H}\sigma_X$ is a* DEL*-generated* ETL*-model.*

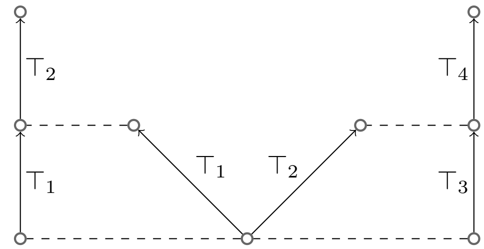

Figure 2. Normalizing Figure 1 with respect to $\emptyset$.

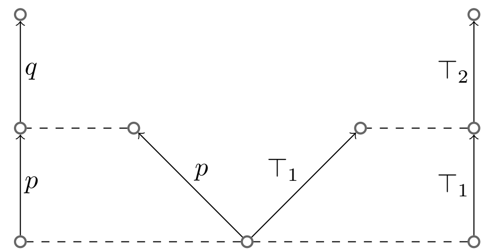

Figure 3. Normalizing Figure 1 with respect to $\{p, q\}$.

Proof. Let $\mathcal{H} = \mathsf{Forest}(\mathcal{M}, \mathsf{p}) = (H, \sim, V)$ and $\mathcal{H}\sigma_X = (H', \sim', V'))$. Let p_0^N be such that for all w in $\mathcal{M}$, $\mathsf{p}_0(w) = \{\sigma \mid w\sigma \in H'\}$. Then $\mathcal{H}\sigma_X = \mathsf{Forest}(\mathcal{M}, \mathsf{p}_0)$. The rest of the proof goes by an argument similar to the proof of Lemma 2.14. QED

Now it is straightforward to show that the normalization with respect to a given X preserves the truth of the formulas in which only the event operators from X occur.

Proposition 3.7 (Normalization) *Let X be an e-closed subset of $\mathbb{E}$. Then, for every DEL-generated model $\mathcal{H}$ and every formula φ in $\mathcal{L}_{DEL}(X)$ (the fragment of $\mathcal{L}_{DEL+P}$ that only allows the event models in X),*

$$\mathcal{H}, h \models \varphi \textit{ iff } \mathcal{H}\sigma_X, \sigma_X(h) \models \varphi.$$

Proof. We proceed by induction on φ. The base and boolean cases are clear. For the knowledge modality case, assume $\mathcal{H}, h \models K_i\psi$. Then, for all $h \sim h'$, $\mathcal{H}, h' \models \psi$. By IH, $\mathcal{H}\sigma_X, \sigma_X(h') \models \psi$. By Definition 3.4, we have $\mathcal{H}\sigma_X, \sigma_X(h) \models K_i\varphi$. The other direction is similar.

For the event modality, assume that $\mathcal{H}, h \models \langle\epsilon\rangle\psi$, where $\epsilon \in X$. Then $\mathcal{H}, h\epsilon \models \psi$. By the IH, $\mathcal{H}\sigma_X, \sigma_X(h\epsilon) \models \psi$. However, since $\epsilon \in X$, we have $\mathcal{H}\sigma_X, \sigma(h)\epsilon \models \psi$. This gives $\mathcal{H}\sigma_X, \sigma_X(h) \models \langle\epsilon\rangle\psi$, as desired. The other direction is similar. QED

Note that, if we also replaced the pointed event models in X that occur in the given model, the truth of the formulas might not be preserved, since the truth definitions of the event model operator explicitly refer to given event models. To see this, suppose $\mathcal{H}, h\epsilon \models \langle\epsilon\rangle\varphi$. If we replaced ϵ in the model with the pointed event model ϵ' of the same type, but distinct from ϵ, $\langle\epsilon\rangle\varphi$ cannot be true by definition, simply because $\epsilon \neq \epsilon'$.

4. Extending TDEL with the Past Modality

One fact about TDEL is that it only has forward-looking operators $\langle\epsilon\rangle$. However, given that, in TDEL, we have the forest structres that encodes all successive stages of update by event models, we can naturally think about the operator that states what *was* the case prior to a given temporal point. In this section, we extend TDEL with a past-looking modality P_ϵ with $\epsilon \in \mathbb{E}$. This extension will be called TDEL+P. Also, given an e-closed subset X of $\mathbb{E}$, we denote the corresponding fragment of TDEL+P by TDEL+P(X).

Let $\mathcal{H} = (\mathsf{H}, \sim, V)$ be an ETL-model generated from an epistemic model and a state-dependent protocol. The semantics of the operator P_ϵ is defined as follows:

$$\mathcal{H}, h \models P_\epsilon\varphi \text{ iff } \exists h' \text{ such that } h = h'\epsilon \text{ and } \mathcal{H}, h' \models \varphi.$$

The dual of P_ϵ is denoted by $\hat{P}_\epsilon$. The reading of P_ϵ is "the event ϵ has happened, before which φ". The dual $\hat{P}_\epsilon$ reads as "Before the event ϵ, φ".

Let t_{PAL} be the type of event models consisting of single reflexive events. Below we show that, given an e-closed subset X of $\mathbb{E}$ such that X is a union of a finite number of types including t_{PAL}, TDEL+P(X) is axiomatizable. For this, we first observe that the normalization results hold for TDEL+P(X).

Proposition 4.1 *Let Y be an e-closed subset of $\mathbb{E}$. Then, for every DEL-generated model $\mathcal{H}$ and every formula φ in* TDEL+P*(X),*

$$\mathcal{H}, h \models \varphi \textit{ iff } \mathcal{H}\sigma_Y, \sigma_Y(h) \models \varphi.$$

Proof. We proceed by induction on φ. The cases other than P_ϵ are as in Lemma 3.7. Thus, assume $\mathcal{H}, h \models P_\epsilon\psi$. Then there must be some h' such that $h'\epsilon$ and $\mathcal{H}, h' \models \psi$. By the IH, $\mathcal{H}\sigma_Y, \sigma_Y(h') \models \psi$. Since $\epsilon \in Y$, $\sigma_Y(h'\epsilon) = \sigma(h\prime)\epsilon$. Thus, $\mathcal{H}\sigma_Y, \sigma_Y(h'\epsilon) \models P_\epsilon\psi$. The other direction is similar. QED

To present the axiomatization of TDEL+P, we need some definitions.

Definition 4.2 Given a formula φ, the *past depth* $d(\varphi)$ of the formula φ is defined as follows:

- $d(p) = 0$ for p propositional.
- $d(\neg\varphi) = d(\varphi)$
- $d(\varphi \wedge \psi) = max\{d(\varphi), d(\psi)\}$
- $d(K_i\varphi) = d(\varphi)$
- $d(\langle\epsilon\rangle\varphi) = d(\varphi) - 1$
- $d(P_\epsilon\varphi) = \max(d(\varphi), 0) + 1$

$\triangleleft$

The intuition behind this definition is that if a formula has a depth n, we would have to go n-steps into the past from the current point of the ETL-tree in order to verify it. Thus, the final clause reflects the intended meaning. Had the definition instead been $d(P_\epsilon\varphi) = d(\varphi) + 1$, this would not have worked for, $P_{(\mathcal{E}_1, e_1)}\langle\mathcal{E}_2, e_2\rangle\langle\mathcal{E}_3, e_3\rangle P$. That definition would mistakenly have set the past depth as -1 instead of 1.

Let X be a union of a finite number of types such that $t_{PAL} \subseteq X$, so X is a class of event models.

Definition 4.3 Given a finite set Σ of expressions in $\mathcal{L}_{\mathsf{TDEL+P}}$ and a type t, define $\mathbb{E}(\Sigma) := \bigcup_{\varphi\in\Sigma}\mathbb{E}(\varphi)$. Also denote by PRE_Σ the conjunction of the preconditions of the events in $\mathbb{E}(\Sigma)$. $\triangleleft$

Definition 4.4 Given a type $t \subseteq X$, let $\mathcal{E}^t_\Sigma$ be a distinguished event of the type t in which the precondition of each event is the tautologous formula of the form $Pre_\Sigma \vee \neg Pre_\Sigma$. The role of $\mathcal{E}^t_\Sigma$ is to pick up one event model of the type t, whose precondition is tautologous and whose pointed event model is not in Σ. The form of the precondition is to prevent the pointed event model formed by $\mathcal{E}^t_\Sigma$ from being in Σ. $\triangleleft$

Definition 4.5 Further, define the set $N_X(\Sigma)$ by:

$$N_X(\Sigma) := \{(\mathcal{E}^t, e) \mid t \subseteq X \text{ is a type and } e \text{ in } \mathcal{E}^t_\Sigma\}.$$

$\triangleleft$

Here, given the definition of $\mathcal{E}^t_\Sigma$, there are infinitely many event models that can be specified as $\mathcal{E}^t_\Sigma$, since there are infinitely many event models of the type t in which the preconditions of events are $Pre_\Sigma \vee \neg Pre_\Sigma$. By definition, isomorphic event models are distinct when they consist of distinct events. Therefore, clearly, there are infinitely many pair-wise *disjoint* sets defined to be $N_X(\Sigma)$ as defined above, depending on which event model will be taken as $\mathcal{E}^t_\Sigma$.

Let $A_1, A_2, \ldots$ be an infinite sequence of such sets, i.e. (1) A_i is of the form defined by $N_X(\Sigma)$ and (2) A_i, A_j are disjoint for every i, j. Define $N^n_X(\Sigma)$ be the union of $A_1, \ldots, A_n$. Clearly, $N^n_X(\Sigma)$ is finite, since A_i is finite for all i and $N^n_X(\Sigma)$ is a finite union of such sets.

Definition 4.6 Also, given a finite set Σ of expressions and a formula φ, define $\epsilon_{\Sigma}^{\top}$ to be an pointed event model in t_{PAL} in which the precondition of the event in the model is $PRE_{\Sigma} \vee PRE_{\Sigma}$. Given the form of the precondition in the definition, $\epsilon_{\Sigma}^{\top}$ does not occur in Σ. ◁

Definition 4.7 The axiomatization of TDEL+P extends that of epistemic logic with necessitation for $[\epsilon]$ and $\hat{P}_{\epsilon}$ and the following axioms and inference rules:

F5 $\langle\epsilon\rangle P_{\epsilon'}\varphi \to \bot$ if $\epsilon \neq \epsilon'$

F6 $\langle\epsilon\rangle P_{\epsilon}\varphi \leftrightarrow \langle\epsilon\rangle\top \wedge \varphi$

A3 $P_{\epsilon}(\varphi \to \psi) \to (P_{\epsilon}\varphi \to P_{\epsilon}\psi)$

R(X) If $\vdash [\epsilon_1] \dots [\epsilon_{d(\varphi)}]\varphi$ for all $\epsilon_1 \dots \epsilon_{d(\varphi)}$ such that, for all k $(1 \leq k \leq d(\varphi))$, $\epsilon_k \in \mathbb{E}(\varphi) \cup N_X^{d(\varphi)}(\mathbb{E}(\varphi) \cup \{\epsilon_{\mathbb{E}(\varphi)}^{\top}\}$, then $\vdash \varphi$.

◁

Note that $\mathbb{E}(\varphi) \cup N_X^n(\mathbb{E}(\varphi)) \cup \{\epsilon_{\mathbb{E}(\varphi)}^{\top}\}$ is finite. Also, to show the soundness of **R(X)**, it suffices to show the following:

Lemma 4.8 *If φ is satisfiable, then $\langle\epsilon_1\rangle \dots \langle\epsilon_{d(\varphi)}\rangle\varphi$ is satisfiable for some sequence $\epsilon_1 \dots \epsilon_{d(\varphi)}$ of the specified form in* R(X).

To show this lemma, we need some definitions. Let p be a state-dependent protocol on $\mathcal{M}$.

Definition 4.9 Given $n \in \mathbb{N}$, we define a local protocol $\mathsf{p}_{n<}$ on $\mathcal{M}^{n,\mathsf{p}}$ so that $p_{n<}(w\sigma_1 \dots \sigma_n) = \{\tau \mid w\sigma_1 \dots \sigma_n\tau \in \mathsf{p}(w)$ where $w \in Dom(\mathcal{M})\}$. ◁

Given an ETL-model Forest$(\mathcal{M}, \mathsf{p})$, the model Forest$(\mathcal{M}^{n,\mathsf{p}}, \mathsf{p}_{n<})$ can be seen as a submodel of Forest$(\mathcal{M}, \mathsf{p})$ that describes what happens in Forest$(\mathcal{M}, \mathsf{p})$ after the $n+1$-th stage, with the histories up to the $n+1$-th stage taken as the elements of the base epistemic model.

Now we prove Lemma 4.8. The idea behind the proof is as follows. Assuming $\mathcal{H}, h \models \varphi$, we first apply the normalization method based on Proposition 4.1. Then, if φ is satisfied in the model at a sufficiently long history (i.e. strictly longer than $d(\varphi)$), then we can satisfy $\langle\epsilon_1\rangle \dots \langle\epsilon_{d(\varphi)}\rangle\varphi$ by tracing the history using the truth definition of the future operator. If any ϵ_i in the sequence is not of the form specified in $R(X)$, then in the model $\mathcal{H}$ we can replace it with an event model of the same type with tautologous preconditions. Such a replacement does not affect the structure of the model, and $\langle\epsilon_1\rangle \dots \langle\epsilon_{d(\varphi)}\rangle\varphi$ will be satisfied at the corresponding node in the resulted model.

However, if the history is not long enough, then we construct a new model from the original, by lifting the roots of the trees with a sequence of single reflexive event models $\epsilon_{\mathbb{E}(\varphi)}^{\top}$ with the tautologous precondition. The new model preserves the structures above the sequence of such events and there is a sufficiently long history at which φ is satisfied. The preservation result follows because iteratively performing single reflexive events with tautologous preconditions (uniformly at every world) keeps the structure of the original model unchanged.

To illustrate this, consider the evaluation of the formula $\varphi = P_{\sigma}\neg P_{\tau}\top$, with past depth 2, in Figure 4. Notice that we can satisfy this formula at world $w\sigma$ in Figure 4, even though $\mathsf{len}(w\sigma) = 2$. To obtain a length of 3 for the history at which the formula in question is satisfied, we add a public announcement with a tautologous precondition, $\epsilon_{\varphi}^{\top}$. This is represented in Figure 5. We now proceed to the proof.

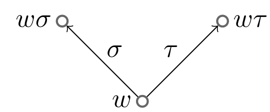

Figure 4. A formula with depth 2 can be satisfied at $w\sigma$. This is a case in which we need to extend the history.

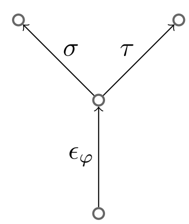

Figure 5. Extending the history with ϵ_{φ}.

Proof. Let $\mathcal{H}, h \models \varphi$. Apply Proposition 4.1 by setting $Y := \mathbb{E}(\varphi)$. Then we obtain $\mathcal{H}\sigma_Y, \sigma_Y(h) \models \varphi$.

Assume $\mathsf{len}(h) > d(\varphi)$. Then for some $g, \epsilon_1, \dots \epsilon_{d(\varphi)}$, $h = g\epsilon_1 \dots \epsilon_{d(\varphi)}$. In $\mathcal{H}\sigma_Y$, for every $\sigma_Y(\epsilon_i) \notin Y$ $(1 \leq i \leq d(\varphi))$, replace $\sigma_Y(\epsilon_i)$ with an isomorphic event model $\epsilon \in N_X(\mathbb{E}(\varphi))$. Given that the preconditions of the event models are tautologous, such a model transformation does not affect the truth value of φ. That is, denoting by $\mathcal{H}'$ and h' the model and the history (corresponding to h) that are obtained by the replacements, we have $\mathcal{H}', h' \models \varphi$. By $\mathsf{len}(h) = \mathsf{len}(h') > d(\varphi)$ and the construction of h', we have some g' and $\epsilon'_1, \dots, \epsilon'_n$ such that $\mathcal{H}', g' \models\rangle\epsilon'_1\rangle \dots \langle\epsilon'_{d(\varphi)}\rangle\varphi$, where $\epsilon'_1, \dots, \epsilon'_{d(\varphi)}$ are of the specified form in $R(X)$.

Thus, assume that $\mathsf{len}(h) \leq d(\varphi)$. Let $k := d(\varphi) - \mathsf{len}(h) + 1$ (the length that we want to add to the history). Let ϵ_0 be $\epsilon_{\mathbb{E}(\varphi)}^{\top}$. Also denote by ϵ_0^k the sequence of k ϵ_0's.

Now let $\mathcal{M} = (W, \sim, V)$. Construct a local protocol p^+ on $\mathcal{M}$ so that $\mathsf{p}^+(w)$ is the set obtained by taking the closure under finite prefix on $\{\epsilon_0^k\sigma \mid \sigma \in \mathsf{p}(w)\}$. Then, by these constructions, it is the case that for all σ (possibly empty):
$\mathsf{Forest}(\mathcal{M}^k, \mathsf{p}^+_{e_0^k<}), (w\epsilon_0^k)\sigma \models \varphi$ iff
$\mathsf{Forest}(\mathcal{M}, \mathsf{p}), w\sigma \models \varphi$
where w is in $\mathcal{M}$. Thus, if we have, for all σ,
$\mathsf{Forest}(\mathcal{M}^k, \mathsf{p}^+_{e_0^k<}), (w\epsilon_0^k)\sigma \models \varphi$ iff
$\mathsf{Forest}(\mathcal{M}, \mathsf{p}^+), w\epsilon_0^k\sigma \models \varphi$.
The desired claim follows. For we can proceed as in the case of $\mathsf{len}(h) > d(\varphi)$, given that
$\mathsf{len}(w\theta\tau) = \mathsf{len}(w) + [d(\varphi) - \mathsf{len}(w\tau) + 1] + \mathsf{len}(\tau) = d(\varphi) + 1$ where $h = w\tau$.

We prove this by showing that, for all σ and formulas ψ,
$\mathsf{Forest}(\mathcal{M}^k, \mathsf{p}^+_{e_0^k<}), w\epsilon_0^k\sigma \models \psi$ iff
$\mathsf{Forest}(\mathcal{M}, \mathsf{p}^+), w\epsilon_0^k\sigma \models \psi$.

The proof is by a straightforward induction. We will only do the past-modality case. The left-to-right direction follows immediately by the IH. So assume the RHS. If h is non-empty, then by the IH we are done. If h is empty, then since $\sigma \neq \epsilon_\varphi$ by definition, we have a contradiction with the RHS. This completes the proof. QED

The completeness proof can be given based on the Henkin-style construction given for TDEL above. Let $\mathcal{H}^{can}$ be the ETL-model constructed from the set of TDEL+P maximally consistent sets in the same way as in TDEL. The lemma for the canonical model that corresponds to Lemma 2.10 can be shown in the same way. Now, we show the truth lemma stated as follows:

Lemma 4.10 (Truth Lemma) *For every formula φ and $h \in \mathcal{H}^{can}$ such that $\mathsf{len}(h) > d(\varphi)$,*

$$\varphi \in \lambda(h) \text{ iff } \mathcal{H}^{can}, h \models \varphi$$

Proof. The boolean and knowledge modality cases are given in the same way as Lemma 2.13 above, so we will only consider the past modality case. Let $h = h'\sigma$ for some $\mathsf{len}(h) \geq d(\varphi) + 1$, where $\sigma \in \mathbb{E}$. Let φ be $P_\tau\psi$.

Assume then that $P_\tau\psi \in \lambda(h)$. By the definition of canonical model, $\langle\sigma\rangle P_\tau\psi \in \lambda(h')$. If $\sigma \neq \tau$, then by F5, $\bot \in \lambda(h')$, which contradicts the consistency of $\lambda(h')$. Thus, assume $\sigma = \tau$. Then, by F6, we have $\psi \in \lambda(h')$. By the IH, $\mathcal{H}^{can}, h' \models \psi$ (note $\mathsf{len}(h') \geq d(\psi) + 1$). Since $h'\sigma \in \mathcal{H}^{can}$ and $\sigma = \tau$, the truth definition implies that $\mathcal{H}^{can}, h \models P_\tau\psi$.

For the other direction, assume that $\mathcal{H}^{can}, h \models P_\tau\psi$. By the truth definition, we have $\sigma = \tau$, and also $\mathcal{H}, h' \models \psi$. By the IH, we have $\psi \in \lambda(h')$. And by the construction of the canonical model, we have $\langle\sigma\rangle\top \in \lambda(h')$. Thus, by F6, we have $\langle\sigma\rangle P_\sigma\psi \in \lambda(h')$, which by construction implies that $P_\sigma\psi \in \lambda(h)$. QED

We can also prove the lemma corresponding to Lemma 2.14 in the same way. Now, to conclude our proof of the completeness result, we need to prove the following theorem.

Theorem 4.11 TDEL+P *is complete with respect to $\mathbb{F}_{sd}(\mathbb{E})$.*

Proof. Let φ be consistent. Then $\langle\sigma_1\rangle \ldots \langle\sigma_{d(\varphi)}\rangle\varphi$ is consistent for some $\sigma_1 \ldots \sigma_{d(\varphi)} \in \mathbb{E}^*$. For suppose otherwise. Then for every $\sigma_1 \ldots \sigma_{d(\varphi)} \in \mathbb{E}^*$, $\langle\sigma_1\rangle \ldots \langle\sigma_{d(\varphi)}\rangle$ is inconsistent and thus $\vdash [\sigma_1] \ldots [\sigma_{d(\varphi)}]\neg\varphi$. By R, $\vdash \neg\varphi$. This contradicts the consistency of φ. Thus $\langle\sigma_1\rangle \ldots \langle\sigma_{d(\varphi)}\rangle\varphi$ is consistent for some $\sigma_1 \ldots \sigma_{d(\varphi)}$. Let $\theta = \langle\tau_1\rangle \ldots \langle\tau_{d(\varphi)}\rangle\varphi$ be one of those formulas. Since θ is consistent, by Lindenbaum's Lemma, we have a maximally consistent set containing it. Note that $d(\theta) = 0$. Thus, by the truth lemma, there is some history h of length 1 such that $\mathcal{H}^{can}, h \models \theta$. This gives us the result that $\mathcal{H}^{can}, h\tau_1 \ldots \tau_{d(\varphi)} \models \varphi$. QED

The reason that we cannot conclude the result immediately from the truth lemma and the analogue of Lemma 2.14 is that we are not sure that, given a formula of depth n, we have a maximal consistent set that contains φ, which is assigned to a history long enough to apply truth lemma. This fact is guaranteed by R, as is seen in the above argument.

5. Philosophical Connections and Applications

Although the addition of a past operator to the temporal framework may seem trivial, it turns out that the resulting increase in expressive power might have several significant applications. The interaction between past and future in an epistemic context can be found in thinking about agency—more specifically, in trying to formulate a definition of an agent's intention—as well as in learning.

Both of these seem at first glance to be forward looking ideas. For instance, intending seems to refer only to something we plan to do in the future. And learning seems to have to do with an update of our state of knowledge. But notice that if we intend to bring something about, it can't already have been the case (since we can't intend to do something that's already been done). And if we want to learn something, we can't already know it. Thus, expressing both of these ideas requires talking about a *change* in our epistemic states. It is not too difficult to come up with a sentence using only the future modality and the static language stating that I am about to learn that φ, or that I do not now know φ, but will after it is announced:

$$\langle !\varphi\rangle K_i\varphi \wedge \neg K_i\varphi$$

Alternately, we can use this formalism to capture our intuitions about what is learned by a public announcement of a formula φ. For what we learn is not necessarily that φ is now the case, but rather than φ was the case before the announcement. So our general formulation of what an agent learns by a public announcement can be expressed by the formula

$$[!\varphi]K_i P_{!\varphi}\varphi.$$

So in order to say that I have in fact learned φ, I need to refer back to the past. Otherwise, all I will be able to say is that I now know φ. But the fact that I now know φ tells me nothing about whether or not I knew it in the past. Thus, in order to claim that I have learned φ, because of some event ϵ I really need to say that I now know φ, but did not know it before ϵ took place:

$$P_\epsilon \neg K_i\varphi \wedge K_i\varphi.$$

The fact that a past modality is required to express that a state of affairs has changed means that it is also related to the idea of a successful update [11]. We can call a public announcement successful when the formula announced is true after the update, and unsuccessful when the formula announced becomes false. For instance, in the familiar Muddy Children example, the announcement by all the children that they do not know their state becomes false afterwards.

Another example of announcements which result in unsuccessful updates are Moore sentences, such as $p \wedge \neg K_i p$, or "p is the case, but i doesn't know it." For after that is announced, i will know that p is the case, and the original formula will become false. So as above, all we know is that $p \wedge \neg K_i p$ was true before it was announced. So even though the formula $K_i(p \wedge \neg K_i p)$ remains inconsistent in epistemic logic, the formula

$$K_i P_{!(p\wedge\neg K_i p)}(p \wedge \neg K_i p)$$

is satisfiable in dynamic epistemic logic, for instance, in a model like the one given in Figure 6.

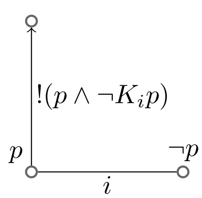

Figure 6. The public announcement of a Moore sentence. At the updated world, it is the case that $K_i P_{!(p\wedge\neg K_i p)}(p \wedge \neg K_i p)$.

So although an agent can never know that p is the case, but she herself does not know it, she can know that it *once was true* that p was the case and she then did not know it.

Now, we might think that the opposite of learning is forgetting, and wonder if this too is something that can be formalized by our models. After all, if we can express that an agent learned that φ after ϵ took place by saying $P_\epsilon \neg K_i\varphi \wedge K_i\varphi$, perhaps we could express that after ϵ, an agent forgot that φ by moving the negation:

$$P_\epsilon K_i\varphi \wedge \neg K_i\varphi.$$

But even though this sentence is expressible, the logic itself does not yet allow for a general way to model agents who can forget. For in the current models, such a sentence would only be satisfiable for a limited class of φ. For instance, it could never be true for a proposition letter. Since we have persistence for proposition letters across updates, once an agent knows that p, he can never forget it after an event. The reason for this is the fact that updates only ever erase uncertainties between worlds, or maintain existing uncertainties. In order to model forgetting, we would require an update mechanism that allowed for adding uncertainties between worlds which were not previously present. There are several different options for implementing such a mechanism, which are beyond the scope of this paper to discuss. However, this avenue seems like another fruitful path to investigate in terms of dynamic epistemic systems with temporal operators.

6. Conclusion

We have shown that, even if we extend the setting of TPAL presented in [8] to the full class of event models, the completeness proof can be given based on the proof given for TPAL in [8]. Also the extension TDEL+P can be axiomatized by the method of normalization for DEL-generated ETL-models.

But these are not the only extensions which suggest themselves for investigation. For instance, in TDEL+P, we only have *labeled* past and future operators. So natural further steps would be to add in an un-indexed past operator, expressing "yesterday", and an un-indexed future operator, expressing "tomorrow". We can look at these operators as quantifying over event models. It turns out that a system TADEL with the "tomorrow" operator can be axiomatized without too many problems, as it can be seen as a generalization of the system TAPAL studied in [4], which has an operator quantifying over public announcements. These results will be presented in forthcoming work by Hoshi, which will demonstrate the way in which the normalization method can be applied to axiomatize TADEL.

Perhaps surprisingly, though, the addition of a "yesterday" operator is not as straightforward, since the normalization method would not work as given. In particular, the method whereby we extend the history with ϵ_φ as illustrated in Figure 5 would not necessarily work for formulas in a language with a "yesterday" operator. For where we can satisfy $P_\sigma \neg P_\tau \top$ in a world with length 2, the formula $P \neg P \top$

can only be satisfied in a world with length 1. So the history could not be lengthened in a world in which the latter was satisfied without changing its truth value.

Other natural extensions include iterated past modality P^*, where the * is the Kleene star operator. In the case of the iterated future modality of the kind, say $\Diamond^*$, "There is some sequence of events after which...", the result in [5] suggests that such an operator results in incompleteness when combined with the common knowledge operator. It is interesting to see if this is also the case for the case of the iterated past-modality P^*.

There are distinct motivations also for considering an extension of TDEL+P together with a common knowledge operator. The considerations raised about learning in the previous section apply just as well to agents' common knowledge after an announcement, since we can also express what becomes common knowledge by the following formula:

$$[!\varphi]C_G P_{!\varphi}\varphi.$$

Further, the relativized common knowledge operator from [9] $C_G(\varphi, \psi)$, which expresses that every G-path which consists exclusively of φ worlds ends in a ψ world, also has a very natural interpretation in past language. One way to paraphrase this operator in natural language is "If φ were announced, it would be common knowledge among G that ψ was the case before the announcement." This is expressible in the past language.

$$C_G(\varphi, \psi) \equiv [!\varphi]C_G P_{!\varphi}\psi$$

Thus, there are many potentially fruitful extensions of the system considered here, which will certainly be the subject of future investigation.

References

[1] A. Baltag, L. Moss, and S. Solecki. The logic of public announcements, common knowledge and private suspicions. In I. Gilboa, editor, *TARK 1998*, number 43-56, 1998.

[2] R. Fagin, J. Halpern, Y. Moses, and M. Vardi. *Reasoning about Knowledge*. Synthese Library. MIT Press, Boston, 1995.

[3] J. Gerbrandy. *Bisimulations on Planet Kripke*. PhD thesis, ILLC, 1999.

[4] T. Hoshi. Logics of public announcement with constrained protocols. LOFT, 2008.

[5] J. Miller and L. Moss. The undecidability of iterated modal relativization. *Studia Logica*, 79:373–407, 2005.

[6] R. Parikh and R. Ramanujam. A knowledge based semantics of messages. *Journal of Logic, Language, and Information*, 12:453–467, 2003.

[7] Y. Shoham and K. Leyton-Brown. Multiagent systems: Algorithmic, game-theoretic, and logical foundations. 2008.

[8] J. van Benthem, J. Gerbrandy, T. Hoshi, and E. Pacuit. Merging frameworks for interaction: DEL and ETL. 2007.

[9] J. van Benthem, J. van Eijck, and B. Kooi. Common knowledge in update logics. In *Proceedings of the 10th Conference on Theoretical Aspects of Rationality and Knowledge*. 2005.

[10] J. van Benthem, J. van Eijck, and B. J. Kooi. Logic of communication and change. *Information and Computation*, 204(11):1620–1662, 2006.

[11] H. van Ditmarsch and B. Kooi. The secret of my success. *Synthese*, 151(2):201–232, 2006.

[12] A. Yap. Dynamic epistemic logic and temporal modality. University of Victoria, 2007.