A Proof of Arrow's Impossibility Theorem

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October 03, 2008

Logical Methods in the Humanities

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Arrow's Impossibility Theorem, Origins and Beyond

Social Choice & Individual Values by Kenneth J. Arrow (1951)

We use a proof in

"Three Brief Proofs of Arrow's Impossibility Theorem." John Geanakoplos. *Economic Theory*, **26**, 2005

Arrow's Impossibility Theorem

Let X be a finite set of Alternative preferences with *at least three elements*.

Assume each individual has a transitive and complete preference over X (ties are allowed).

Consider a society with N individuals, each with a transitive preference over X.

A constitution is a function which associates with each N-tuple (or profile) of preferences a transitive preference called the social preference.

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Arrow's Theorem, Preliminaries

- Transitivity
- Unanimity: if every individual puts α strictly above β, society also puts α strictly above β
- Independence of Irrelevant Alternatives(IIA): If the social relative ranking of two alternatives α and β only depends on *their* relative ranking by every individual
- Dictatorship by individual n if for every pair α and β society strictly prefers α to β whenever n strictly prefers α to β.

Arrow's Theorem

Theorem (Arrow, 1951) Any constitution that respects transitivity, independence of irrelevant alternatives, and unanimity is a dictatorship.

Does not apply to

- Majority voting: transitivity
- Borda count: IIA

Arrow's Theorem: Proof 1 (Geanakoplos, 2005)-I

- Assume $X = \{A, B, \ldots, C\}$
- Lemma: For any profile in which every individual puts alternative B at the very top or the very bottom of his ranking, society must as well.
- Proof: If not then B is in an intermediate position. For example $A \geq B$ and $B \geq C$. By IIA this holds if all the individuals move C above A, because this can be done without changing any A, B or B, C relations; this follows from B being at the extremes of each individual's personal preferences. By unanimity $C > A$ while by transitivity $A \geq C$: contradiction. So B should be either at the top or the bottom of the social preference.

Arrow's Theorem: Proof 1 (Geanakoplos, 2005)-II

- There exists a individual $n^* = n(B)$ that by changing her vote at some profile she can move B from the bottom of the social ranking to the top.
- Let's assume each individual puts B at the bottom of their ranking, by unanimity, the society does the same.
- Now beginning from the individual 1, we start to move B from the bottom of each individual's ranking to the the top. According to the above lemma, B's social ranking will be either at top or the bottom. Assume n^* is the the first individual whose moving B causes the social ranking of B to flip (by unanimity the change will happen the latest with $n^* = N$)

Arrow's Theorem: Proof 1 (Geanakoplos, 2005)-III

- Name the profile of rankings just before the individual n^* moves B profile I, the one after her moving B to the top profile II (in profile I, B is at the bottom of social ranking, in profile II at the top)
- Now we argue that n^* is the dictator over any pair A, C not involving B.
- To see this, let's assume n^*, moves A above B in profile II to have $A >_{n^*} B >_{n^*} C$ and let other individuals $n \neq n^*$ rearrange all their rankings of A and C while keeping Bs in place. By IIA $A > B$ because all A, B relations are like in profile I. The same way $B > C$ because all B, C relations are like profile II. By transitivity society should have $A > C$ agreeing with n^*'s preferences.

Arrow's Theorem: Proof 1 (Geanakoplos, 2005)-IV

- For any other pairs such as A, B, using the same argument as above there is a $n^*(C)$ that is the dictator, but $n^*(B)$ can dictate A, B rankings in profiles I and II, therefore $n^*(C)$ and $n^*(B)$ should be the same.