Decidability of the PAL Substitution Core

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Introduction

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- However, many dynamic logics axiomatized using reduction axioms are not substitution-closed.
- A classic example is Public Announcement Logic (PAL).

Not in the Core...

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Since $[\varphi]\varphi$ is not valid for arbitrary φ, $[p]p$ is not "schematically valid."

It is not in the substitution core.

Not in the Core...

Example

Recall the PAL reduction axioms:

1. $\langle\varphi\rangle p \leftrightarrow (\varphi \wedge p)$
2. $\langle\varphi\rangle(\psi \wedge \chi) \leftrightarrow (\langle\varphi\rangle\psi \wedge \langle\varphi\rangle\chi)$
3. $\langle\varphi\rangle\neg\psi \leftrightarrow (\varphi \wedge \neg\langle\varphi\rangle\psi)$
4. $\langle\varphi\rangle\Diamond\psi \leftrightarrow (\varphi \wedge \Diamond\langle\varphi\rangle\psi)$

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The formula $\langle q\rangle p \leftrightarrow (q \wedge p)$ is valid. However, its substitution instance $\langle p\rangle Kp \leftrightarrow (p \wedge Kp)$ is not valid, so it is not schematically valid.

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Plus: $\langle\varphi\rangle\langle\psi\rangle\chi \leftrightarrow \langle\langle\varphi\rangle\psi\rangle\chi$

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Call a formula **purely epistemic** if every propositional variable and occurrence of $\langle \varphi \rangle$ appears within the scope of a $\Diamond$. Note that if φ is purely epistemic, so is any substitution instance of φ.

In the Core...

Proposition

Formulas 1-3 are schematically valid. In 2 and 3, χ is purely epistemic.

1. $\Box\varphi \rightarrow (\psi \rightarrow \langle\varphi\rangle\psi)$
2. $\langle(\chi \vee \varphi_1) \wedge \varphi_2\rangle\psi \leftrightarrow (\chi \wedge \langle\varphi_2\rangle\psi) \vee (\neg\chi \wedge \langle\varphi_1 \wedge \varphi_2\rangle\psi)$
3. $\langle(\chi \wedge \varphi_1) \vee \varphi_2\rangle\psi \leftrightarrow (\chi \wedge \langle\varphi_1 \vee \varphi_2\rangle\psi) \vee (\neg\chi \wedge \langle\varphi_2\rangle\psi)$

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3. $\langle(\chi \wedge \varphi_1) \vee \varphi_2\rangle\psi \leftrightarrow (\chi \wedge \langle\varphi_1 \vee \varphi_2\rangle\psi) \vee (\neg\chi \wedge \langle\varphi_2\rangle\psi)$

Other schematic validities can be derived from these and the previous.

The Question

Question 1 from "Open Problems in Logical Dynamics":

Is the substitution core of PAL is **decidable**?

van Benthem, J. (2006).
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In Gabbay, D., Goncharov, S., and Zakharyashev, M., editors, *Mathematical Problems from Applied Logic I*, pages 137–192. Springer.

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We have proven the following:

Theorem

The substitution core of single-agent PAL *is decidable.*

Thank you!

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