Summary of *Elastically Deformable Models*

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The "big picture" idea of this paper is to model deformable objects using a differential equation very analogous to the standard mass-spring-damper equation:

$$\frac{\partial}{\partial t}\left(\mu\frac{\partial r}{\partial t}\right)+\gamma\frac{\partial r}{\partial t}+\frac{\delta\varepsilon(r)}{\delta r}=f(r,t) \tag{1}$$

You parameterize your volume, surface, or curve with a; for a volume, it has three components (a_1, a_2, a_3), for a surface, two (a_1, a_2), and for a curve, one (a_1). For example, for a perfect sphere, you could use the standard ρ, ϕ, and θ parameters; for the surface of the sphere, you would just use ϕ and θ.

The function $r(a,t)$ transforms the parameters into a point in 3-D Euclidean space. Its columns are the x, y, and z axes of the Cartesian system, expressed in terms of the chosen parameters. So, for example, if you start (t=0) with a perfect sphere of radius say 2,

$$r(a,t)=r(a_1,a_2,t)=r(\phi,\theta,0)=\left[\begin{array}{ccc} 2\sin(\phi)cos(\theta) & 2\sin(\phi)sin(\theta) & 2\cos(\phi)\end{array}\right]$$

As the sphere starts deforming ($t>0$), there is no longer an analytical form for $r(a,t)$ – or at least not a very simple one – but there is still *some* mapping of parameter points to Euclidean points, and we're going to discretize everything eventually anyway.

The μ in (1) is the mass density, so the first term is a mass times the second time derivative of a position function – just like the $m\frac{d^2}{dt^2}x(t)$ term in the standard mass-spring-damper equation.

The γ in (1) is the damping density, so the second term is a damping coefficient times the first time derivative of a position function – just like the $c\frac{d}{dt}x(t)$ term in the standard mass-spring-damper equation.

The last term on the left-hand side of (1) is analogous to the stiffness term $kx(t)$ in the standard mass-spring-damper equation. They calculate this using the variational derivative of a "functional" $\varepsilon(r)$. (According to an on-line math dictionary, a "functional" is a map from a function space to the complex numbers, which doesn't really shed much light on this for me at least).

In order to calculate this functional, which measures the net instantaneous potential energy of the deformation, they introduce the first and second fundamental forms, which relate changes in parameters to change in Euclidean distance. For a volume, they give the formula

$$dl^2 = \sum_{i,j} G_{ij} da_i da_j \tag{2}$$

(Actually, their equation 2 doesn't square the left side; I'm not quite sure what's up with this. The description I'm giving is consistent with explanations I've found on the web for fundamental forms; I'm not sure if there's a difference in notation, a typo, or something I'm missing, but in any event I don't think it greatly affects the intuition.)

Only the first fundamental form is relevant for a volume; it can be calculated by

$$G_{ij}(r(a)) = \frac{\partial r}{\partial a_i} \cdot \frac{\partial r}{\partial a_j} \tag{3}$$

I don't feel like I really understand any equation until I work out an example, so here are a couple.

First, consider a perfect cube. Then r is just

$$r(a) = \begin{bmatrix} x & y & z \end{bmatrix}$$

The partial derivatives necessary for calculating G are

$$\frac{\partial r}{\partial a_1} = \frac{\partial r}{\partial x} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial r}{\partial a_2} = \frac{\partial r}{\partial y} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

$$\frac{\partial r}{\partial a_3} = \frac{\partial r}{\partial z} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

And so the components of G are

$$g_{11} = \frac{\partial r}{\partial a_1} \cdot \frac{\partial r}{\partial a_1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = 1$$

$$g_{12} = \frac{\partial r}{\partial a_1} \cdot \frac{\partial r}{\partial a_2} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = 0$$

$$g_{13} = \frac{\partial r}{\partial a_1} \cdot \frac{\partial r}{\partial a_3} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = 0$$

$$g_{21} = \frac{\partial r}{\partial a_2} \cdot \frac{\partial r}{\partial a_1} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = 0$$

$$g_{22} = \frac{\partial r}{\partial a_2} \cdot \frac{\partial r}{\partial a_2} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = 1$$

$$g_{23} = \frac{\partial r}{\partial a_2} \cdot \frac{\partial r}{\partial a_3} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = 0$$

$$g_{31} = \frac{\partial r}{\partial a_3} \cdot \frac{\partial r}{\partial a_1} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = 0$$

$$g_{32} = \frac{\partial r}{\partial a_3} \cdot \frac{\partial r}{\partial a_2} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = 0$$

$$g_{33} = \frac{\partial r}{\partial a_3} \cdot \frac{\partial r}{\partial a_3} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = 1$$

Substituting into (2), we get

$$dl^2 = da_1^2 + da_2^2 + da_3^2 = dx^2 + dy^2 + dz^2$$

This is just the standard distance formula; square of the distance equals sum of the squares of the changes on each axis.

As a less trivial example, consider a perfect cylinder. Then, as we remember from cylindrical coordinates, r is the standard

$$r(a) = \begin{bmatrix} \rho\cos(\theta) & \rho\sin(\theta) & z \end{bmatrix}$$

The partials are

$$\frac{\partial r}{\partial a_1} = \frac{\partial r}{\partial \rho} = \left[\begin{array}{ccc} \cos(\theta) & \sin(\theta) & 0 \end{array} \right]$$

$$\frac{\partial r}{\partial a_2} = \frac{\partial r}{\partial \theta} = \left[\begin{array}{ccc} -\rho\sin(\theta) & \rho\cos(\theta) & 0 \end{array} \right]$$

$$\frac{\partial r}{\partial a_3} = \frac{\partial r}{\partial z} = \left[\begin{array}{ccc} 0 & 0 & 1 \end{array} \right]$$

The components of G are

$$g_{11} = \frac{\partial r}{\partial a_1} \cdot \frac{\partial r}{\partial a_1} = \left[\begin{array}{ccc} \cos(\theta) & \sin(\theta) & 0 \end{array} \right] \cdot \left[\begin{array}{ccc} \cos(\theta) & \sin(\theta) & 0 \end{array} \right] = 1$$

$$g_{12} = \frac{\partial r}{\partial a_1} \cdot \frac{\partial r}{\partial a_2} = \left[\begin{array}{ccc} \cos(\theta) & \sin(\theta) & 0 \end{array} \right] \cdot \left[\begin{array}{ccc} -\rho\sin(\theta) & \rho\cos(\theta) & 0 \end{array} \right] = 0$$

$$g_{13} = \frac{\partial r}{\partial a_1} \cdot \frac{\partial r}{\partial a_3} = \left[\begin{array}{ccc} \cos(\theta) & \sin(\theta) & 0 \end{array} \right] \cdot \left[\begin{array}{ccc} 0 & 0 & 1 \end{array} \right] = 0$$

$$g_{21} = \frac{\partial r}{\partial a_2} \cdot \frac{\partial r}{\partial a_1} = \left[\begin{array}{ccc} -\rho\sin(\theta) & \rho\cos(\theta) & 0 \end{array} \right] \cdot \left[\begin{array}{ccc} \cos(\theta) & \sin(\theta) & 0 \end{array} \right] = 0$$

$$g_{22} = \frac{\partial r}{\partial a_2} \cdot \frac{\partial r}{\partial a_2} = \left[\begin{array}{ccc} -\rho\sin(\theta) & \rho\cos(\theta) & 0 \end{array} \right] \cdot \left[\begin{array}{ccc} -\rho\sin(\theta) & \rho\cos(\theta) & 0 \end{array} \right] = \rho^2$$

$$g_{23} = \frac{\partial r}{\partial a_2} \cdot \frac{\partial r}{\partial a_3} = \left[\begin{array}{ccc} -\rho\sin(\theta) & \rho\cos(\theta) & 0 \end{array} \right] \cdot \left[\begin{array}{ccc} 0 & 0 & 1 \end{array} \right] = 0$$

$$g_{31} = \frac{\partial r}{\partial a_3} \cdot \frac{\partial r}{\partial a_1} = \left[\begin{array}{ccc} 0 & 0 & 1 \end{array} \right] \cdot \left[\begin{array}{ccc} \cos(\theta) & \sin(\theta) & 0 \end{array} \right] = 0$$

$$g_{32} = \frac{\partial r}{\partial a_3} \cdot \frac{\partial r}{\partial a_2} = \left[\begin{array}{ccc} 0 & 0 & 1 \end{array} \right] \cdot \left[\begin{array}{ccc} -\rho\sin(\theta) & \rho\cos(\theta) & 0 \end{array} \right] = 0$$

$$g_{33} = \frac{\partial r}{\partial a_3} \cdot \frac{\partial r}{\partial a_3} = \left[\begin{array}{ccc} 0 & 0 & 1 \end{array} \right] \cdot \left[\begin{array}{ccc} 0 & 0 & 1 \end{array} \right] = 1$$

Substituting into (2), we get

$$dl^2 = da_1^2 + \rho^2 da_2^2 + da_3^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2$$

So if there is a change in the radius, the square of the distance is $d\rho^2$; if there is a change in the angle, the square of the distance (arc length) is $\rho^2\theta^2$; and if there is a change in the height, the square of the distance is dz^2, which all check out with basic geometry. (Recall the formula arc length = $r\theta$.)

For a surface, instead of one 3*3 matrix, we require two 2*2 matrices, the first and second fundamental forms. The first fundamental form, G, is a measure of the amount of movement of a surface in the parameter plane, and is calculated as in (3). The second fundamental form, B, is a measure of the change in normal vector and the change of surface position at a surface point as a function of a small movement in the parameter space. It is calculated as

$$B_{ij}(r(a)) = n \cdot \frac{\partial^2 r}{\partial a_i \partial a_j} \tag{4}$$

As an example, consider the surface of a sphere, parameterized by ϕ and θ, with ρ as the (constant) radius. Then r is, from spherical coordinates,

$$r(a) = \begin{bmatrix} \rho\sin(\phi)\cos(\theta) & \rho\sin(\phi)\sin(\theta) & \rho\cos(\phi) \end{bmatrix}$$

The first fundamental form can be calculated, equivalently to (3), as just the Jacobian times the transpose of the Jacobian (try it out for a couple of terms; you do get the same results). The Jacobian is

$$J = \begin{bmatrix} -\rho\sin(\phi) sin(\theta) & \rho\sin(\phi)\cos(\theta) & 0 \\ \rho\cos(\phi) cos(\theta) & \rho\cos(\phi)\sin(\theta) & -\rho\sin(\phi) \end{bmatrix}$$

which can be rewritten as

$$\begin{bmatrix} -y & x & 0 \\ \frac{xz}{\sqrt{x^2+y^2}} & \frac{yz}{\sqrt{x^2+y^2}} & -\sqrt{x^2+y^2} \end{bmatrix}$$

Multiplying this by its transpose yields

$$G = \begin{bmatrix} x^2+y^2 & 0 \\ 0 & r^2 \end{bmatrix}$$

The second fundamental form can be calculated, equivalently to (4), as just dot product of the normal and the Hessian of second derivatives. The Hessian,

$$\begin{bmatrix} \frac{\partial^2 r}{\partial \phi^2} & \frac{\partial^2 r}{\partial \phi \partial \theta} \\ \frac{\partial^2 r}{\partial \theta \partial \phi} & \frac{\partial^2 r}{\partial \theta^2} \end{bmatrix}$$

is

$$\begin{bmatrix} \begin{bmatrix} -x & -y & 0 \end{bmatrix} & \begin{bmatrix} -\frac{yz}{\sqrt{x^2+y^2}} & \frac{xz}{\sqrt{x^2+y^2}} & 0 \end{bmatrix} \\ \begin{bmatrix} -\frac{yz}{\sqrt{x^2+y^2}} & \frac{xz}{\sqrt{x^2+y^2}} & 0 \end{bmatrix} & \begin{bmatrix} -x & -y & -z \end{bmatrix} \end{bmatrix}$$

The normal is just

$$\begin{bmatrix} \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \end{bmatrix}$$

Dotting the normal with the Hessian yields

$$B = \begin{bmatrix} -\frac{x^2+y^2}{r} & 0 \\ 0 & -r \end{bmatrix}$$

Two volumes have the same shape if their first fundamental forms are equivalent; two surfaces have the same shape if their first and second fundamental forms are the same.

I don't know much about "the calculus of variations", but the basic idea is that they approximate the strain energy for elastic bodies by the weighted norms of the differences between the fundamental forms of the deformed body and the fundamental forms of the natural, undeformed body (G^0 and B^0).

$$\varepsilon(r) = \int_\Omega \|G - G^0\|_\alpha^2 + \|B - B^0\|_\beta^2 da_1 da_2$$

So, intuitively, if in the undeformed body, a small change in the parameters results in a small change in distance – in other words, two points close together in parameter space are close together in Euclidean space – but now in the current configuration, a small change in the parameters results in a large change in distance – in other words, two points close together in parameter space have now moved far apart – then there is a lot of deformation going on, resulting in a lot of strain energy.

Finally, the right-hand side of (1) is just the sum of all external forces, such as gravity, spring, viscous (including wind), and collision forces. The collision force is calculated as proportional to $e^{-f(r)}$, where $f(r)$ is the obstacle's "inside/outside" function. So if it is far outside (inside), f(r) is large (small), and the force is small (large). This "force field" method is also used to avoid self-collisions.

They give a simplified form of the variational derivative of the elasticity resistance functional. We need only calculate

$$\alpha_{ij}(a,r) = \eta_{ij}(a)(G_{ij} - G^0_{ij})$$

$$\beta_{ij}(a,r) = \xi_{ij}(a)(B_{ij} - B^0_{ij})$$

The sign of α determines whether the object shrinks (positive) or grows (negative), and the sign of β determines whether it becomes flatter (positive) or more curved (negative).

The η_{ij} and ξ_{ij} are material properties. For example, to make a stretchy rubber sheet, η should be small (so that there is little force resisting changes in length) and ξ zero (so that there is no force resisting changes in curvature). A larger η would be appropriate for more stretch-resistant cloth. Springy metal would have a larger ξ, as it has some resistance to bending. Paper would probably have a fairly large η and a modest ξ.

Section 6.2 shows how the continuous equations we have developed so far can be discretized into a matrix differential equation and how discrete first and second partial derivatives are calculated using finite differences. Section 6.3 uses implicit integration to transform the nonlinear ODE into a linear system. It calculates $r(a,t)$ from 0 to T by taking time steps of Δt. I don't think there is anything very novel in these sections, but they lay out all of the equations, so it could be a good reference should you need to discretize and solve differential equations.