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Generalized Signal Decoupling Problem with Stability for Discrete Time Systems

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Abstract: This paper deals with decoupling problems of unknown, measurable and previewed signals. First the well known solutions of unknown and measurable disturbance decoupling problems are recalled. Then new necessary and sufficient constructive conditions for the previewed signal decoupling problem are proposed. The discrete time case is considered. In this domain previewing a signal by p steps means that the k-th sample of the signal to be decoupled is known p steps in advance.

The main result is to prove that the stability condition for all of the mentioned decoupling problems does not change, i.e. the resolving subspace to be stabilized is the same independently of the type of signal to be decoupled, being it completely unknown (disturbance), measured or previewed.

The problem has been studied through self-bounded controlled invariants, thus minimizing the dimension of the resolving subspace which corresponds to the infimum of a lattice. Note that reduced dimension of the resolving controlled invariant subspace yields to reduce the order of the controller units.

1 Introduction

Disturbance decoupling is a classical problem in control theory. It has been one of the first application considered in the geometric approach framework and has been given attention for more than thirty years. In the first formulation of the disturbance decoupling problem (DDP) [4, 13], disturbance signals are assumed to be unknown and unaccessible. Later Bhattacharyya [8] considered the so called *measured signal decoupling problem* (MSDP) in which signals to be decoupled are considered measurable. The structural conditions for the MSDP to be solved are less restrictive than those for the DDP, while stabilizability conditions are similar.

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In this paper the decoupling control problem is approached in a more general setting. Signals which are known in advance or previewed by a given amount of time are considered. Such problem will be referred to as *previewed signal decoupling problem* (PSDP).

The PSDP has been investigated by Willems [12] who first derived, in the continuous time domain, a necessary and sufficient condition to solve the PSDP with pole placement. This solution was based on the so called proportional-integral-derivative control laws consisting of a feedback of the state system and of a linear combination of signal (to be decoupled) and its time derivatives. The major drawback of these extensions of the disturbance decoupling problem in continuous time domain is that control laws include distributions, hence are not practically implementable. Independently, Imai and Shinozuka [9] proposed a similar necessary and sufficient condition for the PSDP with stability in both discrete and continuous time cases. In [1], Estrada and Malabre proposed a synthesis procedure to solve the PSDP problem with stability using the minimum number of required differentiators for the signal to be decoupled.

Conditions for the PSDP to be solved, given in [12, 9, 1], do not care about dimensionality of the resolving controlled invariant subspace. Furthermore, to the best of our knowledge, the problem of reducing the dimension of the resolving subspace for the PSDP has not been thoroughly investigated in the literature. Note that using controlled invariants of minimal dimensions yields to reduce the order of the controller units and possible state observers.

In this paper a new solution for the PSDP with stability based on a subspace with reduced dimension is proposed. Such dimension optimization is obtained thanks to self-bounded controlled invariants. This is a special class of controlled invariants introduced by Basile and Marro in [7, 11] which enjoys interesting properties, the most important of which is to be a lattice instead of a semi-lattice, hence to admit an infimum other than a supremum.

Moreover this paper provides a unique necessary and sufficient condition for signal decoupling problems with stability independently of the type of signal to be decoupled, being it completely unknown (disturbance), measured or previewed. In other terms it is shown that the proposed resolving subspace for the PSDP problems and the well known resolving subspace of the DDP problem proposed by Basile Marro and Piazzi in [6] are equivalent. The proofs are carried out in a geometric framework and are based on several lattices of self bounded $(A, \mathcal{B})$-controlled invariants.

In this paper discrete-time systems are considered. In such domain, the solution of the PSDP is more elegant and is practically implementable. The structure of the compensator, whereby the signal decoupling of previewed signals is obtained, is discussed. It consists of a preaction and a postaction unit. A new synthesis procedure, based on geometric approach algorithms, is provided. Preliminary results of this work have been presented in [2, 3].

The following notation is used. **R** stands for the field of real numbers. Sets, vector spaces and subspaces are denoted by script capitals like $\mathcal{X}$, $\mathcal{I}$, $\mathcal{V}$, etc.. Since most of the geometric theory of dynamic system herein presented is developed in the vector space $\mathbf{R}^n$, we reserve the symbol $\mathcal{X}$ for the full space, i.e., we assume $\mathcal{X} := \mathbf{R}^n$. Matrices and linear maps are denoted by slanted capitals like A, B, etc., the image and the null space of the generic matrix or linear transformation A by $\mathrm{im}A$ and $\ker A$, respectively, the transpose of the generic real matrix A by A^T, its spectrum by $\sigma(A)$ and its pseudoinverse by $A^\#$.

The reminder of this paper is organized as follows. Section 2 presents the structural conditions for the general PSDP. In Section 3 new necessary and sufficient conditions for the PSDP with stability are stated, in section 4 a synthesis procedure for the decoupling compensator is reported and finally in Section 5 an illustrative example is discussed.

2 Structural conditions for the PSDP

Let us consider the discrete-time system

$$\begin{cases} x(k+1) &= Ax(k) + Bu(k) + Hh(k) \\ y(k) &= Cx(k) \end{cases} \tag{1}$$

where $x \in \mathcal{X}$ (= $\mathbf{R}^n$), $u \in \mathbf{R}^m$, $h \in \mathbf{R}^h$ and $y \in \mathbf{R}^q$ denote the state, the manipulable input, the signal to be decoupled and the regulated output, respectively. In the following the short notations $\mathcal{B} := \mathrm{im} B$, $\mathcal{C} := \ker C$ and $\mathcal{H} := \mathrm{im} H$ will be used.

In this paper we deal with the signal decoupling problem when a certain degree of knowledge for signal $h(k)$ is available. In particular we assume that signal $h(k)$ is previewed, i.e. it is known p steps in advance, or analytically the sample $h(k)$ is known at step $k-p$. Henceforth previewed signal $h(k)$ by p steps will be referred to as p-previewed $h(k)$, or shortly ${}^p h(k)$. Note that measurable disturbance can be thought as 0-previewed signals.

As for the MSDP the input $Bu(k)$ is used to cancel part of the (measured) disturbance when it occurs, in the case of the PSDP, the preview is used to “prepare” the system dynamics to localize signal $h(k)$ on the nullspace of the output matrix C. This is formalized in the following statement.

Problem 1 (Previewed signal decoupling) *Refer to system (1) with zero initial condition and assume that input h is previewed by p instants of time, $p \geq 0$. Determine a control law which, making use of this preview, is able to maintain the output $y(k)$ identically zero.*

Note that in case of non-purely dynamic systems, i.e. when the output equation in (1) is generalized as

$$y(k) = C\,x(k) + D\,u(k) + M\,h(k),$$

the PSDP can be re-casted as in Problem 1 by simply inserting a dummy delay unit at the output and including it in the system equations, thus recovering an equivalent purely dynamic form.

In a geometric framework, the key tool to analyze the structural conditions for the signal decoupling problem, is the well-known [5] algorithm computing $\mathcal{S}^* := \min S(A, \mathcal{C}, \mathcal{B})$, the minimal $(\mathcal{A}, \mathcal{C})$-conditioned invariant containing $\mathcal{B}$, here reported for the reader convenience:

$$\mathcal{S}_0 := \mathcal{B} \tag{2}$$

$$\mathcal{S}_i := \mathcal{B} + A(\mathcal{S}_{i-1} \cap \mathcal{C}). \tag{3}$$

Structural conditions to solve Problem 1 for p-previewed signals are given in the following theorem whose proof can be easily derived from [12, 1].

Theorem 1 *Necessary and sufficient condition for Problem 1 to be solved is that*

$$\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{S}_p. \tag{4}$$

where $\mathcal{V}^* := \max \mathcal{V}(A, \mathcal{B}, \mathcal{C})$ is the maximal controlled invariant contained in the nullspace of $\mathcal{C}$.

Remark 1 *Structural condition (4) in Theorem 1, is similar to that proposed in [12, 1] for the continuous-time case but less restrictive since condition (4) does not consider stability. It is worth* *noting that the case of measurable inputs is accounted for by condition (4). In fact measurable signals corresponds to $p = 0$ and therefore (4) turns into the well known condition [8]*

$$\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}. \tag{5}$$

Similarly the lack of preview yields to the well-known structural condition for unknown signals, i.e.

$$\mathcal{H} \subseteq \mathcal{V}^*. \tag{6}$$

Summarizing, being $\mathcal{V}^* \subseteq \mathcal{V}^* + \mathcal{B} \subseteq \mathcal{V}^* + \mathcal{S}_p$, the larger the preview time the easier it is to solve the decoupling problem.

The following property characterizes the minimum number of preview steps necessary to decouple previewed signals for a given disturbance matrix H.

Property 1 *Consider system (1) and let r be the minimum number of steps necessary to obtain convergence of algorithm for $\mathcal{S}^*$, (2,3). The minimum positive integer $p \leq r$, such that condition (4) holds, corresponds to the minimum number of previewed steps for $h(k)$ needed to decouple signal $Hh(k)$. If for $p = r$ condition (4) is not satisfied, the PSDP has no solution for the given disturbance matrix H.*

If structural condition (4) holds, it is possible to decompose the disturbance effect into two separate parts as follows

$$\begin{aligned} H &= H_V + H_S & (7)\\ \mathcal{H}_V &:= \operatorname{im}(H_V) \subseteq \mathcal{V}^* & (8)\\ \mathcal{H}_S &:= \operatorname{im}(H_S) \subseteq \mathcal{S}_p. & (9) \end{aligned}$$

Matrices H_V and H_S are basic to synthesize the controller solving Problem 1 as discussed in Section 4. Components of $Hh(k)$ lying on $\mathcal{S}_p$ can be canceled through a *preaction unit*. For this purpose note that $\mathcal{S}_p$ can be interpreted as the reachable subspace in p $(p \geq 0)$ steps from $x_0 = 0$, with the state trajectory constrained to lie on the nullspace of the output matrix C in the $(p-1)$-steps interval $[0, p-1]$:

$$Cx(k) = 0 \qquad \text{for } (k = 0, 1, \ldots p-1). \tag{10}$$

In other terms, the *preaction unit*, which is a part of the decoupling controller, exploits the signal preview to cancel $H_S h(k)$, i.e. the part of $Hh(k)$ belonging to $\mathcal{S}_p$ Because of the special reachability subspace $\mathcal{S}_p$, this happens while maintaining the output identically zero.

On the other hand signal $H_V h(k)$ is localized in the nullspace of the output matrix according to standard decoupling techniques [5].

3 Previewed signal decoupling problem with stability

The p–previewed signal decoupling problem with stability is investigated.

Problem 2 (Previewed signal decoupling with stability) *Refer to system (1) with zero initial condition and assume that it is stabilizable and that input h is previewed by p instants of time, $p \geq 0$. Determine a control law which, making use of the preview, is able to maintain the output $y(k)$ identically zero while keeping the state trajectory bounded.*

The Previewed signal decoupling with stability is approached by means of lattices of self-bounded controlled invariants [7, 11]. A special attention is devoted to the dimension of the resolving subspace.

Let us introduce the lattice of all the $(A,\mathcal{B})$-controlled invariants self bounded with respect to $\mathcal{C}$,

$$\Phi = \Phi(\mathcal{B},\mathcal{C}) = \{\mathcal{V} \mid \ A\,\mathcal{V} \subseteq \mathcal{V} + \mathcal{B},\ \mathcal{V} \subseteq \mathcal{C},\ \mathcal{V}^* \cap \mathcal{B} \subseteq \mathcal{V}\} \tag{11}$$

whose infimum is given by

$$\mathcal{V}^* \cap \min\mathcal{S}(A,\mathcal{C},\mathcal{B}),$$

and the lattice of all the $(A,\mathcal{B}+\mathcal{H}_V)$-controlled invariants self bounded with respect to $\mathcal{C}$,

$$\begin{aligned}&\Phi_1 = \Phi(\mathcal{B}+\mathcal{H}_V,\mathcal{C}) = \\ &\{\mathcal{V} \mid \ A\,\mathcal{V} \subseteq \mathcal{V} + \mathcal{B} + \mathcal{H}_V,\ \mathcal{V} \subseteq \mathcal{C},\ \mathcal{V}^* \cap (\mathcal{B}+\mathcal{H}_V) \subseteq \mathcal{V}\}\end{aligned} \tag{12}$$

whose infimum is given by

$$\mathcal{V}_{m1} = \mathcal{V}^* \cap \min\mathcal{S}(A,\mathcal{C},\mathcal{B}+\mathcal{H}_V). \tag{13}$$

Subspace $\mathcal{V}_{m1}$ can be written as in (13) since condition (8) holds and therefore

$$\mathcal{V}^* \equiv \max\mathcal{V}(A,\mathcal{B}+\mathcal{H}_V,\mathcal{C}).$$

The following lemmas hold.

Lemma 1 *The set*

$$\Phi_2 = \{\mathcal{V} \mid \mathcal{V} \in \Phi, \mathcal{H} \subseteq \mathcal{V} + \mathcal{S}_p\} \tag{14}$$

enjoys the following properties:

1. *is a sub-lattice of* Φ*;*

2. $\Phi_2 \equiv \{\mathcal{V} \mid \mathcal{V} \in \Phi, \mathcal{V}^* \cap (\mathcal{H} + \mathcal{S}_p) \subseteq \mathcal{V}\}$*;*

Proof:
(1.) We want to show that given two generic elements $\mathcal{V}_1$ and $\mathcal{V}_2$ of set Φ_2 their sum and intersection still belongs to the same set. Such proof appears trivial for the subspace obtained by summing the two given subspaces. Let's consider now element $\mathcal{V}_1 \cap \mathcal{V}_2$. By assumption, since both $\mathcal{V}_1$ and $\mathcal{V}_2$ belong to Φ_2 it follows that

$$\mathcal{H} \subseteq \mathcal{V}_1 + \mathcal{S}_p \tag{15}$$

$$\mathcal{H} \subseteq \mathcal{V}_2 + \mathcal{S}_p \tag{16}$$

which lead to

$$\mathcal{H} \subseteq (\mathcal{V}_1 + \mathcal{S}_p) \cap (\mathcal{V}_2 + \mathcal{S}_p).$$

By intersecting both terms with $\mathcal{V}^* + \mathcal{S}_p$ we obtain

$$\mathcal{H} \subseteq ((\mathcal{V}_1 + \mathcal{S}_p) \cap (\mathcal{V}_2 + \mathcal{S}_p)) \cap (\mathcal{V}^* + \mathcal{S}_p) \tag{17}$$

since the structural condition (4) holds, and then

$$\mathcal{H} \subseteq (\mathcal{V}^* \cap (\mathcal{V}_1 + \mathcal{S}_p)) \cap (\mathcal{V}^* \cap (\mathcal{V}_2 + \mathcal{S}_p)) + \mathcal{S}_p$$

using the distributive property, being $\mathcal{S}_p$ included in $(\mathcal{V}_1 + \mathcal{S}_p) \cap (\mathcal{V}_2 + \mathcal{S}_p)$. Analogously we get

$$\mathcal{H} \subseteq ((\mathcal{V}_1 \cap \mathcal{V}^*) + (\mathcal{V}^* \cap \mathcal{S}_p)) \cap ((\mathcal{V}_2 \cap \mathcal{V}^*) + (\mathcal{V}^* \cap \mathcal{S}_p)) + \mathcal{S}_p$$

and finally

$$\mathcal{H} \subseteq (\mathcal{V}_1 \cap \mathcal{V}_2) + \mathcal{S}_p$$

being $\mathcal{V}_1$ and $\mathcal{V}_2$ both included in $\mathcal{V}^*$ and both including $\mathcal{V}^* \cap \mathcal{S}_p$.

(2.)
$(\Rightarrow)$

$$\mathcal{V} \in \Phi_1 \Rightarrow \mathcal{H} \subseteq \mathcal{V} + \mathcal{S}_p \Rightarrow \mathcal{H} + \mathcal{S}_p \subseteq \mathcal{V} + \mathcal{S}_p$$

and therefore intersecting both members with $\mathcal{V}^*$ we obtain

$$\mathcal{V}^* \cap (\mathcal{H} + \mathcal{S}_p) \subseteq \mathcal{V}^* \cap (\mathcal{V} + \mathcal{S}_p) = \mathcal{V} + (\mathcal{V}^* \cap \mathcal{S}_p) = \mathcal{V}$$

being $(\mathcal{V}^* \cap \mathcal{S}_p) \subseteq (\mathcal{V}^* \cap \mathcal{S}^*)$ which is the infimum of Φ and therefore is contained in all $\mathcal{V} \in \Phi$.

$(\Leftarrow)$

$$\mathcal{V}^* \cap (\mathcal{H} + \mathcal{S}_p) \subseteq \mathcal{V}$$

summing $\mathcal{S}_p$ to both members we obtain

$$\mathcal{S}_p + (\mathcal{V}^* \cap (\mathcal{H} + \mathcal{S}_p)) \subseteq \mathcal{V} + \mathcal{S}_p$$

from which using the distributive property we obtain

$$(\mathcal{S}_p + \mathcal{V}^*) \cap (\mathcal{S}_p + \mathcal{H}) \subseteq \mathcal{V} + \mathcal{S}_p$$

and being $\mathcal{H} \subseteq \mathcal{S}_p + \mathcal{H} \subseteq \mathcal{S}_p + \mathcal{V}^*$ we obtain

$$(\mathcal{S}_p + \mathcal{H}) \subseteq \mathcal{V} + \mathcal{S}_p$$

and that

$$\mathcal{H} \subseteq \mathcal{V} + \mathcal{S}_p$$

■

Lemma 2 *The infimum of Φ_2 is given by*

$$\mathcal{V}_{m2} = \mathcal{V}^* \cap \min \mathcal{S}(A, \mathcal{V}^*, \mathcal{S}_p + \mathcal{H}). \tag{18}$$

Proof:
The proof will be developed in two steps:

(A) Any element of Φ_2 contains $\mathcal{V}_{m2} = \mathcal{V}^* \cap \mathcal{S}_2^*$ where

$$\mathcal{S}_2^* = \min \mathcal{S}(A, \mathcal{V}^*, \mathcal{S}_p + \mathcal{H}); \tag{19}$$

(B) $\mathcal{V}^* \cap \mathcal{S}_2^*$ is an element of Φ_2

Step *(A)*: Consider the sequence that defines $\mathcal{S}_2^*$:

$$\mathcal{Z}'_0 := \mathcal{S}_p + \mathcal{H} \tag{20}$$
$$\mathcal{Z}'_i := \mathcal{S}_p + \mathcal{H} + A\,(\mathcal{Z}'_{i-1} \cap \mathcal{V}^*) \quad (i=1,\ldots) \tag{21}$$

Let $\mathcal{V}$ be a generic element of Φ_2, so that

$$A\,\mathcal{V} \subseteq \mathcal{V} + \mathcal{B}\,, \quad \mathcal{V} \supseteq \mathcal{V}^* \cap \mathcal{B}.$$

We proceed by induction: clearly

$$\mathcal{Z}'_0 \cap \mathcal{V}^* \subseteq \mathcal{V}$$

since by assumption $\mathcal{V}^* \cap (\mathcal{S}_p + \mathcal{H}) \subseteq \mathcal{V}$, and from

$$\mathcal{Z}'_{i-1} \cap \mathcal{V}^* \subseteq \mathcal{V}$$

it follows that

$$A(\mathcal{Z}'_{i-1} \cap \mathcal{V}^*) \subseteq A\mathcal{V} \subseteq \mathcal{V} + \mathcal{B}$$

since $\mathcal{V}$ is an $(A, \mathcal{B})$-controlled invariant. Adding $\mathcal{S}_p + \mathcal{H}$ to both members yields

$$\mathcal{S}_p + \mathcal{H} + A\,(\mathcal{Z}'_{i-1} \cap \mathcal{V}^*) \subseteq \mathcal{V} + \mathcal{S}_p + \mathcal{H}$$

where the left term of the last inclusion is by definition subspace $\mathcal{Z}'_i$ and, by intersecting with $\mathcal{V}^*$, we finally obtain

$$\mathcal{Z}'_i \cap \mathcal{V}^* \subseteq (\mathcal{V} + (\mathcal{S}_p + \mathcal{H})) \cap \mathcal{V}^* = \mathcal{V} + (\mathcal{S}_p + \mathcal{H}) \cap \mathcal{V}^* = \mathcal{V}$$

which completes the induction argument and the proof of Step A.

Step *(B)*: From the following statements

1. $\mathcal{S}_1^* \cap \mathcal{V}^*$ is an $(A, \mathcal{B})$-controlled invariant contained in $\mathcal{C}$;

2. $\mathcal{S}_1^* \cap \mathcal{V}^*$ is self bounded with respect to $\mathcal{C}$

3. $\mathcal{H} \subseteq (\mathcal{S}_1^* \cap \mathcal{V}^*) + \mathcal{S}_p$

it follows that $\mathcal{V}^* \cap \mathcal{S}_2^*$ is an element of Φ_2.

To prove statement 1 note that

$$A\,\mathcal{V}^* \subseteq \mathcal{V}^* + \mathcal{B}$$
$$A\,(\mathcal{S}_1^* \cap \mathcal{V}^*) \subseteq \mathcal{S}_1^*$$

which simply expresses $\mathcal{V}^*$ as an $(A, \mathcal{B})$-controlled invariant and $\mathcal{S}_1^*$ as an $(A, \mathcal{V}^*)$-conditioned invariant. By intersection it follows that

$$A\,(\mathcal{S}_1^* \cap \mathcal{V}^*) \subseteq \mathcal{S}_1^* \cap (\mathcal{V}^* + \mathcal{B}) = \mathcal{S}_1^* \cap \mathcal{V}^* + \mathcal{B}$$

being $\mathcal{B} \subseteq \mathcal{S}_1^*$. Then $\mathcal{S}_1^* \cap \mathcal{V}^*$ is an $(A, \mathcal{B})$-controlled invariant contained in $\mathcal{C}$.

To prove statement 2 note that

$$\mathcal{V}^* \cap \mathcal{B} \subseteq \mathcal{V}^* \cap \mathcal{S}_1^*.$$

Finally, to prove statement 3 note that being $\mathcal{S}_p \subseteq \mathcal{S}_1^*$ and $\mathcal{H} \subseteq \mathcal{S}_1^*$ it follows that

$$\mathcal{H} \subseteq (\mathcal{V}^* \cap \mathcal{S}_p) + \mathcal{S}_1^* = (\mathcal{V}^* \cap \mathcal{S}_1^*) + \mathcal{S}_p.$$

■

Lemma 3 *The set*

$$\Phi_3 = \{\mathcal{V} \,|\, \mathcal{V} \in \Phi, \mathcal{H}_V \subseteq \mathcal{V} + \mathcal{S}_p\} \tag{22}$$

enjoys the following properties:

1. *is a sub-lattice of* Φ*;*

2. $\Phi_3 \equiv \{\mathcal{V} \,|\, \mathcal{V} \in \Phi, \mathcal{V}^* \cap (\mathcal{H}_V + \mathcal{S}_p) \subseteq \mathcal{V}\}$*;*

3. *the infimum of* Φ_3 *is given by*

$$\mathcal{V}_{m3} = \mathcal{V}^* \cap \min\mathcal{S}(A, \mathcal{V}^*, \mathcal{S}_p + \mathcal{H}_V) \tag{23}$$

4. $\Phi_3 \equiv \Phi_2$*, i.e.* $\mathcal{V}_{m2} \equiv \mathcal{V}_{m3}$

5. $\Phi_3 \subseteq \Phi_1$

6. $\mathcal{V}_{m1} \in \Phi_3$

7. $\mathcal{V}_{m1} \equiv \mathcal{V}_{m2} \equiv \mathcal{V}_{m3}$

Proof:
(1., 2. and 3.): From (8), it follows

$$\mathcal{H}_V \subseteq \mathcal{V}^* + \mathcal{S}_p$$

and therefore proofs of Properties 1 and 2 are analogous to those of Properties 1, 2 in Lemma 1 while proof of Property 3. is similar to that of Lemma 2
(4.) This property follows from Property 2 of this statement, from Lemma 1 and from eqs.(7) and (9).
(5.) Let $\mathcal{V} \in \Phi_3$ it follows

- $A\mathcal{V} \subseteq \mathcal{V} + \mathcal{B} + \mathcal{H}_V$ since $\mathcal{V}$ is an $(A, \mathcal{B})$-controlled invariant
- $\mathcal{V} \subseteq \mathcal{C}$
- $\mathcal{V}^* \cap (\mathcal{B} + \mathcal{H}_V) \subseteq \mathcal{V}^* \cap (\mathcal{S}_p + \mathcal{H}_V) \subseteq \mathcal{V}$

then

$$\mathcal{V} \in \Phi_3 \Rightarrow \mathcal{V} \in \Phi_1.$$

(6.) First of all note that $A\mathcal{V}_{m1} \subseteq \mathcal{V}_{m1} + \mathcal{B}$ since

$$\begin{array}{rll} A\mathcal{V}_{m1} = & A((\mathcal{V}^* \cap \mathcal{C}) \cap \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V)) & \subseteq \\ & A\mathcal{V}^* \cap A(\mathcal{C} \cap \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V)) & \subseteq \\ & (\mathcal{V}^* + \mathcal{B}) \cap \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V) & = \\ & (\mathcal{V}^* \cap \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V)) + \mathcal{B} & = \\ & \mathcal{V}_{m1} + \mathcal{B} & , \end{array}$$

moreover $\mathcal{V}_{m1} \subseteq \mathcal{C}$ and $\mathcal{V}^* \cap \mathcal{B} \subseteq \mathcal{V}_{m1}$, then $\mathcal{V}_{m1} \in \Phi$. Finally, note that $\mathcal{H}_V \subseteq \mathcal{V}_{m1} + \mathcal{S}_p$. In fact, being

$$\mathcal{S}_p \subseteq \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V),$$

it holds

$$\begin{aligned} \mathcal{V}_{m1} + \mathcal{S}_p &= (\mathcal{V}^* \cap \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V)) + \mathcal{S}_p \\ &= (\mathcal{V}^* + \mathcal{S}_p) \cap \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V) \end{aligned}$$

and $\mathcal{H}_V$ is included in both subspaces of the latter intersection.
(7.) Implied by Properties 4, 5 and 6. ■

Before stating the main result, let us recall a fundamental property (proved in [5]) of self bounded subspaces:

Remark 2 *Let $\bar{\mathcal{V}}$ and $\mathcal{V}$ be a couple of any $(A,\mathcal{B})$-controlled invariant subspaces self bounded with respect to $\mathcal{C}$, see eq. (11), such that $\mathcal{V} \subseteq \bar{\mathcal{V}}$. Let F be a matrix such that $(A+BF)\bar{\mathcal{V}} \subseteq \bar{\mathcal{V}}$. Then $(A+BF)\mathcal{V} \subseteq \mathcal{V}$ holds.*

Theorem 2 *The signal decoupling problem with stability for the p-previewed signal ${}^p h(k)$ stated in Problem 2 is solvable if and only if the structural condition (4) is satisfied and*

$$\mathcal{V}_{m1} = \mathcal{V}^* \cap \min \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V),$$

defined in (13), is internally stabilizable.

Proof:
As shown in Theorem 1 the purpose of the preaction is to cancel, at the generic time instant k, the component of $Hh(k)$ on $\mathcal{S}_p$ in order to force the state dynamics (excited by the other component $H_V h(t)$) on a subspace $\mathcal{V}$ satisfying the following properties:

1. $\mathcal{V}$ is an $(A,\mathcal{B})$ controlled invariant included in $\mathcal{C}$;

2. $\mathcal{V}$ is such that $\mathcal{H} \subseteq \mathcal{V} + \mathcal{S}_p$;

3. $\mathcal{V}$ is internally stabilizable.

We will now prove the necessity of the statement, i.e. that if a subspace $\mathcal{V}$ exists that solves Problem 2 (with stability) then $\mathcal{V}_{m1}$ is internally stabilizable. Being $\mathcal{V}_{m1} = \mathcal{V}_{m2}$ (Lemma 3), henceforth we will refer to $\mathcal{V}_{m2}$. Consider subspace

$$\bar{\mathcal{V}} := \mathcal{V} + \mathcal{R}_{\mathcal{V}^*}$$

where $\mathcal{V}$ is a subspace satisfying Properties 1, 2 and 3 and $\mathcal{R}_{\mathcal{V}^*} = \mathcal{V}^* \cap \mathcal{S}^*$ represents the constrained reachability subspace on $\mathcal{C}$. It is clear that $\bar{\mathcal{V}}$ satisfies Properties 1, 2 and 3 because

- $\bar{\mathcal{V}}$ is an $(A,\mathcal{B})$ controlled invariant contained in $\mathcal{C}$, being the sum of two controlled invariants contained in $\mathcal{C}$;
- $\mathcal{H} \subseteq \bar{\mathcal{V}} + \mathcal{S}_p$, being $\mathcal{H} \subseteq \mathcal{V} + \mathcal{S}_p$;
- $\bar{\mathcal{V}}$ is internally stabilizable, being the sum of two internally stabilizable subspaces.

Subspace $\bar{\mathcal{V}}$ is an element of Φ_2 defined in (14), since $\mathcal{V}^* \cap \mathcal{B} \subseteq \mathcal{R}_{\mathcal{V}^*} \subseteq \bar{\mathcal{V}}$. Being $\bar{\mathcal{V}}$ internally stabilizable a state feedback matrix F exists that stabilizes such subspace. Because of Remark 2, such matrix stabilizes every subspace $\mathcal{V} \in \Phi_2$ included in $\bar{\mathcal{V}}$, and therefore also its infimum $\mathcal{V}_{m2}$ being all of these subspaces self bounded.

For the sufficiency part, simply note that if $\mathcal{V}_{m1}$ is internally stabilizable than it satisfies Properties 1, 2 and 3 at once. ■

Regarding the dimension of the resolving subspace, it can be easily shown that

$$\mathcal{V}_m \subseteq \mathcal{V}_g, \tag{24}$$

where $\mathcal{V}_g$ is the resolving subspace defined in [12]. In fact, since $\mathcal{R}_{\mathcal{V}^*} \subseteq \mathcal{V}_g$, from proof of Theorem 2 it follows that

$$\bar{\mathcal{V}} = \mathcal{V}_g + \mathcal{R}_{\mathcal{V}^*} = \mathcal{V}_g \in \Phi_2$$

whose infimum is $\mathcal{V}_m$.

The second main contribution of this paper consists in unifying stability condition for disturbance decoupling problem, measurable signal (disturbance) decoupling problem, reported in [5], and the more general previewed signal decoupling problem.

Theorem 3 *The signal decoupling problem with stability for the p-previewed signal ${}^p h(k)$ stated in Problem 2 is solvable if and only if the structural condition (4) is satisfied and*

$$\mathcal{V}_m := \mathcal{V}^* \cap \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}), \tag{25}$$

is internally stabilizable.

Proof: The proof is based on Theorem 2 and on the equivalence

$$\mathcal{V}_{m1} \equiv \mathcal{V}_m \tag{26}$$

which is proved in the following.

Being

$$\mathcal{H}_S \subseteq \mathcal{S}_p \subseteq \min\mathcal{S}(A, \mathcal{C}, \mathcal{B}) \subseteq \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V),$$

from (7)

$$\mathcal{B} + \mathcal{H} \subseteq \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V), \tag{27}$$

$$\mathcal{B} + \mathcal{H}_V \subseteq \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}). \tag{28}$$

Now, let us prove that

$$\min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}) \subseteq \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V). \tag{29}$$

by applying induction arguments to the subspace sequence defining $\min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H})$, whose i–element is $\mathcal{Z}'_i$. From (27), one gets

$$\mathcal{Z}'_0 = \mathcal{B} + \mathcal{H} \subseteq \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V).$$

Assume that

$$\mathcal{Z}'_{i-1} \subseteq \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V),$$

being $\min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V)$ an $(A, \mathcal{C})$-conditioned invariant containing $\mathcal{B} + \mathcal{H}$, it follows that

$$\mathcal{B} + \mathcal{H} + A(\mathcal{Z}'_{i-1} \cap \mathcal{C}) \subseteq \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V).$$

Similarly, starting from (28), it is possible to prove that

$$\min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V) \subseteq \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}). \tag{30}$$

Finally from (29) and (30) , one gets

$$\min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_V) = \min\mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H})$$

and equivalence (26) is proven. ■

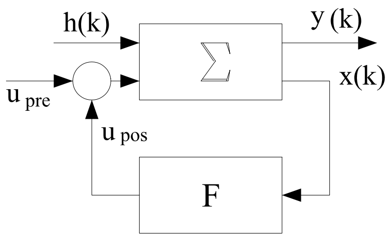

Figure 1: Postaction $u_{\text{pos}}(k)$ is synthesized through a state feedback matrix F.

Remark 3 *Necessary and sufficient conditions for disturbance decoupling problem (DDP), measurable and previewed signal decoupling problem (MSDP and PSDP) with stability only differ for structural condition (6), (5) and (4), respectively, but not for the stability condition which is unique and corresponds to the internal stabilizability of controlled invariant* $\mathcal{V}_m$ *(25).*

4 Synthesis procedure for the controller solving the PSDP

Assume that necessary and sufficient conditions of Theorem 2 or Theorem 3 hold. Due to equation (7) it is always possible to decompose the effect of $Hh(k)$ on system dynamics into two separate parts. The effect of $H_V h(k)$ can be decoupled using a postaction unit, being $\mathcal{H}_V \subseteq \mathcal{V}^*$. The effect of $H_S h(k)$ can be nulled using a preaction unit, being $\mathcal{H}_S \subseteq \mathcal{S}_p$.

4.1 Postaction unit

Due to (7) system (1) can be rewritten as

$$\begin{cases} x(k+1) &= Ax(k) + Bu(k) + H_V h(k) + H_S h(k) \\ y(k) &= Cx(k) \end{cases} \tag{31}$$

where $u(k) = u_{\text{pos}}(k) + u_{\text{pre}}(k)$. The purpose of the postaction unit is to decouple $\mathcal{H}_v$. It is an easy matter to show that $\mathcal{H}_V \subseteq \mathcal{V}_{m1}$. Applying Theorem 2, a stabilizing state feedback matrix F exists such that the state trajectory excited by $H_V h(k)$ evolve onto $\mathcal{V}_{m1} \in \mathcal{C}$. Therefore the postaction unit is simply given by

$$u_{\text{pos}}(k) = Fx(k) \tag{32}$$

as shown in Fig. 1.

It is worth noting that, since the system dynamics evolve on a known subspace, the postaction unit can also be implemented as a feedforward unit.

4.2 Preaction unit

In order to design the preaction unit, subspace $\mathcal{S}_p$ must be interpreted as a special reachability subspace.

Property 2 *Subspace* $\mathcal{S}_p$ *corresponds to the set of states reachable in* p $(p \geq 0)$ *steps from initial condition* $x_0 = 0$*, with the state trajectory constrained to evolve onto* $\mathcal{C}$ *in the preceding* p*-steps interval* $[0, p-1]$.

Analytically, matrices Ω_0, Ω_1 ... Ω_p *exist such that*

$$\mathcal{S}_p = \mathrm{im} \begin{bmatrix} B & AB & A^2B & \cdots & A^pB \end{bmatrix} \begin{bmatrix} \Omega_0 \\ \Omega_1 \\ \Omega_2 \\ \vdots \\ \Omega_p \end{bmatrix}$$

and

$$\begin{bmatrix} CB & CAB & \cdots & CA^{p-1}B \\ 0 & CB & \cdots & CA^{p-2}B \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & CB \end{bmatrix} \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \vdots \\ \Omega_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Taking into account postaction (32), system (31) can be rewritten as

$$x(k+1) = A_f x(k) + B u_{\text{pre}}(k) + H_s h(k) + H_v h(k) \tag{33}$$

where

$$A_f := A + BF.$$

To decouple the effects of $H_S h(k)$ on system dynamics (33), a preaction unit is built as

$$u_{\text{pre}}(k) = \sum_{l=0}^{p} \Phi(l) h(k+l) \tag{34}$$

where gains of the preaction unit are computed as

$$\begin{bmatrix} \Phi_0 \\ \Phi_1 \\ \vdots \\ \Phi_p \end{bmatrix} = M^{\#} \begin{bmatrix} -H_s \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{35}$$

being

$$M = \begin{bmatrix} B & A_f B & A_f^2 B & \cdots & A_f^p B \\ 0 & CB & CA_f B & \cdots & CA_f^{p-1} B \\ 0 & 0 & CB & \cdots & CA_f^{p-2} B \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & CB \end{bmatrix}. \tag{36}$$

Consistency of (35) is guaranteed by Property 2 and by inclusion (9).

The preaction unit in eq. (34) is reported in Fig. 2. Preaction consists of a p-step FIR system which, previewing the signal $h(k)$ p-steps in advance, is able to prepare system dynamics to cancel component $H_S h(k)$ when it presents as input to the system (at the time instant k).

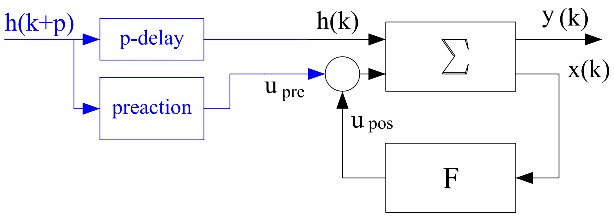

Figure 2: The decoupling compensator consists of a preaction unit and a postaction unit.

5 An illustrative example

Let us consider the previewed signal decoupling problem for system (1) with

$$A = 0.1 \begin{bmatrix} 1 & 2 & 1 & -1 & -2 \\ 0 & -1 & 2 & 1 & 1 \\ 0 & 3 & 1 & -1 & -1 \\ 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0.5774 \\ 0 & 0 & -0.5774 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5774 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Algorithms for $\mathcal{V}^*$ and $\mathcal{S}^*$ converge to

$$\mathcal{V}^* = \text{im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{S}^* = \text{im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0.5774 \\ 0 & 0 & -0.5774 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5774 \end{bmatrix}.$$

Note that

$$\mathcal{S}_1 = \mathcal{S}^*$$

and that

$$\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{S}_1 = \mathcal{V}^* + \mathcal{S}^*.$$

Being

$$\mathcal{H} \not\subseteq \mathcal{V}^* + \mathcal{B},$$

$p = 1$ is the minimum number of previewed steps necessary to solve the decoupling problem.

According to (7), matrix H is divided in two matrices

$$H_V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H_S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.5774 \\ 0 & 0 & -0.5774 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5774 \end{bmatrix}.$$

The resolving subspace is evaluated as

$$\mathcal{V}_{m1} = \mathcal{V}_{m2} = \text{im} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which results internally stabilizable.

It is worth noting that that subspace $\mathcal{V}_g^*$ proposed in [12] has, in this case, a dimension which is double of $\mathcal{V}_{m1}$.

Postaction unit is given by the state feedback $u_{\text{pos}} = Fx(k)$ where

$$F = \begin{bmatrix} -0.1 & 0 & 0 & 0.1 & 0 \\ -0.1 & 0 & 0 & -0.2 & 0 \end{bmatrix}$$

while for preaction unit we obtain from (35) and (36)

$$\Phi_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -5.7735 \end{bmatrix}, \quad \Phi_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Simulation are executed by applying a signal

$$h(k) = \begin{bmatrix} h_1(k) & h_2(k) & h_3(k) \end{bmatrix}^T$$

where $h_1(k)$, $h_2(k)$ and $h_3(k)$ are band-limited white noises, as reported in Fig.3.

The two components of postaction and preaction signals $u_{\text{pos}}(k)$ are reported in Fig. 4 and Fig. 5, respectively. Preaction-postaction compensator perfectly decouples signal $h(k)$ which has a 1-step preview and maintains the state trajectory bounded.

6 Conclusions

A new solution for general signal decoupling problems with stability has been proposed. It is based on two necessary and sufficient constructive conditions, one is structural in nature while the other deals with the stability requirement.

The problem has been approached through self-bounded controlled invariants, thus allowing to reduce the dimension of the resolving subspace which corresponds to the infimum of a lattice.

It has been shown that same conditions for decoupling problem with stability to be solved apply independently of the type of signal to be decoupled, being it completely unknown (disturbance), measured or previewed. In other terms each problem specializes only in its structural condition. The stability condition is the same for the DDP, MSDP and PSDP, since the resolving subspace, whose internal stabilizability needs to be checked, is the infimum of the same lattice.

Finally a new easily implementable synthesis procedure has been proposed and used to solve a simple example.

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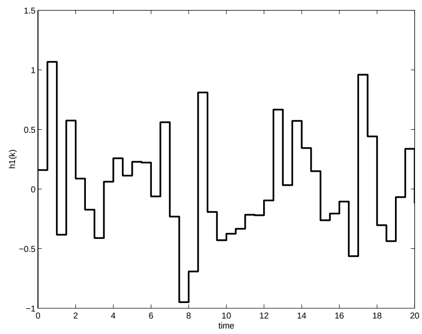
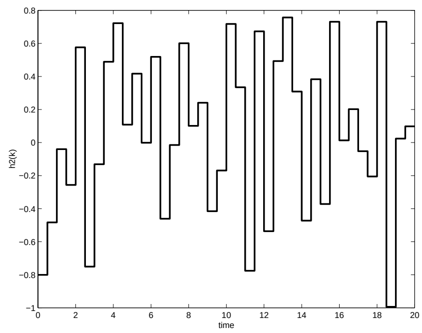
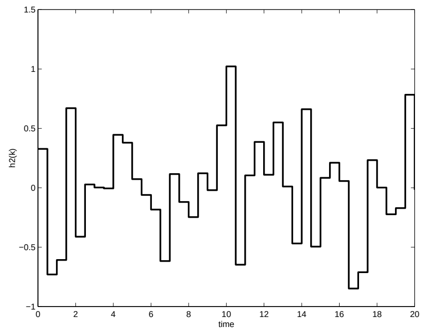


$h_1(k)$



$h_2(k)$



$h_3(k)$

Figure 3: The three components of 1-previewed signal $h(k)$.

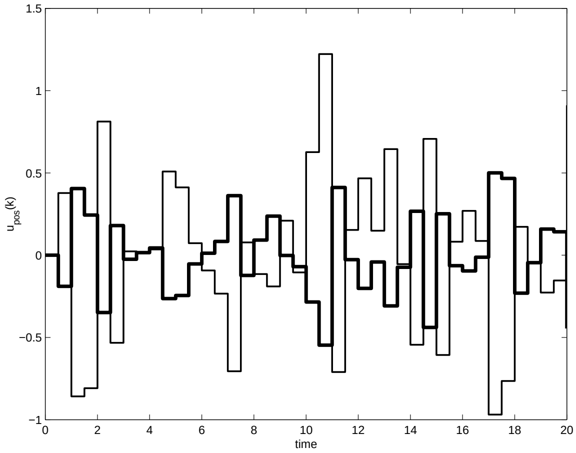


Figure 4: The two components of postaction control $u_{\text{pos}}(k)$.

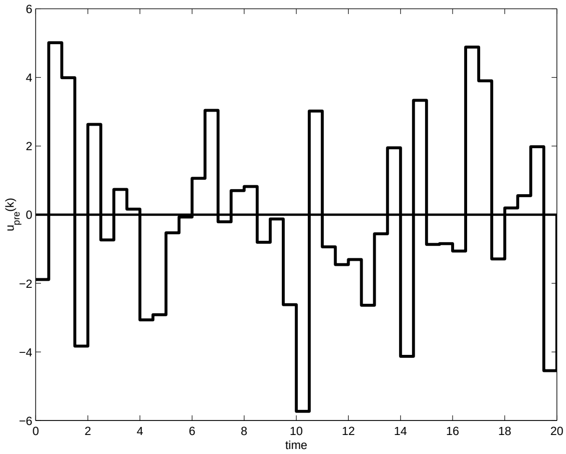


Figure 5: The two components of preaction control $u_{\text{pre}}(k)$.