Random Numbers

- In many applications, want to be able to generate random numbers, permutations, etc.
  - Since computers are deterministic, *true* randomness does not exist
  - Settle for <u>pseudo-randomness</u>: sequence of numbers that *looks* random, but is deterministically generated
  - Most random number generators use a "linear congruential generator" (LCG):
    - Start with a seed number X_0
    - Next "random" number given by: $X_{n+1} = (aX_n + c) \bmod m$
    - Effectiveness is *very* sensitive to choice of *a*, *c*, and *m*
    - Note: the sequence of random numbers must eventually cycle

The `rand()` Function

- In C/C++, there exists a function for generating pseudo-random numbers: `int rand(void)`
  - Part of `<stdlib.h>` library
  - Returns integer between 0 and `RAND_MAX`, inclusive
    - Values supposed to be "uniformly" distributed in range
    - `RAND_MAX` guaranteed to be at least 32767 (often larger)
  - In most implementations, uses LCG
  - Seeded using: `void srand(unsigned int seed)`
    - Often set to current system time: `srand(time(NULL));`
  - For our purposes, we accept approximation: `(rand()/((double)RAND_MAX + 1))` ~ Uni[0, 1)

Shuffling Deck of Cards

- Want to generate a random permutation of set
  - E.g., completely shuffle a deck of cards
  - Want all permutations to be equally likely

```
void shuffle(int arr[], int n) {
   for(int i = n - 1; i > 0; i--) {
      double u = uniformRand(0, 1);  // u in [0, 1)
      // pick one of "remaining" i positions
      int pos = (int)((i + 1) * u);
      swap(arr[i], arr[pos]);
   }
}
```

Bad, But Common, Shuffle

- Common mistake in creating random permutation

```
void badShuffle(int arr[], int n) {
   for(int i = 0; i < n; i++) {
      double u = uniformRand(0, 1);  // u in [0, 1)
      // pick any position
      int pos = (int)(n * u);
      swap(arr[i], arr[pos]);
   }
}
```

  - Has n^n execution paths, but only n! permutations
  - Consider n = 3:
    - Can generate $3^3 = 27$ possible execution paths
    - But, only 3! = 6 possible permutations
    - 27 / 6 = 4.5 (not integer), so not all permutations equally likely!

Another Good Way to Shuffle

- Common (easy) way to shuffle

```
void shuffle(int arr[], int n) {
   double *keys = new double[n];
   for(int i = 0; i < n; i++) {
      keys[i] = uniformRand(0, 1);  // u in [0, 1)
   }
   SortUsingKeys(arr, keys, n);
   delete[] keys;
}
```

  - Pros: all permutations equally likely, easy to code
  - Cons: O(n log n) due to sort vs. O(n) for our first method

Generating Distributions

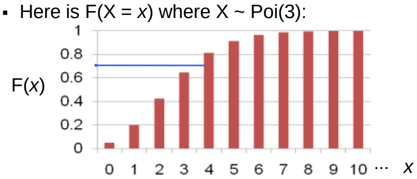
- Given ability to generate numbers ~ Uni(0, 1)
  - How can we generate other distributions?
  - First method we consider is "Inverse Transform"
  - Want to simulate a *continuous* distribution function F
  - Let U ~ Uni(0, 1)
  - Define $X = F^{-1}(U)$ (inverse: $F^{-1}(a) = b \Leftrightarrow F(b) = a$)
  - Note: $P(X \le x) = P(F^{-1}(U) \le x) = P(U \le F(x)) = F(x)$
  - Thus, X will have distribution F
    - Can use method for discrete distributions with some modification

Continuous Inverse Transform

- Need to invert distribution function
  - Want to generate exponential distribution: X ~ Exp(λ)
  - CDF: $F(X = x) = 1 - e^{-\lambda x}$ where $x \geq 0$
  - To compute inverse, let $F(X) = 1 - e^{-\lambda x} = u$, solve for x:
    $e^{-\lambda x} = 1 - u \Leftrightarrow -\lambda x = \log(1 - u) \Leftrightarrow x = -\log(1 - u)/\lambda$
  - So, $F^{-1}(U = u) = x = -\log(1 - u)/\lambda$
  - Since U ~ Uni(0, 1), also have (1 – U) ~ Uni(0, 1)
  - Simplify: $X = F^{-1}(U) = -\log(U)/\lambda$
- Note: closed-form inverse may not always exist
  - Normal distribution doesn't have closed-form inverse

Discrete Inverse Transform

- Recall form of CDF for discrete distribution:
  - Here is F(X = *x*) where X ~ Poi(3):



  - Generate U ~ Uni(0, 1) (e.g., *u* = 0.7)
  - As *x* increases, determine first F(*x*) ≥ U (e.g., *x* = 4)
  - Return that value of *x*

Bring on the Code

- Discrete inverse transform
  - Assumes PMF, p(x), available for distribution modeled
  - Here, assume distribution over non-negative integers
    - E.g., Bernoulli, Binomial, Poisson, many others

```
int discreteInverseTransform() {
   double u = uniformRand(0, 1);  // u in [0, 1)
   int x = 0;
   double F_so_far = p(x);
   while (F_so_far < u) {
      x++;
      F_so_far += p(x);
   }
   return x;
}
```

Rejection Filtering

- Want to simulate random variable X with PDF f(x)
  - Assume we can simulate Y with PDF g(y)
    - where Y has same range as X

```
double rejectionFilter() {
   while (true) {
      double u = uniformRand(0, 1);  // u in [0, 1)
      double y = randomValueFromDistributionOfY();
      if (u <= f(y)/(c * g(y))) return y;
   }
}
```

where constant c ≥ f(y)/g(y) for all y

  - Number iterations of loop ~ Geo(1/c)
  - Proof of correctness in Ross, Chap. 10.2.2

Generating Normal Random Variable

- Want to simulate random variable Z ~ N(0, 1)

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad \text{where} \quad -\infty < z < \infty$$

  - PDF for |Z|: $f(z) = \frac{2}{\sqrt{2\pi}} e^{-z^2/2} \quad \text{where} \quad 0 \leq z < \infty$
  - Will simulate using Y ~ Exp(1) (from inverse transform)
    - PDF: $g(y) = e^{-y}$ where $0 \leq y < \infty$

$$\frac{f(x)}{g(x)} = \sqrt{\frac{2}{\pi}} e^{-(x^2-2x)/2} = \sqrt{\frac{2}{\pi}} e^{-(x^2-2x+1)/2+1/2} = \sqrt{\frac{2e}{\pi}} e^{-(x-1)^2/2} \leq \sqrt{\frac{2e}{\pi}} = c$$

  - So, we obtain: $\frac{f(x)}{c \cdot g(x)} = e^{-(x-1)^2/2}$

Applying Rejection Filtering

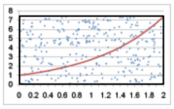
- Note: $\frac{f(x)}{c \cdot g(x)} = e^{-(x-1)^2/2}$ $\quad c = \sqrt{\frac{2e}{\pi}} \approx 1.32$

```
double rejectionFilterNormal() {
   while (true) {
     double u = uniformRand(0, 1);  // u in [0, 1)
     // Y ~ Exp(1) using inverse transform method
     double y = -ln(uniformRand(0, 1));
     if (u <= exp(-((y - 1)*(y - 1)) / 2)) return y;
   }
}

double normal() {
   double x = rejectionFilterNormal();
   double u = uniformRand(0, 1);
   if (u < 0.5) return (x);
   else return (-x);
}
```

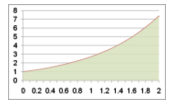
Computing Integrals

- Given ability to generate numbers ~ Uni(0, 1)
  - Want to compute (approximate) value of an integral
  - Useful when integral has no closed form
    - Or may have closed form, but don't know how to derive it
  - Basic idea
    - Consider graph of function
    - Throw "darts" at graph
    - Determine percentage below function
    - Multiply by area into which you threw darts
      - Square: (domain of integral) x (maximum value of function)
  - This is called "Monte Carlo Integration"
    - Named after area in Monaco known for its casinos



Monte Carlo Integration

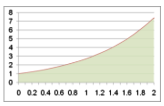
- Consider integral: $\int_0^2 e^x dx$




```
double integrate() {
   // number of "darts" to throw
   int NUM_POINTS = 1000000;
   int numBelowFn = 0;
   for(int i = 0; i < NUM_POINTS; i++) {
      double x = uniformRand(0, 1) * 2.0;
      double y = uniformRand(0, 1) * exp(2.0);
      if (y < exp(x)) numBelowFn++;
   }
   double fraction = (double)numBelowFn/NUM_POINTS;
   return (2.0 * exp(2.0) * fraction);
}
```

How Well Does This Do?

- Consider integral: $\int_0^2 e^x dx$



  - Analytically:

$$\int_0^2 e^x dx = e^x \Big|_0^2 = e^2 - 1 \approx 6.389$$

  - Monte Carlo Integration:

NUM_POINTS	Computed value (to 4 significant digits)
10	5.911
100	6.650
1,000	6.872
10,000	6.239
100,000	6.387
1,000,000	6.391
10,000,000	6.388

Computing Statistics Via Simulation

- Recall example:

```
int Recurse()
{
   int x = randomInt(1, 3); // Equally likely values
   if (x == 1) return 3;
   else if (x == 2) return (5 + Recurse());
   else return (7 + Recurse());
}
```

  - Wanted to compute E[Y]
    - Analytically, derived E[Y] = 15
  - Can approximate by simulation: run algorithm many times, compute average

How Well Does This Do?

- Recall example:

```
int Recurse()
{
   int x = randomInt(1, 3); // Equally likely values
   if (x == 1) return 3;
   else if (x == 2) return (5 + Recurse());
   else return (7 + Recurse());
}
```

  - Simulation

# of runs	Average value (to 5 significant digits)
10	16.800
100	15.080
1,000	15.920
10,000	14.844
100,000	14.977
1,000,000	14.999