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Symmetric Primal-Dual Approximate Linear Programming for Factored MDPs

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ABSTRACT

A weakness of classical Markov decision processes is that they scale very poorly due to the flat state-space representation. Factored MDPs address this representational problem by exploiting problem structure to specify the transition and reward functions of an MDP in a compact manner. However, in general, solutions to factored MDPs do not retain the structure and compactness of the problem representation, forcing approximate solutions, with approximate linear programming (ALP) emerging as a very promising MDP-approximation technique. To date, most ALP work has focused on the primal-LP formulation, while the dual LP, which forms the basis for solving constrained Markov problems, has received much less attention. We show that a straightforward linear approximation of the dual optimization variables is problematic, because some of the required computations cannot be carried out efficiently. Nonetheless, we develop a composite approach that symmetrically approximates the primal and dual optimization variables (effectively approximating both the objective function and the feasible region of the LP) that is computationally feasible and suitable for solving constrained MDPs. We empirically show that this new ALP formulation also performs well on unconstrained problems.

1. INTRODUCTION

Classical methods for solving Markov decision processes (e.g., Puterman, 1994), based on dynamic and linear programming, scale very poorly because of the *flat* state space, which subjects them to the curse of dimensionality (Bellman, 1961), where model size grows exponentially with the number of problem features. Fortunately, many MDPs are well-structured, making possible compact *factored* MDP representations (Boutilier, Dearden, & Goldszmidt, 1995) that model the state space as a cross product of state features, represent the transition function as a dynamic Bayesian network, and assume the reward function can be expressed as a linear combination of several functions, represented compactly on the state features.

However, well-structured problems do not always lead to well-structured solutions (Koller & Parr, 1999; Dolgov & Durfee, 2004a), which precipitates the need for approximate solutions. Approximate linear programming (ALP) (Schweitzer & Seidmann, 1985; de Farias & Van Roy, 2003) is a promising approach, with principled foundations and efficient solution techniques (de Farias & Van Roy, 2003, 2004; Guestrin, Koller, Parr, & Venkataraman, 2003; Patrascu, Poupart, Schuurmans, Boutilier, & Guestrin, 2002). However, ALP work has mostly focused on the *primal* LP, defined on the space of value functions, and significantly less effort has been invested in approximating the *dual* LP, which operates on occupation measures (state-visitation frequencies) and serves as the foundation for solving *constrained MDPs* (Altman, 1999; Kallenberg, 1983; Dolgov & Durfee, 2004b).

Developing efficient solutions to factored MDPs with constraints requires an approximate version of the dual LP, and there are two obvious ways to achieve this: take the Dual of the Approximated version of the primal LP (*DALP*), or Approximate the Dual LP (*ADLP*). The former formulation, the DALP, was considered and analyzed by Guestrin (2003). A weakness of this approach (detailed in Section 3) is that it scales exponentially with the induced width of the associated cluster graph, which can be very large (especially for constrained MDPs, where the cost functions increase the interactions between state features).

The second, ADLP approach instead approximates the dual LP directly. Unfortunately, as we demonstrate in Section 3, linear approximations of the optimization variables do not interact with the dual LP as well as they do with the primal, because the constraint coefficients cannot be computed efficiently. To address this, in Section 4, we develop a *composite ALP* that symmetrically approximates both the primal and the dual optimization coordinates (the value function and the occupation measure), which is

equivalent to approximating both the objective functions and the feasible regions of the LPs. This method provides an efficient approximation to constrained MDPs, and also performs well on unconstrained problems, as we empirically show in Section 5.

As viewed from the latter perspective of solving unconstrained problems, a contribution of this work is that it extends the suite of currently-available ALP techniques by the composite-ALP approach, which has the following useful properties. First, it allows for complete control of the quality-versus-complexity tradeoff in its approximation of the constraint set, as opposed to other methods where the objective function is approximated, but the feasible region is represented exactly (e.g., Guestrin, 2003; Guestrin et al., 2003). As such, the composite ALP is beneficial for domains, where the number of constraints required to exactly represent the feasible region grows exponentially (which frequently occurs in MDPs with costs and constraints); there, our approach trades quality for efficiency, compared to more exact methods. Second, compared to other methods that do approximate the feasible region, the benefit of our approach is that in some domains it might be easier to choose good basis functions for the approximation than it is to find good values for other approximation parameters (e.g., a sampling distribution over the constraint set as proposed by de Farias and Van Roy (2003)).

2. Background and Related Work

A discrete-time, infinite-horizon, discounted MDP (e.g., Puterman, 1994) can be described as $\langle \mathcal{S}, \mathcal{A}, P, R, \gamma \rangle$, where $\mathcal{S} = \{i\}$ is the finite set of system states, $\mathcal{A} = \{a\}$ is a finite set of actions, $P : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \mapsto [0, 1]$ is the transition function (P_{iaj} is the probability of moving into state j upon executing action a in state i), $R : \mathcal{S} \times \mathcal{A} \mapsto \mathbb{R}$ defines the bounded rewards (R_{ia} is the reward for executing action a in state i), and $\gamma \in [0, 1)$ is the discount factor (a unit reward received at time τ is aggregated into the total reward as γ^τ).

A solution to such an MDP is a stationary, deterministic policy, and the key to obtaining it is to compute the optimal value function v, which, for every state, defines the total expected discounted reward of the optimal policy. Given the optimal value function, the optimal policy is to act greedily with respect to it. The optimal value function can be obtained, for example, using the following minimization LP, which is often called the *primal LP* of an MDP (e.g., Puterman, 1994):

$$\min \sum_i \alpha_i v_i \;\Big|\; v_i \geq R_{ia} + \gamma \sum_j P_{iaj} v_j \quad \forall i \in \mathcal{S}, a \in \mathcal{A} \tag{1}$$

where α is an arbitrary strictly positive distribution over the state space ($\alpha_i > 0$). This LP has $|\mathcal{S}|$ optimization variables and $|\mathcal{S}||\mathcal{A}|$ constraints. The problem can also be formulated as an equivalent *dual LP* with $|\mathcal{S}||\mathcal{A}|$ variables and $|\mathcal{S}|$ constraints:

$$\max \sum_{i,a} R_{ia} x_{ia} \;\Bigg|\; \begin{array}{ll} \sum_a x_{ja} - \gamma \sum_{i,a} x_{ia} P_{iaj} = \alpha_j, & \forall i \in \mathcal{S}; \\ x_{ia} \geq 0, & \forall i \in \mathcal{S}, a \in \mathcal{A}, \end{array} \tag{2}$$

where x is the *occupation measure* (x_{ia} is the discounted number of executions of action a in state i), $x_i = \sum_a x_{ia}$ is the total expected discounted number of visits to state i, and the constraints in (eq. 2) ensure the conservation of flow through each state. Given a solution to (eq. 2), the optimal policy can be computed as:

$$\pi_{ia} = x_{ia} / \sum_a x_{ia} = x_{ia} / x_i. \tag{3}$$

The dual LP is well-suited for the addition of constraints (Altman, 1999). Given a set of T cost functions C^t, $t \in [1, T]$, where each cost function is defined similarly to the rewards: $C^t : \mathcal{S} \times \mathcal{A} \mapsto \mathbb{R}$, the problem of maximizing the total expected reward subject to constraints on total costs can be formulated as an LP by augmenting (eq. 2) with linear constraints

$$\sum_{i,a} C^t_{ia} x_{ia} \leq \widehat{c}^t, \quad \forall t \in [1, T], \tag{4}$$

where $\widehat{c}^t$ is the upper bound on cost of type t. As discussed in Section 3, adding such costs to the model has a negative effect on the complexity of factored approximations, as it aids in the propagation of dependencies.

2.1 Factored MDPs

The classical MDP model requires an enumeration of all possible system states and thus scales very poorly. To combat this problem, a compact MDP representation has been proposed (Boutilier et al., 1995) that defines the state space as the cross-product of the state features: $\mathcal{S} = z_1 \times z_2 \ldots z_N$, and uses a factored transition function and an additively-separable reward function.

The transition function is specified as a two-layer dynamic Bayesian network (DBN) (Dean & Kanazawa, 1989), with the current state features viewed as the parents of the next time-step features:

$$P_{iaj} = P(\mathbf{z}(j)|\mathbf{z}(i), a) = \prod_{n=1}^{N} p_n(z_n(j)|a, \mathbf{z}_{p_n}(i)), \tag{5}$$

where $\mathbf{z}(\cdot)$ is the instantiation of all $\mathcal{Z}$ features correspond-

ing to a state, $z_n(\cdot)$ denotes the value of the n^{th} state feature of a state, and $\mathbf{z}_{p_n}(\cdot)$ is the instantiation of the set of features $\mathcal{Z}_{p_n}$ that are the parents of z_n in the transition DBN. Likewise, in the rest of the paper, we will use $\mathcal{Z}_\varphi$ to refer to the set of features in the domain of function φ, and $\mathbf{z}_\varphi$ to refer to an instantiation of these features. The reward function for a factored MDP is compactly defined as

$$R_{ia} = \sum_{m=1}^{M} r_m(\mathbf{z}_{r_m}(i), a), \tag{6}$$

where $\mathbf{z}_{r_m}(\cdot)$ is an instantiation of a subset of state features $\mathcal{Z}_{r_m} \subseteq \mathcal{Z}$ that are in the domain of the m^{th} local reward function r_m.

For a constrained MDP, we can define factored cost functions analogously to the reward function:

$$C^t_{ia} = \sum_{m=1}^{M} c^k_m(\mathbf{z}_{c^k_m}(i), a). \tag{7}$$

Clearly, this factored representation is only beneficial if the local transition, reward, and cost functions have small domains, i.e., they each depend only on a small subset of all state features $\mathcal{Z}$.

2.2 Primal Approximation (ALP)

Approximate linear programming (Schweitzer & Seidmann, 1985; de Farias & Van Roy, 2003) lowers the dimensionality of the primal LP (eq. 1) by restricting the optimization to the space of value functions that are linear combination of a predefined set of K basis functions h:

$$v_i = v(\mathbf{z}(i)) = \sum_{k=1}^{K} h_k(\mathbf{z}_{h_k}(i)) w_k, \tag{8}$$

where $h_k(\mathbf{z}_{h_k})$ is the k^{th} basis function defined on a small subset of the state features $\mathcal{Z}_{h_k} \subset \mathcal{Z}$, and w are the new optimization variables. For the approximation to be computationally effective, the domain of each basis function has to be small ($|\mathcal{Z}_{h_k}| \ll |\mathcal{Z}|$).

As a notational convenience, we can rewrite the above as $v = Hw$, where H is a $|\mathcal{S}| \times |w|$ matrix composed of basis functions h_k.[1]

Thus, LP (eq. 1) becomes:

$$\min \alpha^T H w \;\Big|\; AHw \geq r, \tag{9}$$

where we define the constraint matrix $A_{ia,j} = \delta_{ij} - \gamma P_{iaj}$ (where δ_{ij} is the Kronecker delta, $\delta_{ij} = 1 \Leftrightarrow i = j$).

For this method to be effective, we need to be able to efficiently compute the objective function $\alpha^T H$ and the constraints AH, which can be done as described in (Guestrin et al., 2003). Consider a factored initial distribution:

$$\alpha_i = \prod_m \mu_m(\mathbf{z}_{\mu_m}(i)),$$

where, as usual, the domain of each factor μ_m is taken to be small ($|\mathcal{Z}_{\mu_m}| \ll |\mathcal{Z}|$). Then, the objective function can be computed as:[2]

$$\begin{aligned}(\alpha^T H)_k &= \sum_i \alpha_i H_{ik} = \sum_i \prod_m \mu_m(\mathbf{z}_{\mu_m}(i)) h_k(\mathbf{z}_{h_k}(i)) \\ &= \sum_{\mathbf{z}'} \prod_{m'} \mu_{m'}(\mathbf{z}'_{\mu_{m'}}) h_k(\mathbf{z}'_{h_k}),\end{aligned}$$

where $\mathbf{z}'$ iterates over all features in the union of the domain of h_k and the domains of those $\mu_{m'}$ that have a non-zero intersection with the domain of h_k: $\mathbf{z}' = \{\mathbf{z}_{\mu_m} \cup \mathbf{z}_{h_k} : \mathcal{Z}_{\mu_m} \cap \mathcal{Z}_{h_k} \neq \varnothing\}$, because all μ_m that do not have any variables in common with h_k factor out and their sum is 1 (since it is a sum of a probability distribution over its domain). This computation is illustrated in the following example.

Example 1: Consider a state space $\mathcal{S} = z_1 \times z_2 \times z_3$, a set of basis functions $H = [h_1(z_1), h_2(z_1, z_2), h_3(z_3)]$, and an initial distribution $\alpha = \mu_1(z_1)\mu_2(z_2)\mu_3(z_2, z_3)$. Then,

$$\begin{aligned}(\alpha^T H)_1 &= \sum_{z_1,z_2,z_3} \mu_1(z_1)\mu_2(z_2)\mu_3(z_2,z_3)h_1(z_1) \\ &= \sum_{z_1} \mu_1(z_1)h_1(z_1) \sum_{z_2,z_3} \mu_2(z_2)\mu_3(z_2,z_3) \\ &= \sum_{z_1} \mu_1(z_1)h_1(z_1),\end{aligned}$$

which can be computed efficiently by summing over all values of z_1 instead of $z_1 \times z_2 \times z_3$. Similarly, both $(\alpha^T H)_2$ and $(\alpha^T H)_3$ can be computed by summing over $z_2 \times z_3$. □

The constraint coefficients in (eq. 9) can also be computed efficiently:

$$\begin{aligned}(AH)_{ia,k} &= \sum_j A_{ia,j} h_k(j) = \sum_j (\delta_{ij} - \gamma P_{iaj}) h_k(j) \\ &= \sum_{\mathbf{z}} \delta(\mathbf{z}(i), \mathbf{z}) h_k(\mathbf{z}) - \gamma \sum_{\mathbf{z}} P(\mathbf{z}|\mathbf{z}(i), a) h_k(\mathbf{z}).\end{aligned}$$

The first sum can be computed efficiently, because it is

[1] While using this notation, it is important to keep in mind that the exponentially-sized H would never be explicitly written out, because each column is a basis function that can be represented compactly.

[2] Here and below the general expression is followed by a simple example; some readers might find it beneficial to switch this order.

simply $h_k(\mathbf{z}_{h_k}(i))$, since δ is nonzero only at $\mathbf{z}(j) = \mathbf{z}(i)$. The second term can also be computed efficiently, since $P(\mathbf{z}(j)|\mathbf{z}(i), a)$ is a factored probability distribution (just like in the case of α above). The following example illustrates the computation.

Example 2: Consider $\mathcal{S}$ and H as in Example 1 and the transition model (with actions omitted):

$$P(z_1', z_2', z_3'|z_1, z_2, z_3) = p_1(z_1'|z_1)p_2(z_2'|z_1, z_2)p_2(z_3'|z_3)$$

Then, the second term in AH, shown for $k = 3$, becomes:

$$\begin{aligned}
&\sum_j P_{iaj} h_3(j) = \sum_j P(z_1'(j), z_2'(j), z_3'(j)|z_1, z_2, z_3)h_3(j) \\
&= \sum_{z_1', z_2', z_3'} p_1\big(z_1'|z_1(i)\big)p_2\big(z_2'|z_1(i), z_2(i)\big)p_3\big(z_3'|z_3(i)\big)h_3(z_1') \\
&= \sum_{z_3'} p_3\big(z_3'|z_3(i)\big)h_3(z_3') \sum_{z_1', z_2'} p_1\big(z_1'|z_1(i)\big)p_2\big(z_2'|z_1(i)\big) \\
&= \sum_{z_3'} p_3\big(z_3'|z_3(i)\big)h_3(z_3')
\end{aligned}$$

which can be efficiently computed by summing over z_3'. □

The primal ALP described above reduces the number of optimization variables from $|\mathcal{S}|$ to $|w| = K$, and, as just illustrated, the coefficients of the objective function and every constraint row can be computed efficiently. However, the number of rows in the constraint matrix remains exponential at $|\mathcal{S}||\mathcal{A}|$, so the ALP has to undergo some additional transformation (or approximation) to become feasible. To address this issue, several techniques have been proposed, such as sampling (de Farias & Van Roy, 2004) and exploiting problem structure (Guestrin et al., 2003).

2.3 Dual of Primal Approximation

The primal ALP (eq. 9) operates on the value function coordinates, and is thus not well-suited for addition of costs and constraints (defined on the occupation measure). Guestrin (Guestrin, 2003) considers the dual of (eq. 9) that can be used for formulating constrained problems:

$$\max r^T x \Big| H^T A^T x = \alpha. \tag{10}$$

This LP has $|\mathcal{S}||\mathcal{A}|$ variables (occupation-measure) and $|w| = K$ constraints (approximated flow conservation). The exact occupation measure can be represented more compactly by using *marginal occupation measures* (or marginal visitation frequencies (Guestrin, 2003)), which define the occupation measure over subsets of the state features (their domains are defined by the transition, reward, and basis functions). However, assuring global consistency of the marginal occupation measures requires expanding their domains, making the complexity exponential in the size of the induced width of the *cluster graph* (a graph with a vertex per variable and edges between variables that appear in one function). For some domains, the induced width is large, especially for constrained MDPs, where the cost functions introduce additional edges into the cluster graph. Guestrin (2003) also suggests an interesting further approximation of the DALP, where global consistency of marginal occupation measures is not guaranteed. To date, this approximation has not been carefully investigated, but is potentially promising. Another implication of the DALP approach is that the number of constraints grows with the number of primal basis functions and their domains (the more functions, and the bigger their domains, the larger the induced width of the cluster graph).

The new approach that we propose independently controls the number of optimization variables (via the primal basis) and the number of constraints (via the dual basis), thus providing an effective approximation method for problems with large induced graph widths.

3. Approximation of the Dual LP

Another way to construct an ALP suitable for constrained problems is to approximate the variables of the dual LP (eq. 2) using the primal ALP techniques. The focus of this section is on the negative result that shows that this approximation, by itself, is not computationally feasible, but the analysis of this section also paves the way for the approximation presented in Section 4. By straightforwardly applying the techniques from the primal ALP, we could restrict the optimization in (eq. 2) to a subset of the occupation measures that belong to a certain basis $Q = [q_l],\ l \in [1, L]$, reducing the number of optimization variables from $|\mathcal{S}||\mathcal{A}|$ to $|y| = L$:

$$\max r^T Q y \big| A^T Q y = \alpha, \quad Qy \geq 0. \tag{11}$$

For this approximation to be practical, we need to efficiently compute the objective function $r^T Q$ and the constraint matrix $A^T Q$, as well as deal with the exponential number of constraints. The objective-function coefficients can be computed efficiently:

$$\begin{aligned}
(r^T Q)_l &= \sum_{i,a} (r^T)_{ia} Q_{ia,l} = \sum_{i,a} \sum_{m=1}^{M} r_m\big(\mathbf{z}_{r_m}(i)\big) q_l(\mathbf{z}_{q_l}) \\
&= \beta \sum_{m=1}^{M} \Big[\sum_{\mathbf{z}_{r_m} \bigcup \mathbf{z}_{q_l}} r_m\big(\mathbf{z}_{r_m}(i)\big) q_l(\mathbf{z}_{q_l}) \Big],
\end{aligned}$$

where $\beta = |\mathcal{Z} \setminus (\mathcal{Z}_{r_m} \bigcup \mathcal{Z}_{q_l})|$ is the normalization constant that is the size of the domain not included in the summation.

Each of the M terms above can be efficiently computed by summing over the state variables in the union $\mathcal{Z}_{r_m} \bigcup \mathcal{Z}_{q_l}$. Unfortunately, the same is not true for the constraint coefficients, and therein lies the biggest problem of the dual ALP:

$$(A^T Q)_{j,l} = \sum_{i,a} \delta_{ij} q_l(i,a) - \sum_{i,a} \gamma P_{iaj} q_l(i,a) \qquad (12)$$

The first term can be calculated efficiently, as in the case of the primal ALP, since it is simply $q_l(\mathbf{z}_{h_k}(j))$. However, the second term presents problems, as demonstrated below.

Example 3: Consider $\mathcal{S}$ and P as in the previous examples. The problematic second term in (eq. 12), for $Q = [q_1(z_1, z_2, a), q_2(z_2, a), q_3(z_3, a)]$, becomes ($l = 3$, with actions a omitted for brevity):

$$\sum_{z_1,z_2,z_3} q_3(z_3) p_1(z_1'(j)|z_1) p_2(z_2'(j)|z_1, z_2) p_3(z_3'(j)|z_3)$$
$$= \sum_{z_3} q_3(z_3) p_3(z_3'(j)|z_3) \sum_{z_1,z_2} p_1(z_1'(j)|z_1) p_2(z_1'(j)|z_2, z_2)$$

and computing the last term requires summing over the whole state space $z_1 \times z_2 \times z_3$. □

This example demonstrates the critical difference between the primal and dual ALPs, due to the difference between the left- and the right-hand-side operators $A(\cdot)$ and $(\cdot)A$, used in the primal and dual ALPs, respectively. The former can be computed efficiently, because $\sum_a P(a|b) = 1$ and their product drops out of the computation, while the latter cannot, since a product of terms of the form $\sum_b P(a|b)$ is hard to compute efficiently. Therefore, the drawback of the dual ALP (eq. 11) is that it has an exponential number of constraints, and computing the coefficients for *each one* of them scales exponentially.

4. COMPOSITE ALP

The ADLP (eq. 11) approximates the dual variables x, which is equivalent to approximating the feasible region of the primal ALP (eq. 9); the primal does the opposite. We can combine the two by applying the dual approximation $x = Qy$ to the DALP (eq. 10):

$$\max r^T Qy \left| \begin{array}{l} H^T A^T Qy = H^T \alpha, \\ Qy \geq 0. \end{array} \right. \qquad (13)$$

This ALP still has an exponential number ($|\mathcal{S}||\mathcal{A}|$) of constraints in $Qy \geq 0$, but this can be resolved in several ways. These constraints can be reformulated using the non-serial dynamic programming approach (Bertele & Brioschi, 1972) (analogously to its application in (Guestrin et al., 2003)), yielding an equivalent, but smaller, constraint set. Another approach is to simply restrict attention to non-negative basis functions Q and replace the constraints with a stricter condition $y \geq 0$ (introducing another source of approximation error). We will adopt the latter approach (which works quite well, as shown by our experiments), leading to the following LP:

$$\max r^T Qy \left| \begin{array}{l} H^T A^T Qy = H^T \alpha, \\ y \geq 0. \end{array} \right. \qquad (14)$$

The above gives the dual form of the composite ALP. The equivalent primal form is:

$$\min \alpha^T Hw \Big| Q^T AHw \geq Q^T r. \qquad (15)$$

The primal form has K variables (one per primal basis function h_k) and L constraints (one per dual basis function q_l); the dual form is the opposite. Thus, the composite ALP combines the efficiency gains of the primal and the dual approximations. However, as in the case of the primal and the dual ALPs, the usefulness of the composite ALP is contingent upon our ability to efficiently compute the coefficients of its objective function and constraints. The objective functions of the two forms are the same as in the primal and the dual ALPs, respectively, so both can be computed efficiently as described in the earlier sections.

Thus, the important question is whether the constraint coefficients can be computed efficiently. A first glance at the constraints conveys some pessimism, because of the term AQ, which was the stumbling block in the dual ALP. However, despite that, the computation can be carried out efficiently if we apply the primal approximation first and then the dual approximation to the result. Consider the primal approximation:

$$(AH)_{ia,k}$$
$$= h_k\big(\mathbf{z}_{h_k}(i)\big) - \gamma \sum_{\mathbf{z}_{h_k}} \prod_{n: z_n \in \mathcal{Z}_{h_k}} p_n\big(z_n | \mathbf{z}_{p_n}(i), a\big) h_k(\mathbf{z}_{h_k})$$
$$= h_k(\mathbf{z}_{h_k}) - \psi(\mathbf{z}_{\psi_k}, a),$$

where we introduced ψ_k to refer to the second term, which is a compact function whose domain is the union of the DBN parents of all features that are in the domain of the k^{th} basis function: $\mathcal{Z}_{\psi_k} = \bigcup_{n: z_n \in \mathcal{Z}_{h_k}} \mathcal{Z}_{p_n}$. Applying the dual approximation to the result:

$$\big(Q^T (AH)\big)_{l,k} = \sum_{i,a} q_l(i,a) \Big(h_k(\mathbf{z}_{h_k}(i)) - \psi(\mathbf{z}_{\psi_k}, a) \Big)$$
$$= \sum_{\mathbf{z}_{h_k} \bigcup \mathbf{z}_{q_l}} q_l(\mathbf{z}_{q_l}, a) h_k(\mathbf{z}_{h_k}) - \sum_{\mathbf{z}_{\psi_k} \bigcup \mathbf{z}_{q_l}, a} q_l(\mathbf{z}_{q_l}, a) \psi(\mathbf{z}_{\psi_k}, a)$$

The first term can be computed efficiently be summing over

$\mathcal{Z}_{h_k} \bigcup \mathcal{Z}_{q_l}$, the union of the domains of the k^{th} primal and the l^{th} dual basis function. The second term is obtained by summing over the action space and $\mathcal{Z}_{\psi_k} \bigcup \mathcal{Z}_{q_l} = \big(\bigcup_{n:z_n \in \mathcal{Z}_{h_k}} \mathcal{Z}_{p_n}\big) \bigcup \mathcal{Z}_{q_l}$, the domain of q_l dual basis function and the union of the DBN parents of all features in the domain of h_k. Therefore, the coefficients of the composite constraint matrix can be computed efficiently by summing over relatively small domains (assuming the domains of all basis functions are small and the transition DBN is well-structured).

Example 4: Consider $\mathcal{S}$, P, and H as in the previous examples. Then, for $k = 3$, we have:

$$(AH)_{ia,3} = h_3\big(z_3(i)\big) - \gamma \sum_{z_3'} P\big(z_3'|z_3(i), a\big) h_3(z_3')$$

Thus, $\mathcal{Z}_{\psi_3} = \{z_3\}$, and ψ_3 can be computed efficiently by summing over z_3'. Multiplying by Q, we get for $l = 2$:

$$(Q^T AH)_{2,3} = \sum_{i,a} (Q^T)_{2,ia} (AH)_{ia,3}$$

$$= \sum_{z_2, z_3} q_2(z_2) h_3(z_3) - \gamma \sum_{z_2, z_3} q_2(z_2) \sum_{z_3'} P\big(z_3'|z_3, a\big) h_3(z_3')$$

which can be computed by summing over $z_2 \times z_3$. □

Another important issue to consider is the feasibility and boundedness of the composite ALPs (eq. 15) and (eq. 14). All ALPs that approximate only the optimization variables (primal ALP (eq. 9); its dual, DALP (eq. 10); the dual approximation, ADLP (eq. 11)) are bounded, because the approximation limits the search to a subset of possible solutions. Feasibility of the primal ALP (eq. 9) can also be ensured by adding a constant to H (de Farias & Van Roy, 2003). In the case of the composite ALP, where both the feasible region and the objective function are approximated, guaranteeing boundedness and feasibility is slightly more complicated.

Proposition 1: The primal form of the composite ALP (eq. 15) is feasible for any dual basis $Q \geq 0$ and any primal basis H that contains a constant function $h_k(\mathbf{z}_{h_k}) = 1$.

Proof: By the results of de Farias and Van Roy (de Farias & Van Roy, 2003), the primal ALP (eq. 9) is feasible whenever the primal basis H contains a constant. Call a feasible solution to the primal ALP w^*. By definition, w^* satisfies

$$AHw^* \geq r.$$

Then, for any $Q \geq 0$, $Q^T AHw^* \geq Q^T r$ also holds, meaning that (eq. 15) also has a feasible solution. □

In other words, introducing a dual approximation Q only enlarges the feasible region of the primal form of the composite ALP (eq. 15), thus guaranteeing its feasibility. Unfortunately, (eq. 15) is not, in general, bounded, because the dual basis Q might contain too few constraints. Intuitively, to bound (eq. 15), we need at least as many constraints as optimization variables. Therefore, an important question is: Given a primal basis H, how big must the dual basis Q be to ensure the boundedness of the primal form (eq. 15), or, equivalently, the feasibility of the dual form (eq. 14)?

Proposition 2: For any primal basis H ($|\mathcal{S}| \times K$), there exists a dual basis Q ($|\mathcal{S}||\mathcal{A}| \times L$), such that the number of dual basis functions does not exceed the number of primal functions ($L \leq K$), and the dual form of the composite ALP (eq. 14) is feasible for H and Q.

Proof: A flat set of constraints $A^T x = \alpha$ is always feasible, thus there also always exists a solution to

$$H^T A^T x = H^T \alpha.$$

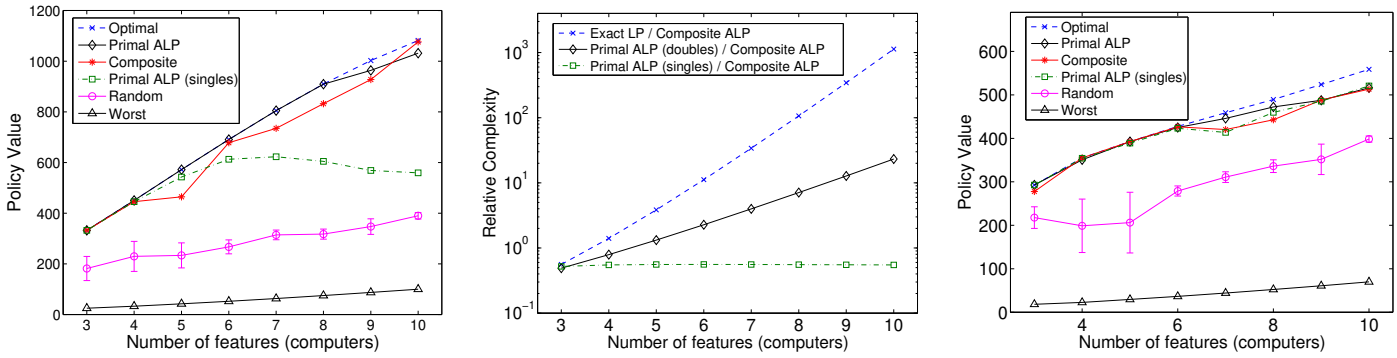
Let $\text{rank}(H^T A^T) = m \leq K$. Then, let us reorder rows and columns such that the upper-left $m \times m$ corner of $H^T A^T$ is non-singular. Let the dual basis contain m linearly-independent functions and reorder the rows of Q such that the top m rows are also linearly independent. Then, $H^T A^T Q$ will be $K \times m$, with a non-singular $m \times m$ matrix in the upper-left corner, with the remaining rows (and right-hand sides $H^T \alpha$), their linear combinations. Thus, the resulting system $H^T A^T Qy = H^T \alpha$ will have a solution. □

Therefore, for any primal basis H with K functions, there exists a dual basis Q with $L \leq K$ functions, which guarantees feasibility of the dual form of the composite ALP (eq. 14). By standard properties of LPs, this ensures the boundedness of the primal form (eq. 14), thus assuring a feasible and bounded solution to both. Intuitively, (eq. 14) has more variables than equations ($L > K$), thus it is usually feasible. So, from the practical standpoint, ensuring boundedness and feasibility of the composite ALPs is not difficult (when $L > K$, all but the most degenerate systems are underconstrained), which was confirmed by our experiments where using a meaningful dual basis with several times more functions than the primal (but on the same order of magnitude), resulted in a feasible (eq. 14).

For factored MDPs with cost functions (eq. 7) and cost constraints (eq. 4), we can add such constraints to the dual form of the composite ALP (eq. 14):

$$\max r^T Qy \left| \begin{array}{l} H^T A^T Qy = H^T \alpha, \\ CQy \leq \widehat{C}, \\ y \geq 0. \end{array} \right. \tag{16}$$

The coefficients of each of the T rows of the constraint $CQy \leq \widehat{C}$ can be computed efficiently, in exactly the



(a) (b) (c)

Fig. 1. Quality, unidirectional ring (a); Relative efficiency (b); Quality, bidirectional ring (c)

same way as the reward function $r^T Qy$, since each row of the constraint matrix C defines the cost function of type $t \in [1, T]$, which is assumed to be additively-decomposable (eq. 7), just like the reward function r.

5. Experimental Evaluation

The main driving force behind the composite ALP was to construct an efficient ALP, suitable for constrained MDPs, but the approach can certainly also be applied to unconstrained problems. Therefore, since there is a wider variety of algorithms for unconstrained MDPs, we focus on unconstrained domains in our analysis. This ignores one of the advantages of the composite ALP, but gives a more direct and clear comparison to other methods.

We evaluated the composite ALP on several domains, including the "SysAdmin" problem (Guestrin et al., 2003), the results for which we report here. The domain involves a network of n computers, each of which can fail with a probability that depends on the status of neighboring computers. The state of the system is defined by n binary features, where each feature defines the status of one computer. At each time step, the decision-maker can reboot a computer and receives a reward that is proportional to the number of computers that are up and running.

Figure 1a compares the values of policies for a problem with a unidirectional-ring network The values of policies obtained by the following methods are given: 1) Optimal, 2) Primal ALP[3] with basis functions over all pairs of features (quadratic), 3) Composite ALP with the same primal basis, and dual basis over neighbor triplets (linear), 4) Primal ALP with basis functions over single features (linear), 5) Random policy, 6) Worst policy. We performed no optimizations of the bases, and only used very simple functions, such as constants, binary indicators, and identity matrices. The plot shows the actual values of policies (not the value functions), which is a more accurate metric, as constraint approximation in the composite ALP can lead to unrealizable value functions.[4]

Figure 1b shows the efficiency gains of the composite ALP, relative to the exact LP and the two primal ALPs variations. A problem where each pair of variables appears in at least one function has an induced width that equals the total number of state variables. Thus, the composite ALP achieves exponential speedup, compared to a primal ALP or a DALP with a basis set defined on all pairs of features, but without a significant loss in quality (Figure 1a). The complexity of the composite ALP in these experiments roughly matches the complexity of the primal ALP (or the DALP) with basis functions over single features (Figure 1b), but the composite produces noticeably better policies (Figure 1a). Of course, for more-structured problems, basis functions over single features might suffice, for the more symmetric bidirectional-ring network (Figure 1c).

A successful deployment of the composite ALP hinges on the ability to construct good basis functions. The problem of basis selection is outside the scope of this paper,[5] but our preliminary experimental investigation suggests that, for many domains, simple and intuitive basis functions perform quite well. We performed experiments on domains with weakly-coupled tasks, using a very simple and intuitive approach to basis function construction: in the spirit of

[3]Note that, for unconstrained problems, the primal ALP (eq. 9) is equivalent to the DALP (eq. 10).

[4]Given the same primal basis, the composite ALP will, in general, produce lower-quality solutions than the primal ALP, because it also approximates the feasible region. The data point in Figure 1a, corresponding to 10 computers, is unusual. The value function of the composite ALP maps to a better policy than a more accurate value function of the primal ALP.

[5]For a discussion of the complexity and heuristics for basis selection for the primal ALP, see (Patrascu et al., 2002).

(Poupart, Boutilier, Patrascu, & Schuurmans, 2002), we used optimal solutions (v^* and x^*) to subproblems with small combinations of subtasks as basis functions (H and Q) for the original problems. Our preliminary results indicate that this approach is promising and yields high-quality approximations.

6. Discussion, Conclusions, and Future Work

Our main motivation in this work has been to develop a tractable approximation to constrained MDPs, for which exact solutions are predominantly based on the dual LP (eq. 2). The sole previous ALP formulation based on the dual LP is Guestrin's DALP (Guestrin, 2003). As discussed in Section 3, DALP unfortunately scales exponentially with the induced width of the cluster graph, which can be quite large, especially for constrained problems. We have presented the composite ALP approach as a more tractable yet still effective alternative that approximates both the optimization variables and the feasible regions of the LPs, symmetrically handling both the primal and dual variables. The composite ALP can also be effective in solving unconstrained MDPs, as we have empirically shown in Section 5. Overall, our experiments confirm the intuition behind composite ALPs: if the objective function is approximated, then using the exact feasible region can be wasteful. In the future, we would also like to establish more definitive quality bounds for the approach.

An alternative feasible-region approximation technique, which statistically samples the constraint set, was proposed by de Farias and Van Roy (2003). However, applying this idea to the dual formulation is problematic, since computing the coefficients for a *given* constraint in the ADLP (eq. 11) is computationally difficult, as demonstrated in Section 3. For unconstrained problems, a careful comparison of the constraint sampling scheme to the composite ALP is an interesting direction for future work, but a direct comparison is difficult, because, even given the same primal basis H, the performance of the two algorithms can vary greatly depending on the choice of constraint-approximation parameters (sampling distribution and the dual basis Q). Another possible way of approximating dual LPs for problems with large induced cost-network widths is to use the DALP (eq. 11) with marginal occupation measures that are not globally consistent, as suggested by Guestrin (2003). This idea (and its comparison to our composite ALP) deserves future study.

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