Balls, Urns, and the Supreme Court

- Supreme Court case: Berghuis v. Smith

If a group is underrepresented in a jury pool, how do you tell?

- Article by Erin Miller – Friday, January 22, 2010
- Thanks to (former CS109er) Josh Falk for this article

Justice Breyer [Stanford Alum] opened the questioning by invoking the binomial theorem. He hypothesized a scenario involving **"an urn with a thousand balls, and sixty are red, and nine hundred forty are black, and then you select them at random… twelve at a time."** According to Justice Breyer and the binomial theorem, if the red balls were black jurors then **"you would expect… something like a third to a half of juries would have at least one black person"** on them.

- Justice Scalia's rejoinder: "We don't have any urns here."

Justice Breyer Meets CS109

- Should model this combinatorially (X ~ HypGeo)
  - Ball draws not independent trials (balls not replaced)
- Exact solution:

  P(draw 12 black balls) = $\binom{940}{12} \Big/ \binom{1000}{12} \approx 0.4739$

  P(draw ≥ 1 red ball) = 1 – P(draw 12 black balls) ≈ 0.5261
- Approximation using Binomial distribution
  - Assume P(red ball) constant for every draw = 60/1000
  - X = # red balls drawn. X ~ Bin(12, 60/1000 = 0.06)
  - P(X ≥ 1) = 1 – P(X = 0) ≈ 1 – 0.4759 = 0.5240

In Breyer's description, should actually expect just over half of juries to have at least one black person on them

Demo

From Discrete to Continuous

- So far, all random variables we saw were *discrete*
  - Have finite or countably infinite values (e.g., integers)
  - Usually, values are binary or represent a count
- Now it's time for *continuous* random variables
  - Have (uncountably) infinite values (e.g., real numbers)
  - Usually represent measurements (arbitrary precision)
    - Height (centimeters), Weight (lbs.), Time (seconds), etc.
- Difference between how *many* and how *much*
- Generally, it means replace $\sum_{x=a}^{b} f(x)$ with $\int_a^b f(x)dx$

Continuous Random Variables

- *X* is a **Continuous Random Variable** if there is function $f(x) \geq 0$ for $-\infty \leq x \leq \infty$, such that:

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

- *f* is a Probability Density Function (PDF) if:

$$P(-\infty < X < \infty) = \int_{-\infty}^{\infty} f(x)dx = 1$$

Probability Density Functions

- Say *f* is a **Probability Density Function** (PDF)

$$P(-\infty < X < \infty) = \int_{-\infty}^{\infty} f(x)dx = 1$$

  - $f(x)$ is **not** a probability, it is probability/units of X
  - Not meaningful without some subinterval over X

$$P(X = a) = \int_a^a f(x)dx = 0$$

  - Contrast with Probability Mass Function (PMF) in discrete case: $p(a) = P(X = a)$

    where $\sum_{i=1}^{\infty} p(x_i) = 1$ for X taking on values x_1, x_2, x_3, ...

Cumulative Distribution Functions

- For a continuous random variable X, the **Cumulative Distribution Function** (CDF) is:

$$F(a) = P(X < a) = P(X \le a) = \int_{-\infty}^{a} f(x)dx$$

- Density f is derivative of CDF F: $f(a) = \frac{d}{da}F(a)$
- For continuous f and small ε:

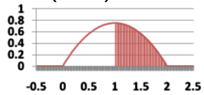
$$P(a - \tfrac{\varepsilon}{2} \le X \le a + \tfrac{\varepsilon}{2}) = \int_{a-\varepsilon/2}^{a+\varepsilon/2} f(x)dx \approx \varepsilon f(a)$$

  - So, ratio of probabilities can still be meaningful:
    - P(X = 1)/P(X = 2) ≈ (ε f(1))/(ε f(2)) = f(1)/f(2)

Simple Example

- X is continuous random variable (CRV) with PDF:

$$f(x) = \begin{cases} C(4x - 2x^2) & \text{when } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$



  - What is C?

$$\int_0^2 C(4x - 2x^2)dx = 1 \;\Rightarrow\; C\left(2x^2 - \frac{2x^3}{3}\right)\Bigg|_0^2 = 1$$

$$C\left(\left(8 - \frac{16}{3}\right) - 0\right) = 1 \;\Rightarrow\; C\frac{8}{3} = 1 \;\Rightarrow\; C = \frac{3}{8}$$

  - What is P(X > 1)?

$$\int_1^{\infty} f(x)dx = \int_1^2 \frac{3}{8}(4x - 2x^2)dx = \frac{3}{8}\left(2x^2 - \frac{2x^3}{3}\right)\Bigg|_1^2 = \frac{3}{8}\left[\left(8 - \frac{16}{3}\right) - \left(2 - \frac{2}{3}\right)\right] = \frac{1}{2}$$

Disk Crashes

- X = days of use before your disk crashes

$$f(x) = \begin{cases} \lambda e^{-x/100} & x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

  - First, determine λ to have actual PDF
    - Good integral to know: $\int e^u du = e^u$

$$1 = \int \lambda e^{-x/100} dx = -100\lambda \int \frac{-1}{100} e^{-x/100} dx = -100\lambda e^{-x/100}\Big|_0^{\infty} = 100\lambda \;\Rightarrow\; \lambda = \frac{1}{100}$$

  - What is P(50 < X < 150)?

$$F(150) - F(50) = \int_{50}^{150} \frac{1}{100} e^{-x/100} dx = -e^{-x/100}\Big|_{50}^{150} = -e^{-3/2} + e^{-1/2} \approx 0.383$$

  - What is P(X < 10)?

$$F(10) = \int_0^{10} \frac{1}{100} e^{-x/100} dx = -e^{-x/100}\Big|_0^{10} = -e^{-1/10} + 1 \approx 0.095$$

Expectation and Variance

For discrete RV X:	For continuous RV X:
$E[X] = \sum_x x\, p(x)$	$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$
$E[g(X)] = \sum_x g(x)\, p(x)$	$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$
$E[X^n] = \sum_x x^n\, p(x)$	$E[X^n] = \int_{-\infty}^{\infty} x^n\, f(x)\, dx$

For both discrete and continuous RVs:

$$E[aX + b] = aE[X] + b$$

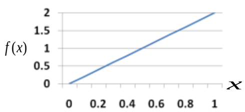
$$\text{Var}(X) = E[(X - \mu)^2] = E[X^2] - (E[X])^2$$

$$\text{Var}(aX + b) = a^2 \text{Var}(X)$$

Linearly Increasing Density

- X is a continuous random variable with PDF:

$$f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$



  - What is E[X]?

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_0^1 2x^2 dx = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3}$$

  - What is Var(X)?

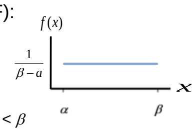
$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^1 2x^3 dx = \frac{1}{2}x^4\Big|_0^1 = \frac{1}{2}$$

$$Var(X) = E[X^2] - (E[X])^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$$

Uniform Random Variable

- X is a **Uniform Random Variable**: X ~ Uni(α, β)
  - Probability Density Function (PDF):

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \alpha \le x \le \beta \\ 0 & \text{otherwise} \end{cases}$$



    - Sometimes defined over range $\alpha < x < \beta$
  - $P(a \le x \le b) = \int_a^b f(x)dx = \frac{b - a}{\beta - \alpha}$ (for $\alpha \le a \le b \le \beta$)
  - $E[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_{\alpha}^{\beta} \frac{x}{\beta - \alpha} dx = \frac{x^2}{2(\beta - \alpha)}\Big|_{\alpha}^{\beta} = \frac{\beta^2 - \alpha^2}{2(\beta - \alpha)} = \frac{\alpha + \beta}{2}$
  - $Var(X) = \frac{(\beta - \alpha)^2}{12}$

Fun with the Uniform Distribution

- X ~ Uni(0, 20)

$$f(x) = \begin{cases} \frac{1}{20} & 0 \le x \le 20 \\ 0 & \text{otherwise} \end{cases}$$

  - P(X < 6)?

$$P(x < 6) = \int_0^6 \frac{1}{20} dx = \frac{6}{20}$$

  - P(4 < X < 17)?

$$P(4 < x < 17) = \int_4^{17} \frac{1}{20} dx = \frac{17}{20} - \frac{4}{20} = \frac{13}{20}$$

Riding the Marguerite Bus

- Say the Marguerite bus stops at the Gates bldg. at 15 minute intervals (2:00, 2:15, 2:30, etc.)
  - Passenger arrives at stop uniformly between 2-2:30pm
  - X ~ Uni(0, 30)
- P(Passenger waits < 5 minutes for bus)?
  - Must arrive between 2:10-2:15pm or 2:25-2:30pm

$$P(10 < X < 15) + P(25 < x < 30) = \int_{10}^{15} \tfrac{1}{30} dx + \int_{25}^{30} \tfrac{1}{30} dx = \frac{5}{30} + \frac{5}{30} = \frac{1}{3}$$

- P(Passenger waits > 14 minutes for bus)?
  - Must arrive between 2:00-2:01pm or 2:15-2:16pm

$$P(0 < X < 1) + P(15 < x < 16) = \int_0^1 \tfrac{1}{30} dx + \int_{15}^{16} \tfrac{1}{30} dx = \frac{1}{30} + \frac{1}{30} = \frac{1}{15}$$

When to Leave For Class

- Biking to a class on campus
  - Leave *t* minutes before class starts
  - X = travel time (minutes). X has PDF: $f(x)$
  - If early, incur cost: c/min. If late, incur cost: k/min.

$$\text{Cost: } C(X,t) = \begin{cases} c(t - X) & \text{if } x < t \\ k(X - t) & \text{if } x \ge t \end{cases}$$

  - Choose *t* (when to leave) to minimize E[C(*X*, *t*)]:

$$E[C(X,t)] = \int_0^\infty C(X,t)\ f(x)dx = \int_0^t c(t-x)\ f(x)dx + \int_t^\infty k(x-t)\ f(x)dx$$

Minimization via Differentiation

- Want to minimize w.r.t. *t*:

$$E[C(X,t)] = \int_0^t c(t-x)\ f(x)dx + \int_t^\infty k(x-t)\ f(x)dx$$

  - Differentiate E[C(*X*, *t*)] w.r.t. *t*, and set = 0 (to obtain *t**):
    - Leibniz integral rule:

$$\frac{d}{dt}\int_{f_1(t)}^{f_2(t)} g(x,t)dx = \frac{df_2(t)}{dt} g(f_2(t),t) - \frac{df_1(t)}{dt} g(f_1(t),t) + \int_{f_1(t)}^{f_2(t)} \frac{\partial g(x,t)}{\partial t} dx$$

$$\frac{d}{dt} E[C(X,t)] = c(t-t) f(t) + \int_0^t cf(x)dx - k(t-t) f(t) - \int_t^\infty kf(x)dx$$

$$0 = cF(t^*) - k[1 - F(t^*)] \ \Rightarrow\ F(t^*) = \frac{k}{c+k}$$