The Questions of Our Time

- Y is a non-negative continuous random variable
  - Probability Density Function: $f_Y(y)$
  - Already knew that:

$$E[Y] = \int_{-\infty}^{\infty} y \, f_Y(y) \, dy$$

  - But, did you know that:

$$E[Y] = \int_{0}^{\infty} P(Y > y) \, dy \text{ ?!?}$$

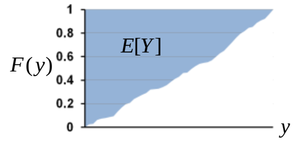
  - No, I didn't think so...
  - Analogously, in the discrete case, where $X = 1, 2, \ldots, n$

$$E[X] = \sum_{i=1}^{n} P(X \ge i)$$

Life Gives You Lemmas, Make Lemma-nade!

- A lemma in the home or office is a good thing

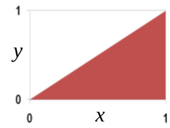
$$E[Y] = \int_{0}^{\infty} P(Y > y) \, dy = \int_{0}^{\infty} (1 - F(y)) \, dy$$



- Proof:

$$\int_{y=0}^{\infty} P(Y > y) \, dy = \int_{y=0}^{\infty} \int_{x=y}^{\infty} f_Y(x) dx \, dy$$

$$= \int_{x=0}^{\infty} \left(\int_{y=0}^{x} dy \right) f_Y(x) dx = \int_{x=0}^{\infty} x \, f_Y(x) dx = E[Y]$$



Discrete Joint Mass Functions

- For two discrete random variables X and Y, the **Joint Probability Mass Function** is:

$$p_{X,Y}(a,b) = P(X = a, Y = b)$$

- Marginal distributions:

$$p_X(a) = P(X = a) = \sum_y p_{X,Y}(a, y)$$

$$p_Y(b) = P(Y = b) = \sum_x p_{X,Y}(x, b)$$

- Example: X = value of die D_1, Y = value of die D_2

$$P(X = 1) = \sum_{y=1}^{6} p_{X,Y}(1, y) = \sum_{y=1}^{6} \frac{1}{36} = \frac{1}{6}$$

A Computer (or Three) in Every House

- Consider households in Silicon Valley
  - A household has C computers: C = X Macs + Y PCs
  - Assume each computer equally likely to be Mac or PC

$$P(C = c) = \begin{cases} 0.16 & c = 0 \\ 0.24 & c = 1 \\ 0.28 & c = 2 \\ 0.32 & c = 3 \end{cases}$$

Y \ X	0	1	2	3	$p_Y(y)$
0	0.16	0.12	0.07	0.04	0.39
1	0.12	0.14	0.12	0	0.38
2	0.07	0.12	0	0	0.19
3	0.04	0	0	0	0.04
$p_X(x)$	0.39	0.38	0.19	0.04	1.00

Marginal distributions

Continuous Joint Distribution Functions

- For two continuous random variables X and Y, the **Joint Cumulative Probability Distribution** is:

$$F_{X,Y}(a,b) = F(a,b) = P(X \le a, Y \le b) \quad \text{where } -\infty < a, b < \infty$$

- Marginal distributions:

$$F_X(a) = P(X \le a) = P(X \le a, Y < \infty) = F_{X,Y}(a, \infty)$$

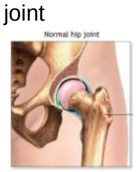
$$F_Y(b) = P(Y \le b) = P(X < \infty, Y \le b) = F_{X,Y}(\infty, b)$$

- Let's look at one:

Demo

Joint

- This is a joint

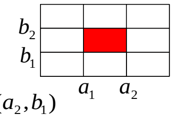


- A joint is not a mathematician
  - It did not start doing mathematics at an early age
  - It is not the reason we have "joint distributions"
  - And, no, Charlie Sheen does not look like a joint
    - But he does have them…
    - He also has **joint** custody of his children with Denise Richards

Computing Joint Probabilities

- Let $F_{X,Y}(x, y)$ be joint CDF for X and Y

$$\begin{aligned}
\mathrm{P}(X > a, Y > b) &= 1 - P((X > a, Y > b)^c) \\
&= 1 - P((X > a)^c \cup (Y > b)^c) \\
&= 1 - P((X \le a) \cup (Y \le b)) \\
&= 1 - (P(X \le a) + P(Y \le b) - P(X \le a, Y \le b)) \\
&= 1 - F_X(a) - F_Y(b) + F_{X,Y}(a, b)
\end{aligned}$$



$$\mathrm{P}(a_1 < X \le a_2, b_1 < Y \le b_2)$$
$$= F(a_2, b_2) - F(a_1, b_2) + F(a_1, b_1) - F(a_2, b_1)$$

Jointly Continuous

- Random variables X and Y, are **Jointly Continuous** if there exists PDF $f_{X,Y}(x, y)$ defined over $-\infty < x, y < \infty$ such that:

$$\mathrm{P}(a_1 < X \le a_2, b_1 < Y \le b_2) = \int_{a_1}^{a_2} \int_{b_1}^{b_2} f_{X,Y}(x, y)\, dy\, dx$$

- Cumulative Density Function (CDF):

$$F_{X,Y}(a,b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(x, y)\, dy\, dx \qquad f_{X,Y}(a,b) = \frac{\partial^2}{\partial a\, \partial b} F_{X,Y}(a,b)$$

- Marginal density functions:

$$f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a, y)\, dy \qquad f_Y(b) = \int_{-\infty}^{\infty} f_{X,Y}(x, b)\, dx$$

Imperfection on a Disk

- Disk surface is a circle of radius R
  - A single point imperfection uniformly distributed on disk

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\pi R^2} & \text{if } x^2 + y^2 \le R^2 \\ 0 & \text{if } x^2 + y^2 > R^2 \end{cases} \quad \text{where } -\infty < x, y < \infty$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy = \frac{1}{\pi R^2} \int_{x^2 + y^2 \le R^2} dy = \frac{1}{\pi R^2} \int_{y=-\sqrt{R^2 - x^2}}^{\sqrt{R^2 - x^2}} dy = \frac{2\sqrt{R^2 - x^2}}{\pi R^2}$$

$$f_Y(y) = \frac{2\sqrt{R^2 - y^2}}{\pi R^2} \quad \text{where} -R \le y \le R, \text{ by symmetry}$$

  - Distance to origin: $D = \sqrt{X^2 + Y^2}$, $P(D \le a) = \frac{\pi a^2}{\pi R^2} = \frac{a^2}{R^2}$

$$E[D] = \int_0^R P(D > a) da = \int_0^R (1 - \frac{a^2}{R^2}) da = \left(a - \frac{a^3}{3R^2} \right)\Bigg|_0^R = \frac{2R}{3}$$

Welcome Back the Multinomial!

- Multinomial distribution
  - *n* independent trials of experiment performed
  - Each trial results in one of *m* outcomes, with respective probabilities: $p_1, p_2, \ldots, p_m$ where $\sum_{i=1}^{m} p_i = 1$
  - X_i = number of trials with outcome *i*

$$P(X_1 = c_1, X_2 = c_2, \ldots, X_m = c_m) = \binom{n}{c_1, c_2, \ldots, c_m} p_1^{c_1} p_2^{c_2} \ldots p_m^{c_m}$$

where $\sum_{i=1}^{m} c_i = n$ and $\binom{n}{c_1, c_2, \ldots, c_m} = \frac{n!}{c_1! c_2! \cdots c_m!}$

Hello Die Rolls, My Old Friend…

- 6-sided die is rolled 7 times
  - Roll results: 1 one, 1 two, 0 three, 2 four, 0 five, 3 six

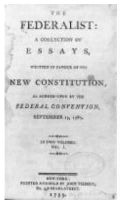
$$P(X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 2, X_5 = 0, X_6 = 3)$$
$$= \frac{7!}{1!1!0!2!0!3!} \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^1 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^3 = 420 \left(\frac{1}{6}\right)^7$$

- This is generalization of Binomial distribution
  - Binomial: each trial had 2 possible outcomes
  - Multinomial: each trial has *m* possible outcomes

Probabilistic Text Analysis

- Ignoring order of words, what is probability of any given word you write in English?
  - P(word = “the”) > P(word = “transatlantic”)
  - P(word = “Stanford”) > P(word = “Cal”)
  - Probability of each word is just multinomial distribution
- What about probability of those same words in someone else’s writing?
  - P(word = “probability” | writer = you) >
    P(word = “probability” | writer = non-CS109 student)
  - After estimating P(word | writer) from known writings, use Bayes Theorem to determine P(writer | word) for new writings!

Old and New Analysis



- Authorship of "Federalist Papers"
  - 85 essays advocating ratification of US constitution
  - Written under pseudonym "Publius"
    - Really, Alexander Hamilton, James Madison and John Jay
  - Who wrote which essays?
    - Analyzed probability of words in each essay versus word distributions from known writings of three authors

- Filtering Spam
  - P(word = "Viagra" | writer = you)
    << P(word = "Viagra" | writer = spammer)