Summary of

Real-Time Fluid Dynamics for Games

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Review of Vector Operators

The del operator is basically a vector of differential operators for each of the Cartesian coordinates:

$$\bigtriangledown = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \tag{1}$$

Del applied to a scalar yields the gradient vector:

$$\bigtriangledown f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} \tag{2}$$

Del dotted with a vector yields the divergence, a scalar:

$$\bigtriangledown \cdot \mathbf{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \tag{3}$$

Del crossed with a vector yields the curl, a vector. The way to calculate it can be remembered easily by writing it as the 2 by 3 vector shown here, and getting each column of the result as the product of the upper left to lower right diagonal minus the product of the upper right to lower left diagonal of the other two columns:

$$\bigtriangledown \times \mathbf{f} = \begin{bmatrix} \frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \\ \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{bmatrix} \tag{4}$$

The $(\mathbf{a} \cdot \bigtriangledown)$ operator can be applied to a scalar:

$$(\mathbf{a} \cdot \bigtriangledown) f = \mathbf{a} \cdot (\bigtriangledown f) \tag{5}$$

or to a vector:

$$(\mathbf{a} \cdot \bigtriangledown) \mathbf{f} = \begin{bmatrix} \mathbf{a} \cdot \bigtriangledown f_1 \\ \mathbf{a} \cdot \bigtriangledown f_2 \\ \mathbf{a} \cdot \bigtriangledown f_3 \end{bmatrix} \tag{6}$$

Del squared is the Laplacian, and is a vector of second derivatives for each Cartesian coordinate:

$$\bigtriangledown^2 = \bigtriangledown \cdot \bigtriangledown = \begin{bmatrix} \frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial y^2} \\ \frac{\partial^2}{\partial z^2} \end{bmatrix} \tag{7}$$

When applied to a scalar, the Laplacian can be calculated as the divergence of the gradient of the scalar (which intuitively makes sense since we're taking derivatives of a derivatives):

$$\bigtriangledown^2 f = (\bigtriangledown \cdot \bigtriangledown) f = \bigtriangledown \cdot (\bigtriangledown f) \tag{8}$$

When applied to a vector, the Laplacian can be calculated as the gradient of the divergence of the vector minus the curl of the curl of the vector ((which again intuitively makes some sense since we're taking derivatives of a derivatives):

$$\bigtriangledown^2 \mathbf{f} = (\bigtriangledown \cdot \bigtriangledown) \mathbf{f} = \bigtriangledown (\bigtriangledown \cdot \mathbf{f}) - \bigtriangledown \times (\bigtriangledown \times \mathbf{f}) \tag{9}$$

Navier-Stokes Equations

As an example, let's say we have a vector field **u**:

$$\mathbf{u} = \begin{bmatrix} x^2 + z \\ y^3 + x^3 \\ z^4 + 2y \end{bmatrix} \tag{10}$$

a density function ρ:

$$\rho = 5x^2 + 3y^3 + 4z^8 \tag{11}$$

a rate of diffusion constant:

$$\kappa = 20 \tag{12}$$

and density sources:

$$S = 3x + 8z^3 \tag{13}$$

The evolution of the density over time can then be calculated using the Navier-Stokes density equation:

$$\frac{\partial \rho}{\partial t} = -\left(\mathbf{u} \cdot \triangledown\right) \rho + \kappa \triangledown^2 \rho + S \tag{14}$$

$$-\mathbf{u} \cdot (\triangledown \rho) + \kappa \left(\triangledown \cdot (\triangledown \rho)\right) + S \tag{15}$$

$$-\begin{bmatrix} x^2 + z \\ y^3 + x^3 \\ z^4 + 2y \end{bmatrix} \cdot \begin{bmatrix} 10x \\ 9y^2 \\ 32z^7 \end{bmatrix} + 20 \left(\triangledown \cdot \begin{bmatrix} 10x \\ 9y^2 \\ 32z^7 \end{bmatrix} \right) + 3x + 8z^3 \tag{16}$$

$$-\left(10x^3 + 10xz + 9y^5 + 9x^3y^2 + 32z^11 + 64yz^7\right) + 20\left(10 + 18y + 224z^6\right) + 3x + 8z^3 \tag{17}$$

$$-10x^3 - 10xz - 9y^5 - 9x^3y^2 - 32z^{11} - 64yz^7 + 200 + 360y + 4480z^6 + 3x + 8z^3 \tag{18}$$

But we also need to know how the velocity field itself evolves over time. Say we have a viscosity (the resistance of the fluid to flowing, or 'diffusing'):

$$v = 10 \tag{19}$$

and source field:

$$\mathbf{f} = \begin{bmatrix} 8z \\ 4y \\ 6x \end{bmatrix} \tag{20}$$

Then the evolution of the velocity field over time can be calculated using the Navier-Stokes velocity equation:

$$\frac{\partial \rho}{\partial t} = -\left(\mathbf{u} \cdot \triangledown\right) \mathbf{u} + v \triangledown^2 \mathbf{u} + \mathbf{f} \tag{21}$$

$$-\begin{bmatrix} \mathbf{u} \cdot \triangledown \mathbf{u_1} \\ \mathbf{u} \cdot \triangledown \mathbf{u_2} \\ \mathbf{u} \cdot \triangledown \mathbf{u_3} \end{bmatrix} + 10 \left(\left(\triangledown \left(\triangledown \cdot \mathbf{u} \right) \right) - \triangledown \times \left(\triangledown \times \mathbf{u} \right) \right) + \mathbf{f} \tag{22}$$

$$-\begin{bmatrix} \begin{bmatrix} x^2+z \\ y^3+x^3 \\ z^4+2y \end{bmatrix} \cdot \begin{bmatrix} 2x \\ 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} x^2+z \\ y^3+x^3 \\ z^4+2y \end{bmatrix} \cdot \begin{bmatrix} 3x^2 \\ 3y^2 \\ 0 \end{bmatrix} \\ \begin{bmatrix} x^2+z \\ y^3+x^3 \\ z^4+2y \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 4z^3 \end{bmatrix} \end{bmatrix} + 10\left(\left(\triangledown\left(2x+3y^2+4z^3\right)\right) - \triangledown \times \begin{bmatrix} \frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+z & y^3+x^3 & z^4+2y \end{bmatrix}\right) + \begin{bmatrix} 8z \\ 4y \\ 6x \end{bmatrix} \tag{23}$$

$$\begin{bmatrix} 2x^3+2xz+z^4+2y \\ 3x^4+3x^2z+3y^5+3x^3y^2 \\ 2y^3+2x^3+4z^7+8yz^3 \end{bmatrix} + 10\left(\begin{bmatrix} 2 \\ 6y \\ 12z^2 \end{bmatrix} - \triangledown \times \begin{bmatrix} 2 \\ -1 \\ 3x^2 \end{bmatrix}\right) + \begin{bmatrix} 8z \\ 4y \\ 6x \end{bmatrix} \tag{24}$$

$$\begin{bmatrix} 2x^3+2xz+z^4+2y \\ 3x^4+3x^2z+3y^5+3x^3y^2 \\ 2y^3+2x^3+4z^7+8yz^3 \end{bmatrix} + 10\left(\begin{bmatrix} 2 \\ 6y \\ 12z^2 \end{bmatrix} - \begin{bmatrix} \frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2 & -1 & 3x^2 \end{bmatrix}\right) + \begin{bmatrix} 8z \\ 4y \\ 6x \end{bmatrix} \tag{25}$$

$$\begin{bmatrix} 2x^3+2xz+z^4+2y \\ 3x^4+3x^2z+3y^5+3x^3y^2 \\ 2y^3+2x^3+4z^7+8yz^3 \end{bmatrix} + 10\left(\begin{bmatrix} 2 \\ 6y \\ 12z^2 \end{bmatrix} - \begin{bmatrix} 0 \\ -6x \\ 0 \end{bmatrix}\right) + \begin{bmatrix} 8z \\ 4y \\ 6x \end{bmatrix} \tag{26}$$

$$\begin{bmatrix} 2x^3+2xz+z^4+2y \\ 3x^4+3x^2z+3y^5+3x^3y^2 \\ 2y^3+2x^3+4z^7+8yz^3 \end{bmatrix} + \begin{bmatrix} 20 \\ 60y+60x \\ 120z^2 \end{bmatrix} + \begin{bmatrix} 8z \\ 4y \\ 6x \end{bmatrix} \tag{27}$$

$$\begin{bmatrix} 2x^3+2xz+z^4+2y+8z+20 \\ 3x^4+3x^2z+3y^5+3x^3y^2+64y+60x \\ 2y^3+2x^3+4z^7+8yz^3+120z^2+6x \end{bmatrix} \tag{28}$$