Two Envelopes Revisited

- The "two envelopes" problem set-up
  - Two envelopes: one contains \$X, other contains \$2X
  - You select an envelope and open it
    - Let Y = \$ in envelope you selected
    - Let Z = \$ in other envelope

    $E[Z \mid Y] = \frac{1}{2}\cdot\frac{Y}{2} + \frac{1}{2}\cdot 2Y = \frac{5}{4}Y$
  - Before opening envelope, think either <u>equally</u> good
    - So, what happened by opening envelope?
  - E[Z | Y] above assumes all values X (where $0 < X < \infty$) are equally likely
    - Note: there are infinitely many values of X
    - So, not true probability distribution over X (doesn't integrate to 1)

Subjectivity of Probability

- Belief about contents of envelopes
  - Since implied distribution over X is not a true probability distribution, what is our distribution over X?
    - *Frequentist*: play game infinitely many times and see how often different values come up.
    - <u>Problem</u>: I only allow you to play the game *once*
  - Bayesian probability
    - Have <u>prior</u> belief of distribution for X (or anything for that matter)
    - Prior belief is a *subjective* probability
      - By extension, ***all*** probabilities are subjective
    - Allows us to answer question when we have no/limited data
      - E.g., probability a coin you've never flipped lands on heads
    - As we get more data, prior belief is "swamped" by data

The Envelope, Please

- *Bayesian*: have prior distribution over X, P(X)
  - Let Y = \$ in envelope you selected
  - Let Z = \$ in other envelope
  - Open your envelope to determine Y
  - If Y > E[Z | Y], keep your envelope, otherwise switch
    - No inconsistency!
  - Opening envelope provides data to compute P(X | Y) and thereby compute E[Z | Y]
  - Of course, there's the issue of how you determined your prior distribution over X…
    - Bayesian: Doesn't matter how you determined prior, but you *must* have one (whatever it is)
    - Imagine if envelope you opened contained \$17.51

Revisting Bayes Theorem

- Bayes Theorem (θ = model parameters, D = data):

"Posterior" "Likelihood" "Prior"

$$P(\theta \mid D) = \frac{P(D \mid \theta)\, P(\theta)}{P(D)}$$

  - <u>Likelihood</u>: you've seen this before (in context of MLE)
    - Probability of data given probability model (parameter θ)
  - <u>Prior</u>: before seeing any data, what is belief about model
    - I.e., what is *distribution* over parameters θ
  - <u>Posterior</u>: after seeing data, what is belief about model
    - After data D observed, have posterior distribution $p(\theta \mid D)$ over parameters θ conditioned on data. Use this to predict new data.
    - Here, we assume prior and posterior distribution have same parametric form (we call them "conjugate")

Computing P(θ | D)

- Bayes Theorem (θ = model parameters, D = data):

$$P(\theta \mid D) = \frac{P(D \mid \theta)\, P(\theta)}{P(D)}$$

- We have prior P(θ) and can compute P(D | θ)
- But how do we calculate P(D)?
  - Complicated answer: $P(D) = \int P(D \mid \theta) P(\theta)\, d\theta$
  - Easy answer: It is does not depend on θ, so ignore it
    - Just a constant that forces P(θ | D) to integrate to 1

P(θ | D) for Beta and Bernoulli

- Prior: θ ~ Beta(*a*, *b*); D = {*n* heads, *m* tails}

$$f_{\theta|D}(\theta = p \mid D) = \frac{f_{D|\theta}(D \mid \theta = p)\, f_\theta(\theta = p)}{f_D(D)}$$

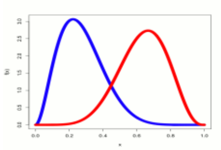
$$= \frac{\binom{n+m}{n} p^n (1-p)^m \cdot \frac{p^{a-1}(1-p)^{b-1}}{C_1}}{C_2} = \frac{\binom{n+m}{n}}{C_1 C_2} p^n (1-p)^m \cdot p^{a-1}(1-p)^{b-1}$$

$$= C_3 p^{n+a-1}(1-p)^{m+b-1}$$

- By definition, this is Beta(*a* + *n*, *b* + *m*)
  - All constant factors combine into a single constant
  - Could just ignore constant factors along the way

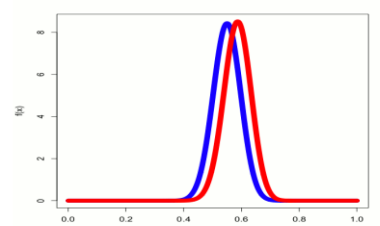
Where'd Ya Get Them P(θ)?

- θ is the probability a coin turns up heads
- Model θ with 2 different priors:
  - $P_1(\theta)$ is Beta(3,8) (blue)
  - $P_2(\theta)$ is Beta(7,4) (red)
- They look pretty different!



- Now flip 100 coins; get 58 heads and 42 tails
  - What do posteriors look like?

It's Like Having Twins



- As long as we collect enough data, posteriors will converge to the correct value!

From MLE to Maximum A Posteriori

- Recall Maximum Likelihood Estimator (MLE) of θ

$$\theta_{MLE} = \arg\max_{\theta} \prod_{i=1}^{n} f(X_i \mid \theta)$$

- Maximum A Posteriori (MAP) estimator of θ:

$$\theta_{MAP} = \arg\max_{\theta}\ f(\theta \mid X_1, X_2, \ldots, X_n) = \arg\max_{\theta} \frac{f(X_1, X_2, \ldots, X_n \mid \theta)\, g(\theta)}{h(X_1, X_2, \ldots, X_n)}$$

$$= \arg\max_{\theta} \frac{\left(\prod_{i=1}^{n} f(X_i \mid \theta)\right) g(\theta)}{h(X_1, X_2, \ldots, X_n)} = \arg\max_{\theta}\ g(\theta) \prod_{i=1}^{n} f(X_i \mid \theta)$$

where g(θ) is prior distribution of θ.

- As before, can often be more convenient to use log:

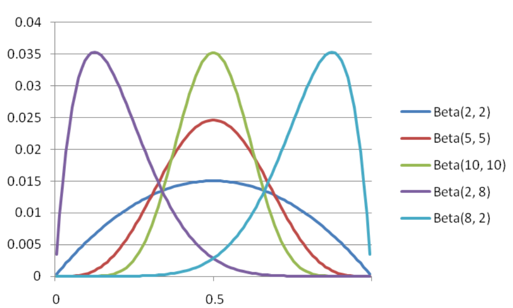
$$\theta_{MAP} = \arg\max_{\theta} \left(\log(g(\theta)) + \sum_{i=1}^{n} \log(f(X_i \mid \theta)) \right)$$

- MAP estimate is the mode of the posterior distribution

Conjugate Distributions Without Tears

- Just for review…
- Have coin with unknown probability θ of heads
  - Our <u>prior</u> (subjective) belief is that θ ~ Beta(*a*, *b*)
  - Now flip coin *k* = *n* + *m* times, getting *n* heads, *m* tails
  - Posterior density: (θ | *n* heads, *m* tails) ~ Beta(*a*+*n*, *b*+*m*)
    - Beta is conjugate for Bernoulli, Binomial, Geometric, and Negative Binomial
  - *a* and *b* are called "hyperparameters"
    - Saw (*a* + *b* – 2) imaginary trials, of those (*a* – 1) are "successes"
  - For a coin you never flipped before, use Beta(*x*, *x*) to denote you think coin likely to be fair
    - How strongly you feel coin is fair is a function of *x*

Mo' Beta



Multinomial is Multiple Times the Fun

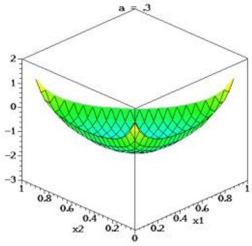
- Dirichlet(a_1, a_2, ..., a_m) distribution
  - Conjugate for Multinomial
    - Dirichlet generalizes Beta in same way Multinomial generalizes Bernoulli/Binomial

$$f(x_1, x_2, \ldots, x_n) = \frac{1}{B(a_1, a_2, \ldots, a_n)} \prod_{i=1}^{n} x_i^{a_i - 1}$$

  - Intuitive understanding of hyperparameters:
    - Saw $\sum_{i=1}^{m} a_i - m$ imaginary trials, with (a_i – 1) of outcome *i*
  - Updating to get the posterior distribution
    - After observing $n_1 + n_2 + \ldots + n_m$, new trials with n_i of outcome *i*...
    - ... posterior distribution is Dirichlet($a_1 + n_1$, $a_2 + n_2$, ..., $a_m + n_m$)

Best Short Film in the Dirichlet Category

- And now a cool animation of Dirichlet(*a*, *a*, *a*)
  - This is actually *log* density (but you get the idea…)



Thanks Wikipedia!

Getting Back to your Happy Laplace

- Recall example of 6-sides die rolls:
  - X ~ Multinomial(p_1, p_2, p_3, p_4, p_5, p_6)
  - Roll n = 12 times
  - Result: 3 ones, 2 twos, 0 threes, 3 fours, 1 fives, 3 sixes
    - MLE: p_1=3/12, p_2=2/12, p_3=0/12, p_4=3/12, p_5=1/12, p_6=3/12
  - Dirichlet prior allows us to pretend we saw each outcome k times before. MAP estimate: $p_i = \frac{X_i + k}{n + mk}$
    - Laplace's "law of succession": idea above with k = 1
    - Laplace estimate: $p_i = \frac{X_i + 1}{n + m}$
    - Laplace: p_1=4/18, p_2=3/18, p_3=1/18, p_4=4/18, p_5=2/18, p_6=4/18
    - No longer have 0 probability of rolling a three!

Good Times With Gamma

- Gamma(α, λ) distribution
  - Conjugate for Poisson
    - Also conjugate for Exponential, but we won't delve into that
  - Intuitive understanding of hyperparameters:
    - Saw α total imaginary events during λ prior time periods
  - Updating to get the posterior distribution
    - After observing n events during next k time periods...
    - ... posterior distribution is Gamma(α + n, λ + k)
    - Example: Gamma(10, 5)
    - Saw 10 events in 5 time periods. Like observing at rate = 2
    - Now see 11 events in next 2 time periods → Gamma(21, 7)
    - Equivalent to updated rate = 3

It's Normal to Be Normal

- Normal(μ_0, σ_0^2) distribution
  - Conjugate for Normal (with unknown μ, known σ^2)
  - Intuitive understanding of hyperparameters:
    - A priori, believe true μ distributed ~ N(μ_0, σ_0^2)
  - Updating to get the posterior distribution
    - After observing n data points...
    - ... posterior distribution is:

$$N\left(\left(\frac{\mu_0}{\sigma_0^2} + \frac{\sum_{i=1}^{n} x_i}{\sigma^2} \right) \Big/ \left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2} \right),\ \left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2} \right)^{-1} \right)$$