Solving signal decoupling problems through self-bounded controlled invariants

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Abstract

This paper deals with decoupling problems of unknown, measurable and previewed signals. First the well known solutions of unknown and measurable disturbance decoupling problems are recalled. Then new necessary and sufficient constructive conditions for the previewed signal decoupling problem are proposed. The discrete time case is considered. In this domain previewing a signal by p steps means that the k-th sample of the signal to be decoupled is known p steps in advance.
The main result is to prove that the stability condition for all of the mentioned decoupling problems does not change, i.e. the resolving subspace to be stabilized is the same independently of the type of signal to be decoupled, being it completely unknown (disturbance), measured or previewed.
The problem has been studied through self-bounded controlled invariants, thus minimizing the dimension of the resolving subspace which corresponds to the infimum of a lattice. Note that reduced dimension on resolving controlled invariant subspace yields to reduce the order of the controller units.

1 Introduction

Disturbance decoupling is a classical problem in control theory. It has been one of the first application considered in the geometric approach framework and has been given attention for more than thirty years. In the first formulation of the disturbance decoupling problem (DDP) [2, 11], disturbance signals are assumed to be unknown and unaccessible. Later Bhattacharyya [6] considered the so called *measured signal decoupling problem* (MSDP) in which signals to be decoupled are considered measurable. The structural conditions for the MSDP to be solved are less restrictive than those for the DDP, while stabilizability conditions are similar.
In this paper the decoupling control problem is approached in a more general setting. Signals which are known in advance or previewed by a given amount of time are considered. Such problem will be referred to as *previewed signal decoupling problem* (PSDP).
The PSDP has been investigated by Willems [10] who first derived, in the continuous time domain, a necessary and sufficient condition to solve the PSDP with pole placement. This solution was based on the so called proportional-integral-derivative control laws consisting of a feedback of the state system and of a linear combination of signal (to be decoupled) and its time derivatives. The major drawback of these extensions of the disturbance decoupling problem in continuous time domain is that control laws include distributions, hence are not practically implementable.
Independently, Imai and Shinozuka [7] proposed a similar necessary and sufficient condition for the PSDP with stability in both discrete and continuous time cases. They also proposed a synthesis procedure of the preaction unit and of the state feedback matrix to solve the PSDP.
Conditions for the PSDP to be solved, given in [10] and [7], do not care about dimensionality of the resolving controlled invariant subspace. Furthermore, to the best of our knowledge, the problem of reducing the dimension of the resolving subspace for the PSDP has not been thoroughly investigated in the literature. Note that using controlled invariants of minimal dimensions yields to reduce the order of the controller units and possible state observers.
Barbagli, Marro and Prattichizzo [1] proposed a new solution for the PSDP with stability based on a subspace with reduced dimension. Such dimension optimization was gained through self-bounded controlled invariants. This is a special class of controlled invariants introduced by Basile and Marro in [5, 9] which enjoys interesting properties, the most important of which is to be a lattice instead of a semi-lattice, hence to admit an infimum other than a supremum.
This paper deals with discrete-time systems. In such domain, the solution to the PSDP is more elegant and is practically implementable. After reviewing the well known concepts of minimum conditioned invariant containing a given subspace, a structural condition for the previewed signal decoupling problem and a condition for the problem with stability are presented.
The main result of this paper consists in proposing a unique necessary and sufficient condition for signal decoupling problems with stability independently of the

type of signal to be decoupled, being it completely unknown (disturbance), measured or previewed.

This paper shows that the resolving subspace proposed in [1] for the PSDP problems and the well known resolving subspace of the DDP problem proposed by Basile, Marro and Piazzi in [4] are equivalent. The proof is carried out in a geometric framework and is based on several lattices of self bounded $(A, \mathcal{B})$-controlled invariants.

The following notation is used. $\mathbf{R}$ stands for the field of real numbers. Sets, vector spaces and subspaces are denoted by script capitals like $\mathcal{X}$, $\mathcal{I}$, $\mathcal{V}$, etc.. Since most of the geometric theory of dynamic system herein presented is developed in the vector space $\mathbf{R}^n$, we reserve the symbol $\mathcal{X}$ for the full space, i.e., we assume $\mathcal{X} := \mathbf{R}^n$. Matrices and linear maps by slanted capitals like A, B, etc., the image and the null space of the generic matrix or linear transformation A by im$A$ and ker$A$ respectively, the transpose of the generic real matrix A by A^T and its spectrum by $\sigma(A)$.

The reminder of this paper is organized as follows. Section 2 presents the structural conditions for the general PSDP. In Section 3 new necessary and sufficient conditions for the PSDP with stability are stated, and finally in Section 4 an illustrative example is presented.

2 Structural conditions for the PSDP

Let us consider the discrete-time system

$$\left\{ \begin{array}{rcl} x(k+1) &=& Ax(k) + Bu(k) + Hh(k) \\ y(k) &=& Cx(k) \end{array} \right. \qquad (1)$$

where $x \in \mathcal{X}$ $(= \mathbf{R}^n)$, $u \in \mathbf{R}^m$, $h \in \mathbf{R}^h$ and $y \in \mathbf{R}^q$ denote the state, the manipulable input, the signal to be decoupled and the regulated output, respectively. In the following the short notations $\mathcal{B} := \mathrm{im}B$, $\mathcal{C} := \ker C$ and $\mathcal{H} := \mathrm{im}H$ will be used.

In this paper we deal with the signal decoupling problem when a certain degree of knowledge of signal $h(k)$ is available. In particular we assume that signal $h(k)$ is previewed, i.e. it is known p steps in advance, or analytically the sample $h(k)$ is known at step $k-p$. Henceforth previewed signal $h(k)$ by p steps will be referred to as p-previewed $h(k)$, or shortly ${}^p h(k)$. Note that measurable disturbance can be thought as 0-previewed signals.

Preview on $h(k)$ is needed in order to "prepare" the system dynamics to localize the disturbance signal on the nullspace of the output matrix C. This is formalized in the following statement.

Problem 1 (Previewed signal decoupling) *Refer to system (1) with zero initial condition and assume that input h is previewed by p instants of time, $p \geq 0$. Determine a control law which, making use of this preview, is able to maintain the output $y(k)$ identically zero.*

In a geometric framework, the key tool to analyze the structural conditions for the signal decoupling problem, is the well-known [3] algorithm computing $\mathcal{S}^* := \min S(A, \mathcal{C}, \mathcal{B})$, the minimal $(\mathcal{A}, \mathcal{C})$-conditioned invariant containing $\mathcal{B}$, here reported for the reader convenience:

$$\mathcal{S}_0 := \mathcal{B} \qquad (2)$$

$$\mathcal{S}_i := \mathcal{B} + A(\mathcal{S}_{i-1} \cap \mathcal{C}). \qquad (3)$$

Structural conditions to solve Problem 1 for p-previewed signals are given in the following theorem.

Theorem 1 *Necessary and sufficient condition for Problem 1 to be solved is that*

$$\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{S}_p. \qquad (4)$$

where $\mathcal{V}^* := \max\mathcal{V}(A, \mathcal{B}, \mathcal{C})$ is the maximal controlled invariant contained in the nullspace of $\mathcal{C}$.

Remark 1 *Structural condition (4) in Theorem 1, is similar to the one proved in [10] for the continuous-time case and in [7] for the discrete time case. However, condition (4) is less restrictive since it does not consider stability. It is worth noting that the case of measurable inputs is accounted for by condition (4). In fact measurable signals corresponds to $p = 0$ and therefore (4) turns into the well known condition*

$$\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}.$$

Similarly the lack of preview yields to the well-known structural condition for unknown signals, i.e.

$$\mathcal{H} \subseteq \mathcal{V}^*.$$

Summarizing, being $\mathcal{V}^* \subseteq \mathcal{V}^* + \mathcal{B} \subseteq \mathcal{V}^* + \mathcal{S}_p$, the larger the preview time of the signal to be decoupled the easier it is to solve the PSDP.

The following property characterizes the minimum number of preview steps necessary to decouple previewed signals for a given disturbance matrix H.

Property 1 *Consider system (1) and let r be the minimum number of steps necessary to obtain convergence of algorithm for $\mathcal{S}^*$, (2,3). The minimum positive integer $p \leq r$ such that condition (4) holds, corresponds to the minimum number of previewed steps for $h(k)$ needed to decouple signal $Hh(k)$. If for $p = r$ condition (4) is not satisfied, the PSDP has no solution for the given disturbance matrix H.*

For a better understanding of Theorem 1, some guess on how to employ the preview to localize previewed signals in the nullspace of the output matrix, is given. A detailed discussion on the controller design can be found in [1].

If structural condition (4) holds, it is possible to decompose the disturbance effect into two separate parts as follows

$$\begin{aligned} H &= H_V + H_S && (5)\\ \mathcal{H}_V &:= \operatorname{im}(H_V) \subseteq \mathcal{V}^* && (6)\\ \mathcal{H}_S &:= \operatorname{im}(H_S) \subseteq \mathcal{S}_p. && (7)\end{aligned}$$

Components of $Hh(k)$ lying on $\mathcal{S}_p$ can be canceled through a *preaction unit*. For this purpose note that $\mathcal{S}_p$ can be interpreted as the reachable subspace in p ($p \geq 0$) steps from $x_0 = 0$, with the state trajectory constrained to lie on the nullspace of the output matrix C in the $(p-1)$-steps interval $[0, p-1]$:

$$Cx(k) = 0 \qquad \text{for} \quad (k = 0, 1, \dots p-1). \tag{8}$$

In other terms, the *preaction unit*, which is a part of the decoupling controller, exploits the signal preview to cancel $H_S h(k)$, i.e. the part of $Hh(k)$ belonging to $\mathcal{S}_p$ Because of the special reachability subspace $\mathcal{S}_p$, this happens while maintaining the output identically zero. On the other hand signal $H_V h(k)$ is localized in the nullspace of the output matrix according to standard decoupling techniques [3].

3 Previewed signal decoupling problem with stability

The p–previewed signal decoupling problem with stability is investigated.

Problem 2 (Previewed signal decoupling with stability) *Refer to system (1) with zero initial condition and assume that it is stabilizable and that input h is previewed by p instants of time, $p \geq 0$. Determine a control law which, making use of the preview, is able to maintain the output $y(k)$ identically zero while keeping the state on a bounded trajectory.*

The Previewed signal decoupling with stability is approached by means of lattices of self-bounded controlled invariants [5, 9]. A special attention is devoted to the dimension of the resolving subspace.
Let us introduce the lattice of all the $(A, \mathcal{B})$-controlled invariants self bounded with respect to $\mathcal{C}$,

$$\Phi = \Phi(\mathcal{B}, \mathcal{C}) = \{\mathcal{V} \mid \; A\,\mathcal{V} \subseteq \mathcal{V} + \mathcal{B},\; \mathcal{V} \subseteq \mathcal{C},\; \mathcal{V}^* \cap \mathcal{B} \subseteq \mathcal{V}\} \tag{9}$$

whose infimum is

$$\mathcal{V}^* \cap \min\mathcal{S}(A, \mathcal{C}, \mathcal{B}),$$

and the lattice of all the $(A, \mathcal{B} + \mathcal{H}_V)$-controlled invariants self bounded with respect to $\mathcal{C}$,

$$\begin{aligned}\Phi_1 &= \Phi(\mathcal{B} + \mathcal{H}_V, \mathcal{C}) = \\ &= \{\mathcal{V} \mid \; A\,\mathcal{V} \subseteq \mathcal{V} + \mathcal{B} + \mathcal{H}_V,\; \mathcal{V} \subseteq \mathcal{C},\; \mathcal{V}^* \cap (\mathcal{B} + \mathcal{H}_V) \subseteq \mathcal{V}\}\end{aligned} \tag{10}$$

whose infimum is

$$\mathcal{V}_{m1} = \mathcal{V}^* \cap \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V). \tag{11}$$

Subspace $\mathcal{V}_{m1}$ can be written as in (11) since condition (6) holds and therefore

$$\mathcal{V}^* \equiv \max\mathcal{V}(A, \mathcal{B} + \mathcal{H}_V, \mathcal{C}).$$

The following lemmas hold.

Lemma 1 *The set*

$$\Phi_2 = \{\mathcal{V} \mid \mathcal{V} \in \Phi, \mathcal{H} \subseteq \mathcal{V} + \mathcal{S}_p\} \tag{12}$$

enjoys the following properties:

1. *is a sub-lattice of Φ;*

2. $\Phi_2 \equiv \{\mathcal{V} \mid \mathcal{V} \in \Phi, \mathcal{V}^* \cap (\mathcal{H} + \mathcal{S}_p) \subseteq \mathcal{V}\}$*;*

3. *the infimum of Φ_2 is*

$$\mathcal{V}_{m2} = \mathcal{V}^* \cap \min\mathcal{S}(A, \mathcal{V}^*, \mathcal{H} + \mathcal{S}_p). \tag{13}$$

Proof: *(Property 1.)* We want to show that given two generic elements $\mathcal{V}_1$ and $\mathcal{V}_2$ of set Φ_2 their sum and intersection still belongs to the same set. Such proof appears trivial for the subspace obtained by summing the two given subspaces. Let's consider now element $\mathcal{V}_1 \cap \mathcal{V}_2$. By assumption, since both $\mathcal{V}_1$ and $\mathcal{V}_2$ belong to Φ_2 it is obvious that

$$\mathcal{H} \subseteq \mathcal{V}_1 + \mathcal{S}_p \tag{14}$$

$$\mathcal{H} \subseteq \mathcal{V}_2 + \mathcal{S}_p \tag{15}$$

which lead to

$$\mathcal{H} \subseteq (\mathcal{V}_1 + \mathcal{S}_p) \cap (\mathcal{V}_2 + \mathcal{S}_p).$$

By intersecting both terms with $\mathcal{V}^* + \mathcal{S}_p$ we obtain

$$\mathcal{H} \subseteq ((\mathcal{V}_1 + \mathcal{S}_p) \cap (\mathcal{V}_2 + \mathcal{S}_p)) \cap (\mathcal{V}^* + \mathcal{S}_p) \tag{16}$$

since the structural condition (4) holds, and then

$$\mathcal{H} \subseteq (\mathcal{V}^* \cap (\mathcal{V}_1 + \mathcal{S}_p)) \cap (\mathcal{V}^* \cap (\mathcal{V}_2 + \mathcal{S}_p)) + \mathcal{S}_p$$

using the distributive property, being $\mathcal{S}_p$ included in $(\mathcal{V}_1 + \mathcal{S}_p) \cap (\mathcal{V}_2 + \mathcal{S}_p)$. Analogously we get

$$\mathcal{H} \subseteq ((\mathcal{V}_1 \cap \mathcal{V}^*) + (\mathcal{V}^* \cap \mathcal{S}_p)) \cap ((\mathcal{V}_2 \cap \mathcal{V}^*) + (\mathcal{V}^* \cap \mathcal{S}_p)) + \mathcal{S}_p$$

and finally

$$\mathcal{H} \subseteq (\mathcal{V}_1 \cap \mathcal{V}_2) + \mathcal{S}_p$$

being $\mathcal{V}_1$ and $\mathcal{V}_2$ both included in $\mathcal{V}^*$ and both including $\mathcal{V}^* \cap \mathcal{S}_p$.

(Property 2.)
$(\Rightarrow)$

$$\mathcal{V} \in \Phi_1 \Rightarrow \mathcal{H} \subseteq \mathcal{V} + \mathcal{S}_p \Rightarrow \mathcal{H} + \mathcal{S}_p \subseteq \mathcal{V} + \mathcal{S}_p$$

and therefore intersecting both members with $\mathcal{V}^*$ we obtain

$$\mathcal{V}^* \cap (\mathcal{H} + \mathcal{S}_p) \subseteq \mathcal{V}^* \cap (\mathcal{V} + \mathcal{S}_p) = \mathcal{V} + (\mathcal{V}^* \cap \mathcal{S}_p) = \mathcal{V}$$

being $(\mathcal{V}^* \cap \mathcal{S}_p) \subseteq (\mathcal{V}^* \cap \mathcal{S}^*)$ which is the infimum of Φ and therefore is contained in all $\mathcal{V} \in \Phi$.

($\Leftarrow$)

$$\mathcal{V}^* \cap (\mathcal{H} + \mathcal{S}_p) \subseteq \mathcal{V}$$

summing $\mathcal{S}_p$ to both members we obtain

$$\mathcal{S}_p + (\mathcal{V}^* \cap (\mathcal{H} + \mathcal{S}_p)) \subseteq \mathcal{V} + \mathcal{S}_p$$

from which using the distributive property we obtain

$$(\mathcal{S}_p + \mathcal{V}^*) \cap (\mathcal{S}_p + \mathcal{H}) \subseteq \mathcal{V} + \mathcal{S}_p$$

and being $\mathcal{H} \subseteq \mathcal{S}_p + \mathcal{H} \subseteq \mathcal{S}_p + \mathcal{V}^*$ we obtain

$$(\mathcal{S}_p + \mathcal{H}) \subseteq \mathcal{V} + \mathcal{S}_p$$

from which obviously

$$\mathcal{H} \subseteq \mathcal{V} + \mathcal{S}_p$$

(Property 3.)
The proof will be developed in two steps:

A. Any element of Φ_2 contains $\mathcal{V}_{m2} = \mathcal{V}^* \cap \mathcal{S}_2^*$ where

$$\mathcal{S}_2^* = \min\mathcal{S}(A, \mathcal{V}^*, \mathcal{H} + \mathcal{S}_p); \qquad (17)$$

B. $\mathcal{V}^* \cap \mathcal{S}_2^*$ is an element of Φ_2

(Step A.) Consider the sequence that defines $\mathcal{S}_2^*$:

$$\mathcal{Z}'_0 \;:=\; \mathcal{S}_p + \mathcal{H} \qquad (18)$$
$$\mathcal{Z}'_i \;:=\; \mathcal{S}_p + \mathcal{H} + A\,(\mathcal{Z}'_{i-1} \cap \mathcal{V}^*) \quad (i=1,\ldots) \qquad (19)$$

Let $\mathcal{V}$ be a generic element of Φ_2, so that

$$A\,\mathcal{V} \subseteq \mathcal{V} + \mathcal{B}\,, \quad \mathcal{V} \supseteq \mathcal{V}^* \cap \mathcal{B}.$$

We proceed by induction: clearly

$$\mathcal{Z}'_0 \cap \mathcal{V}^* \subseteq \mathcal{V}$$

since by assumption $\mathcal{V}^* \cap (\mathcal{S}_p + \mathcal{H}) \subseteq \mathcal{V}$, and from

$$\mathcal{Z}'_{i-1} \cap \mathcal{V}^* \subseteq \mathcal{V}$$

it follows that

$$A(\mathcal{Z}'_{i-1} \cap \mathcal{V}^*) \subseteq A\mathcal{V} \subseteq \mathcal{V} + \mathcal{B}$$

since $\mathcal{V}$ is an $(A, \mathcal{B})$-controlled invariant. Adding $\mathcal{S}_p + \mathcal{H}$ to both members yields

$$\mathcal{S}_p + \mathcal{H} + A\,(\mathcal{Z}'_{i-1} \cap \mathcal{V}^*) \subseteq \mathcal{V} + \mathcal{S}_p + \mathcal{H}$$

where the first term of the last inclusion is by definition subspace $\mathcal{Z}'_i$ and, by intersecting with $\mathcal{V}^*$, we finally obtain

$$\mathcal{Z}'_i \cap \mathcal{V}^* \subseteq (\mathcal{V} + (\mathcal{S}_p + \mathcal{H})) \cap \mathcal{V}^* = \mathcal{V} + (\mathcal{S}_p + \mathcal{H}) \cap \mathcal{V}^* = \mathcal{V}$$

which completes the induction argument and the proof of Step A.

(Step B.) Note that

1. $\mathcal{S}_1^* \cap \mathcal{V}^*$ is an $(A, \mathcal{B})$-controlled invariant contained in $\mathcal{C}$;

2. $\mathcal{S}_1^* \cap \mathcal{V}^*$ is self bounded with respect to $\mathcal{C}$

3. $\mathcal{H} \subseteq (\mathcal{S}_1^* \cap \mathcal{V}^*) + \mathcal{S}_p$

To prove 1. note that

$$A\,\mathcal{V}^* \subseteq \mathcal{V}^* + \mathcal{B}$$
$$A\,(\mathcal{S}_1^* \cap \mathcal{V}^*) \subseteq \mathcal{S}_1^*$$

which simply expresses $\mathcal{V}^*$ as an $(A, \mathcal{B})$-controlled invariant and $\mathcal{S}_1^*$ as an $(A, \mathcal{V}^*)$-conditioned invariant. By intersection it follows that

$$A\,(\mathcal{S}_1^* \cap \mathcal{V}^*) \subseteq \mathcal{S}_1^* \cap (\mathcal{V}^* + \mathcal{B}) = \mathcal{S}_1^* \cap \mathcal{V}^* + \mathcal{B}$$

being $\mathcal{B} \subseteq \mathcal{S}_1^*$. Then $\mathcal{S}_1^* \cap \mathcal{V}^*$ is an $(A, \mathcal{B})$-controlled invariant contained in $\mathcal{C}$.
To prove 2. note that

$$\mathcal{V}^* \cap \mathcal{B} \subseteq \mathcal{V}^* \cap \mathcal{S}_1^*.$$

Finally, to prove 3. note that being $\mathcal{S}_p \subseteq \mathcal{S}_1^*$ and $\mathcal{H} \subseteq \mathcal{S}_1^*$ it follows that

$$\mathcal{H} \subseteq (\mathcal{V}^* \cap \mathcal{S}_p) + \mathcal{S}_1^* = (\mathcal{V}^* \cap \mathcal{S}_1^*) + \mathcal{S}_p.$$

■

Lemma 2 *The set*

$$\Phi_3 = \{\mathcal{V} \,|\, \mathcal{V} \in \Phi, \mathcal{H}_V \subseteq \mathcal{V} + \mathcal{S}_p\} \qquad (20)$$

enjoys the following properties:

1. *is a sub-lattice of* Φ*;*

2. $\Phi_3 \equiv \{\mathcal{V} \,|\, \mathcal{V} \in \Phi, \mathcal{V}^* \cap (\mathcal{H}_V + \mathcal{S}_p) \subseteq \mathcal{V}\}$*;*

3. *the infimum of* Φ_3 *is*

$$\mathcal{V}_{m3} = \mathcal{V}^* \cap \min\mathcal{S}(A, \mathcal{V}^*, \mathcal{H}_V + \mathcal{S}_p) \qquad (21)$$

4. $\Phi_3 \equiv \Phi_2$*, i.e.* $\mathcal{V}_{m2} \equiv \mathcal{V}_{m3}$

5. $\Phi_3 \subseteq \Phi_1$

6. $\mathcal{V}_{m1} \in \Phi_3$

7. $\mathcal{V}_{m1} \equiv \mathcal{V}_{m2} \equiv \mathcal{V}_{m3}$

Proof: *(Property 1., 2. and 3.)*
Note that $\mathcal{H}_V \subseteq \mathcal{V}^* \Rightarrow \mathcal{H}_V \subseteq \mathcal{V}^* + \mathcal{S}_p$ and therefore proofs are analogous to those of properties 1, 2 and 3 in Lemma 1.
(Property 4.) This property follows from properties 2 of this statement and Lemma 1 and from eqs.(5) and (7).
(Property 5.)
Let $\mathcal{V} \in \Phi_3$ it follows

- $A\mathcal{V} \subseteq \mathcal{V} + \mathcal{B} + \mathcal{H}_V$ since $\mathcal{V}$ is an $(A, \mathcal{B})$-controlled invariant
- $\mathcal{V} \subseteq \mathcal{C}$
- $\mathcal{V}^* \cap (\mathcal{B} + \mathcal{H}_V) \subseteq \mathcal{V}^* \cap (\mathcal{S}_p + \mathcal{H}_V) \subseteq \mathcal{V}$

then $\mathcal{V} \in \Phi_3 \Rightarrow \mathcal{V} \in \Phi_1$.
(Property 6.)
First of all note that $A\mathcal{V}_{m1} \subseteq \mathcal{V}_{m1} + \mathcal{B}$ since

$$\begin{array}{rll} A\mathcal{V}_{m1} = & A\big((\mathcal{V}^* \cap \mathcal{C}) \cap \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V)\big) & \subseteq \\ & A\mathcal{V}^* \cap A\big(\mathcal{C} \cap \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V)\big) & \subseteq \\ & (\mathcal{V}^* + \mathcal{B}) \cap \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V) & = \\ & \big(\mathcal{V}^* \cap \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V)\big) + \mathcal{B} & = \\ & \mathcal{V}_{m1} + \mathcal{B} & , \end{array}$$

moreover $\mathcal{V}_{m1} \subseteq \mathcal{C}$ and $\mathcal{V}^* \cap \mathcal{B} \subseteq \mathcal{V}_{m1}$, then $\mathcal{V}_{m1} \in \Phi$. Finally, note that $\mathcal{H}_V \subseteq \mathcal{V}_{m1} + \mathcal{S}_p$. In fact, being $\mathcal{S}_p \subseteq \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V)$, it holds

$$\mathcal{V}_{m1} + \mathcal{S}_p = \big(\mathcal{V}^* \cap \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V)\big) + \mathcal{S}_p = (\mathcal{V}^* + \mathcal{S}_p) \cap \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V)$$

and $\mathcal{H}_V$ is included in both subspaces of the latter intersection.
(Property 7.)
From properties 4, 5 and 6. ■

Lemma 3 *The following equivalence holds*

$$\mathcal{V}_{m1} \equiv \mathcal{V}^* \cap \min\mathcal{S}(A, \mathcal{C}, \mathcal{H} + \mathcal{B}) \tag{22}$$

Proof: Being $\mathcal{H}_S \subseteq \mathcal{S}_p \subseteq \min\mathcal{S}(A, \mathcal{C}, \mathcal{B}) \subseteq \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V)$, from (5)

$$\mathcal{B} + \mathcal{H} \subseteq \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V), \tag{23}$$
$$\mathcal{B} + \mathcal{H}_V \subseteq \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}). \tag{24}$$
$$\tag{25}$$

Now, let us prove that

$$\min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}) \subseteq \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V). \tag{26}$$

by applying induction arguments to the subspace sequence defining $\min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H})$, whose i–element is $\mathcal{Z}'_i$. From (23)

$$\mathcal{Z}'_0 = \mathcal{B} + \mathcal{H} \subseteq \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V),$$

assume that

$$\mathcal{Z}'_{i-1} \subseteq \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V),$$

being $\min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V)$ an $(A, \mathcal{C})$-conditioned invariant containing $\mathcal{B} + \mathcal{H}$, it follows that

$$\mathcal{B} + \mathcal{H} + A(\mathcal{Z}'_{i-1} \cap \mathcal{C}) \subseteq \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V).$$

Similarly, starting from (24), it is possible to prove that

$$\min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V) \subseteq \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}). \tag{27}$$

Finally from (26) and (27)

$$\min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V) = \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H})$$

and the proof ends. ■

Let's recall a fundamental property (proved in [3]) of self bounded subspaces:

Property 2 *Let $\bar{\mathcal{V}}$ and $\mathcal{V}$ be a couple of any $(A, \mathcal{B})$-controlled invariant subspaces self bounded with respect to $\mathcal{C}$ such that $\mathcal{V} \subseteq \bar{\mathcal{V}}$. Let F be a matrix such that $(A + BF)\bar{\mathcal{V}} \subseteq \bar{\mathcal{V}}$. Then $(A + BF)\mathcal{V} \subseteq \mathcal{V}$ holds.*

We are now ready to present the main result.

Theorem 2 *The signal decoupling problem with stability for the p-previewed signal $^p h(k)$ stated in Problem 2 is solvable if and only if the structural condition (4) is satisfied and*

$$\mathcal{V}_m := \mathcal{V}^* \cap \min\mathcal{S}(A, \mathcal{C}, \mathcal{H} + \mathcal{B}) \tag{28}$$

is internally stabilizable.

Proof: As shown in Theorem 1 the purpose of the preaction is to cancel, at the generic time instant k, the component of $Hh(k)$ on $\mathcal{S}_p$ in order to force the state dynamics (excited by the other component $H_V h(t)$) on a subspace $\mathcal{V}$ satisfying the following properties:

1. $\mathcal{V}$ is an $(A, \mathcal{B})$ controlled invariant included in $\mathcal{C}$;

2. $\mathcal{V}$ is such that $\mathcal{H} \subseteq \mathcal{V} + \mathcal{S}_p$;

3. $\mathcal{V}$ is internally stabilizable.

We will now prove the necessity of the statement, i.e. that if a subspace $\mathcal{V}$ exists that solves Problem 2 (with stability) then $\mathcal{V}_m$ is internally stabilizable. Being, from Lemma 3 and 2, $\mathcal{V}_m = \mathcal{V}_{m2}$, henceforth we will refer to $\mathcal{V}_{m2}$. Consider subspace

$$\bar{\mathcal{V}} := \mathcal{V} + \mathcal{R}_{\mathcal{V}^*}$$

where $\mathcal{V}$ is a subspace satisfying Properties 1, 2 and 3 and $\mathcal{R}_{\mathcal{V}^*} = \mathcal{V}^* \cap \mathcal{S}^*$ represents the constrained reachability subspace on $\mathcal{C}$. It is clear that $\bar{\mathcal{V}}$ satisfies Properties 1, 2 and 3 because

- $\bar{\mathcal{V}}$ is an $(A, \mathcal{B})$ controlled invariant contained in $\mathcal{C}$, being the sum of two controlled invariants contained in $\mathcal{C}$;
- $\mathcal{H} \subseteq \bar{\mathcal{V}} + \mathcal{S}_p$, being $\mathcal{H} \subseteq \mathcal{V}$;
- $\bar{\mathcal{V}}$ is internally stabilizable, being the sum of two internally stabilizable subspaces.

Subspace $\bar{\mathcal{V}}$ is an element of Φ_2 defined in (12), since $\mathcal{V}^* \cap \mathcal{B} \subseteq \mathcal{R}_{\mathcal{V}^*} \subseteq \bar{\mathcal{V}}$. Being $\bar{\mathcal{V}}$ internally stabilizable a state feedback matrix F exists that stabilizes such subspace. Because of Property 2, such matrix stabilizes every subspace $\mathcal{V} \in \Phi_2$ included in $\bar{\mathcal{V}}$, and therefore also its infimum $\mathcal{V}_{m2}$ being all of these subspaces self bounded.
For the sufficiency part, simply note that if $\mathcal{V}_m$ is internally stabilizable than it satisfies Properties 1, 2 and 3 at once. ■

Regarding the dimension of the resolving subspace, it can be easily shown that

$$\mathcal{V}_m \subseteq \mathcal{V}_g, \tag{29}$$

where $\mathcal{V}_g$ is the resolving subspace defined in [10]. In fact, since $\mathcal{R}_{\mathcal{V}^*} \subseteq \mathcal{V}_g$, from proof of Theorem 2 it follows that

$$\bar{\mathcal{V}} = \mathcal{V}_g + \mathcal{R}_{\mathcal{V}^*} = \mathcal{V}_g \in \Phi_2$$

whose infimum is $\mathcal{V}_m$.

4 An example

Consider system (1) with

$$A = 0.1 \begin{bmatrix} 1 & 2 & 1 & -1 & -2 \\ 0 & -1 & 2 & 1 & 1 \\ 0 & 3 & 1 & -1 & -1 \\ 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 0.5774 \\ 0 & -0.5774 \\ 0 & 0 \\ 0 & 0.5774 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Being

$$\mathcal{V}^* = \text{im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{S}_1 = \mathcal{S}^* = \text{im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0.5774 \\ 0 & 0 & -0.5774 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5774 \end{bmatrix}$$

structural condition (4) holds with one step of preview. In other terms, being $\mathcal{H}$ not included in $\mathcal{V}^* + \mathcal{B}$, the measurability of signal $h(k)$ is not sufficient to obtain decoupling, but a preview of at least one step is needed to solve Problem 1.

As regards signal decoupling problem with stability (Problem 2), the resolving subspace $\mathcal{V}_{m1}$ is given by

$$\mathcal{V}_{m1} = \text{im} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which results internally stabilizable.

It is worth noting that subspace $\mathcal{V}_g$ proposed in [10, 7] has dimension 2 which is larger than that of $\mathcal{V}_{m1}$.

5 Conclusions

A new solution for general signal decoupling problems with stability has been proposed. It is based on two necessary and sufficient constructive conditions, one is structural in nature while the other deals with the stability requirement.

The problem has been approached through self-bounded controlled invariants, thus allowing to reduce the dimension of the resolving subspace which corresponds to the infimum of a lattice.

It has been shown that same conditions for decoupling problem with stability to be solved apply independently of the type of signal to be decoupled, being it completely unknown (disturbance), measured or previewed. In other terms we showed that conditions for the DDP, MSDP and PSDP with stability to be solved are similar. The resolving subspace whose internal stabilizability needs to be checked is the infimum of the same lattice. Each problem specializes only in its structural condition.

References

[1] F. Barbagli, G. Marro and D. Prattichizzo, "Previewed signal decoupling problem in discrete-time systems", *Proceedings of MTNS 2000*, France, June 2000.

[2] G. Basile and G. Marro, "L'invarianza rispetto ai disturbi studiata nello spazio degli stati" ("Disturbance decoupling considered in the state space"), *Rendiconti della LXX Riunione Annuale AEI*, paper 1-4-01, Rimini, Italy 1969.

[3] G. Basile and G. Marro, *Controlled and Conditioned Invariants in Linear System Theory.* Prentice-Hall, Englewood Cliffs, NJ, 1992.

[4] G. Basile, G. Marro and A. Piazzi, "A new solution to the disturbance localization problem with stability and its dual", *Proceedings of the '84 International AMSE Conference on Modelling and Simulation*, vol. 1.2, pp. 19–27, Athens, 1984.

[5] G. Basile and G. Marro, "Self-bounded controlled invariant subspaces: a straightforward approach to constrained controllability," in *J. Optimiz. Th. Applic.* vol. 38, no. 1, 1982, pp. 71–81.

[6] S.P. Bhattacharyya, "Disturbance rejection in linear systems", *Int. J. Systems Science.*, vol. 5, no. 7, 1974, pp. 931-943.

[7] H. Imai, M. Shinozuka, T.Yamaki, D. Li and M.Kuwanasile, "Disturbance Decoupling by Feedforward and Preview Control", *Journal of Dynamic Systems, Measurement and Control*, vol. 105, no. 11, March 1983, pp. 931-943.

[8] G. Marro and A. Piazzi, "A geometric approach to multivariable perfect tracking," *Proceedings of the 13th World IFAC Congress*, June 30–July 5, 1996, San Francisco (U.S.A.), vol. C, pp. 241–246.

[9] J. M. Schumacher "On a conjecture of Basile and Marro," in *J. Optimiz. Th. Applic.* vol. 41, no. 2, pp. 371-376, 1983.

[10] J.C. Willems, "Feedforward control, PID control laws, and almost invariant subspaces", *Syst. Contr. Lett.*, vol. 1, no. 4, pp. 277-282, 1982.

[11] W.M. Wonham and A.S. Morse, "Decoupling and pole assignment in linear multivariable systems: a geometric approach", *SIAM Journal of Control and Optimization*, vol. 8, no. 1, pp. 1–18, 1970.

[12] W.M. Wonham, *Linear Multivariable Control: A Geometric Approach*, 3rd Edition, Springer-Verlag, New York, 1985.