Inference and Update

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Abstract

We look at two fundamental logical processes, often intertwined in planning and problem solving: inference and update. Inference is an internal process with which we draw new conclusions, uncovering what is implicit in the information we already have. Update, on the other hand, is produced by external communication, usually in the form of announcements and in general in the form of observations, giving us information that might have been not available (even implicitly) to us before. Both processes have received attention from the logic community, usually separately. In this work, we develop a logical language that allows us to describe them together. We present syntax and semantics, as well as a complete logic for the language; we also discuss similarities and differences with other approaches, and we mention some possible ways the work can be extended.

1. Introduction

Consider the following situation, from [19]:

> You are in a restaurant with your parents, and you have ordered three dishes: fish, meat, and vegetarian. Now a new waiter comes back from the kitchen with the three dishes. What the new waiter can do to get to know which dish corresponds to which person ?

The waiter can ask *"Who has the fish?"*; then, he can ask once again *"Who has the meat?"*. Now he does not have to ask anymore: "two questions plus one inference are all that is needed" ([19]). His reasoning involves two fundamental logical processes: inference and update. The main goal of the present work is to develop a framework in which we can express how they work together.

Inference is an *internal* process: the agent revises her own information in search of what can be derived from it. Update, on the other hand, is produced by *external* communication: the agent gets new information via observations. Both are logical processes, both describe dynamics of information, both are used in every day situations, and still, they have been studied separately.

Inference has been traditionally taken as the main subject of study of logic, "... drawing new conclusions as a means of elucidating or 'unpacking' information that is implicit in the given premises"([20]). Among the most important branches, we can mention Hilbert-style proof systems, natural deduction and tableaux. Recent works, like [7, 8] and [13, 12] have incorporated modal logics to the field, representing inference as a non-deterministic step-by-step process.

Update, on the other hand, has been a main subject of what have been called *Dynamic Epistemic Logic*. Works like [16] and [10] turned attention to the effect public announcements have on the knowledge of an agent. Many works have followed them, including the study of more complex actions ([3, 2]) and the effect of announcements over a more wide propositional attitudes (the soft/hard facts of [17], the knowledge/belief of [4, 5]).

In [20], the author shows how these two phenomena fall directly within the scope of modern logic. As he emphasize, "asking a question and giving an answer is just as 'logical' as drawing a conclusion!". Here, we propose a merging of the two traditions. We consider that both processes are equally important in their own right, but so it is their interaction. In this work, we develop a logical language that join inference and update in a natural way. We first present a modal language to describe inference (section 2). After combining it with epistemic logic (section 3), we give a complete axiomatization. Then we incorporate updates, and we give a set of reduction axioms for the operation (section 4).

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Finally, we compare our work with other approaches (section 5) and mention some further work we consider interesting (section 6).

2. Internal process: an inference language

This section presents a logical language to express inference. The language is based on the work of Jago ([13, 12]), but contain some changes that make it more suitable for our purposes. The agent's information is represented as a set of formulas of a given *internal language*, which in our case is the classical propositional language. Inference steps are then represented as binary relations over such sets, allowing us to use a modal language to talk about them.

Definition 2.1 (Facts and rules) Let $\mathcal{P}$ be a set of atomic propositions, and let $\mathcal{F}_{\mathcal{P}}$ denote the classical propositional language based on $\mathcal{P}$.

- Formulas of $\mathcal{F}_{\mathcal{P}}$ are called *facts* over $\mathcal{P}$.
- A tuple of the form $(\{\lambda_1, \ldots, \lambda_n\}, \lambda)$ (for $n \geq 0$), where each λ_i and λ are facts in $\mathcal{F}_{\mathcal{P}}$, is called a *rule* over $\mathcal{F}_{\mathcal{P}}$. A rule will be also represented as $\lambda_1, \ldots, \lambda_n \Rightarrow \lambda$, and the set of rules over $\mathcal{F}_{\mathcal{P}}$ will be denoted by $R_{\mathcal{F}_{\mathcal{P}}}$.

$\triangleleft$

While facts describe situations about the world, rules describe relations between such situations. Intuitively, a rule $\rho = (\{\lambda_1, \ldots, \lambda_n\}, \lambda)$ indicates that if every λ_i is true, so it is λ. The set of facts $\text{prem}(\rho) := \{\lambda_1, \ldots, \lambda_n\}$ is called the *set of premises of* ρ, and the fact $\text{conc}(\rho) := \lambda$ is called the *conclusion of* ρ.

Definition 2.2 (Internal language) Given a set of atomic propositions $\mathcal{P}$, the *internal language over* $\mathcal{P}$, denoted as $\mathcal{I}_{\mathcal{P}}$, is given by the union of facts in $\mathcal{F}_{\mathcal{P}}$ and rules over $\mathcal{F}_{\mathcal{P}}$, that is, $\mathcal{I}_{\mathcal{P}} = \mathcal{F}_{\mathcal{P}} \cup R_{\mathcal{F}_{\mathcal{P}}}$. $\triangleleft$

Elements of $\mathcal{I}_{\mathcal{P}}$ will be called in general *formulas* of $\mathcal{I}_{\mathcal{P}}$. The subindexes indicating the set of atomic propositions will be omitted if no confusion arises.

For expressing how the agent's information evolves through inference steps, a (modal) inference language is defined.

Definition 2.3 (Language $\mathcal{IL}$) Let $\mathcal{A}$ be a set of agents and $\mathcal{P}$ a set of atomic propositions. Formulas φ of the *inference language* $\mathcal{IL}$ are given by

$$\varphi ::= \top \mid I_i\,\gamma \mid \neg\varphi \mid \varphi \vee \psi \mid \langle\rho\rangle_i\,\varphi$$

with $i \in \mathcal{A}$ and γ, ρ formulas of the internal language $\mathcal{I}_{\mathcal{P}}$ with ρ a rule. Formulas of the form $I_i\,\gamma$ express *"the agent i is informed about γ"*, while formulas of the form $\langle\rho\rangle_i\,\varphi$ express *"there is an inference step in which agent i applies the rule ρ and, after doing it, φ is the case"*. $\triangleleft$

The semantic model of $\mathcal{IL}$ is based on a Kripke model: we have a set of worlds and labeled binary relations between them. The main idea is that every world represents the information of the agents at a given stage, while a relation with label $D_{(\rho,i)}$ from a world w to a world w' indicates that the information of agent i at w allows her to perform an inference step with rule ρ, and that the information that results from applying ρ at w is represented by w'. To make formal this intuitive idea, we first need to define what we will understand by the phrases *"the information of i at w allows her to perform an inference step with ρ"* and *"the information that results from applying ρ at w is represented by w'"*. The concepts of *set-matching rule* and *rule-extension of a world* will do the job.

We will use the following abbreviation. Given a universe U, a set $A \subseteq U$ and an element $a \in U$, we denote $A \cup \{a\}$ as $A + a$.

Definition 2.4 (Set-matching rule) Let ρ be a rule in $\mathcal{I}$ and let Γ be a set of formulas of the internal language $\mathcal{I}$. We say that ρ *is* Γ*-matching* (ρ can be applied at Γ) if and only if ρ and all its premises are in Γ, that is, $(\text{prem}(\rho) + \rho) \subseteq \Gamma$. $\triangleleft$

Definition 2.5 (Extension of set of formulas) Let ρ be a rule in $\mathcal{I}$, and let Γ, Γ' be sets of formulas of the internal language $\mathcal{I}$. We say that Γ' *is a* ρ*-extension of* Γ if and only if Γ' is Γ plus the conclusion of ρ, that is, $\Gamma' = \Gamma + \text{conc}(\rho)$. $\triangleleft$

With the notions of Γ-matching rule and ρ-extension of Γ, we can give a formal definition of the models where formulas of $\mathcal{IL}$ are interpreted.

Definition 2.6 (Inference model) Let $\mathcal{A}$ be a set of agents and let $\mathcal{P}$ be a set of atomic propositions. An *inference model* is a tuple $M = (W, D_{(\rho,i)}, Y_i)$ where

- W is a non-empty set of worlds.
- $Y_i : W \to \wp(\mathcal{I}_{\mathcal{P}})$ is the *information set function* for each agent $i \in \mathcal{A}$. It assigns to i a set of formulas of the internal language in each world w.
- $D_{(\rho,i)} \subseteq (W \times W)$ is the *inference relation* for each pair (ρ, i), with ρ a rule in $\mathcal{I}_{\mathcal{P}}$ and i an agent in $\mathcal{A}$. The relation represents the application of a rule, so

if $D_{(\rho,i)}\, ww'$, then ρ is $Y_i(w)$-matching and $Y_i(w')$ is a ρ-extension of $Y_i(w)$.

$\triangleleft$

Note that the definition of $D_{(\rho,i)}$ just states the property any tuple should satisfy in order to be in the relation. The relation is not induced by the property, so it is possible to have two worlds w and w' such that there is a rule ρ that is $Y_i(w)$-matching and $Y_i(w')$ is a ρ-extension of $Y_i(w)$, and still do not have the pair (w, w') in $D_{(\rho,i)}$. One of the goals of the work is to make the basic definitions as general as possible, and then analyze the different concepts of inference and information we can get by asking for extra properties of the inference relation[1] and of the information sets (as we do later for the case of truthful information, that is, knowledge). This allows us to represent agents that are not as powerful reasoners as those represented with classic epistemic logic, and it may play an important role when studying agents with diverse reasoning abilities (cf. the discussion in section 6).

The concepts of *set-matching rule* and *rule-extension of a world* have their *possible world* version. We say that ρ is w-matching for i if it is $Y_i(w)$-matching, and we say that w' is a ρ-extension of w for i if $Y_i(w')$ is a ρ-extension of $Y_i(w)$.

Definition 2.7 Given an inference model $M = (W, D_{(\rho,i)}, Y_i)$ and a world $w \in W$, the relation $\models$ between the pair M, w and $\top$ (the always true formula), negations and disjunctions is given as usual. For the remaining formulas, we have

$$\begin{array}{lll} M, w \models I_i\, \gamma & \textit{iff} & \gamma \in Y_i(w) \\ M, w \models \langle\rho\rangle_i\, \varphi & \textit{iff} & \text{there is } w' \in W \text{ such that} \\ & & D_{(\rho,i)}\, ww' \text{ and } M, w' \models \varphi \end{array}$$

$\triangleleft$

3. The real world: an epistemic inference language

We have a language that express the agent's information and how it evolves through inferences. Still, we cannot talk about the real world or about the agent's uncertainty. In this section, we extend the current language to express those notions.

Syntactically, we extend the inference language with classical epistemic logic. We add basic formulas of the form p (for p an atomic proposition) and we close it under the modal operator P_i (for i an agent).

[1] In fact, the definition of $D_{(\rho,i)}$ restricts inferences to *deductive* ones. Within the proposed framework, it is possible to represent other inference processes, as mentioned in section 6.

Definition 3.1 (Epistemic inference language) Let $\mathcal{A}$ be a set of agents and let $\mathcal{P}$ be a set of atomic propositions. The formulas of the *epistemic inference language* $\mathcal{EI}$ are given by

$$\varphi ::= \top \mid p \mid I_i\, \gamma \mid \neg\varphi \mid \varphi \vee \psi \mid P_i\, \varphi \mid \langle\rho\rangle_i \varphi$$

with $i \in \mathcal{A}$, $p \in \mathcal{P}$ and γ, ρ formulas of the internal language $\mathcal{I}_{\mathcal{P}}$ with ρ a rule. $\triangleleft$

The propositional connectives $\wedge$, $\rightarrow$ and $\leftrightarrow$ are defined as usual; the modal operators K_i and $[\rho]_i$ are defined as the dual of P_i and $\langle\rho\rangle_i$, respectively.

As argued by van Benthem in [18], the operator K_i should be read as a more implicit notion, describing not the information the agent actually has, but the maximum amount of information she can get under her current uncertainty (i.e., without external interaction). In our framework, *explicit* information is represented with formulas of the form $I_i\, \gamma$, indicating that γ is part of the agent's information set; *implicit* information is represented with formulas of the form $K_i\, \varphi$, indicating what the agent can eventually get if she has enough explicit information (i.e., enough formulas and rules) and enough time to perform the adequate inference steps.

Semantically, we combine inference models with classic Kripke models. Each world has two components: information sets containing the facts and rules each agent is informed about, and a valuation indicating the truth value of atomic propositions. We also have two binary relations: the inference one indicating how inference steps modify information sets, and the epistemic one indicating the worlds each agent considers possible.

Definition 3.2 (Epistemic inference model) Let $\mathcal{A}$ be a set of agents and let $\mathcal{P}$ be a set of atomic propositions. An *epistemic inference model* is a tuple $M = (W, \sim_i, D_{(\rho,i)}, V, Y_i)$ where:

- W is a non-empty set of worlds.
- $V : W \rightarrow \wp(\mathcal{P})$ is a *valuation function.*
- $Y_i : W \rightarrow \wp(\mathcal{I}_{\mathcal{P}})$ is the *information set function* for agent i.
- $D_{(\rho,i)}$ is the *inference relation* for each pair (ρ, i), just as in definition 2.6. It satisfies an extra requirement: if $D_{(\rho,i)}\, ww'$, then $V(w) = V(w')$.
- $\sim_i$ is the *epistemic relation* for agent i. The relation satisfy the following property: for all worlds w, w', u, u': if $w \sim_i u$ and $D_{(\rho,i)}\, ww', D_{(\rho,i)}\, uu'$ for some rule ρ, then $w' \sim_i u'$.

◁

We have two new restrictions: one for the inference relation and one relating it with the epistemic relation. It is worthwhile to justify them.

1. The relation $D_{(\rho,i)}$ describes inference, an agent's internal process that changes her information but does not change the real situation. If an agent can go from w to w' by an inference step, w and w' should satisfy the same propositional letters.

2. This property, called *no miracles* in [21] and related with the *no learning* property of [11], reflects the following idea: if two worlds are epistemically indistinguishable and the same rule is applied at both of them, then the resulting worlds should be epistemically indistinguishable too.

Definition 3.3 Given an epistemic inference model $M = (W, \sim_i, D_{(\rho,i)}, V, Y_i)$ and a world $w \in W$, the relation $\models$ between the pair M, w and $\top$, negations and disjunctions is given as usual. For the remaining formulas, we have:

$M, w \models p$	*iff*	$p \in V(w)$
$M, w \models I_i\,\gamma$	*iff*	$\gamma \in Y_i(w)$
$M, w \models P_i\,\varphi$	*iff*	there is $u \in W$ such that $w \sim_i u$ and $M, u \models \varphi$
$M, w \models \langle\rho\rangle_i\,\varphi$	*iff*	there is $w' \in W$ such that $D_{(\rho,i)}\,ww'$ and $M, w' \models \varphi$

We say a formula φ is *valid in a epistemic inference model* M (notation $M \models \varphi$) if $M, w \models \varphi$ for all worlds w in M. We say that φ is *valid in the class of models* $\boldsymbol{M}$ (notation $\mathbf{M} \models \varphi$) if φ is valid in M ($M \models \varphi$) for all M in $\mathbf{M}$. ◁

As it is currently defined, epistemic inference models do not impose any restriction to the information sets: *any* propositional formula of $\mathcal{I}$ can be in *any* information set $Y_i(w)$. We can have non-veridical information sets (if we have $\gamma \in Y_i(w)$ and $M, w \not\models \gamma$ for some $w \in W$) describing situations where the information of the agent is not true, or even inconsistent ones (if we have γ and $\neg\gamma$ in $Y_i(w)$ for some $w \in W$), describing situations where her information is contradictory.

In the present work we focus on a special class of models: those in which the information sets of the agents describe *knowledge*. We ask for the epistemic relation to be an equivalence one, and we ask for all formulas of an information set to be true at the correspondent world [2].

Definition 3.4 (Class $\mathbf{EI}_K$) The class of epistemic inference models $\mathbf{EI}_K$ contains exactly those models in which each $\sim_i$ is an equivalence relation and for every world $w \in W$, if $\gamma \in Y_i(w)$ then $M, w \models \gamma$. The following table summarize the properties of models in this class.

P1	$D_{(\rho,i)}\,ww'$ implies ρ is w-matching and w' is a ρ-extension of w (for i).
P2	If $D_{(\rho,i)}\,ww'$, then w and w' satisfy the same propositional letters.
P3	If $D_{(\rho,i)}\,ww'$, $D_{(\rho,i)}\,uu'$ and $w \sim_i u$ for some rule ρ, then $w' \sim_i u'$.
P4	$\sim_i$ is an equivalence relation.
P5	$\gamma \in Y_i(w)$ implies $M, w \models \gamma$.

◁

Our first result is a syntactic characterization of formulas of $\mathcal{EI}$ that are valid on models of $\mathbf{EI}_K$. Non-defined concepts, like a (modal) logic, Λ-consistent / inconsistent set and maximal Λ-consistent set (for a normal modal logic Λ) are completely standard, and can be found in chapter 4 of [6].

Definition 3.5 (Logic EI_K) The logic EI_K is the smallest set of formulas of $\mathcal{EI}$ that is created from the set of axioms [3] and a set of rules of table 1. ◁

Axioms		
P	All propositional tautologies	
E-K	$K_i\,(\varphi \to \psi) \to (K_i\,\varphi \to K_i\,\phi)$	
E-Dual	$P_i\,\varphi \leftrightarrow \neg K_i\,\neg\varphi$	
I-K	$[\rho]_i\,(\varphi \to \psi) \to ([\rho]_i\,\varphi \to [\rho]_i\,\psi)$	
I-Dual	$\langle\rho\rangle_i\,\varphi \leftrightarrow \neg[\rho]_i\,\neg\varphi$	
T	$\varphi \to P_i\,\varphi$	
4	$P_i\,P_i\,\varphi \to P_i\,\varphi$	
B	$\varphi \to K_i\,P_i\,\varphi$	
A1	$[\rho]_i\,I_i\,\mathrm{conc}(\rho)$	
A2	$\langle\rho\rangle_i\,\top \to I_i\,(\mathrm{prem}(\rho) + \rho)$	
A3	$I_i\,\gamma \to [\rho]_i\,I_i\,\gamma$	
A4	$\langle\rho\rangle_i\,I_i\,\gamma \to I_i\,\gamma$	with $\gamma \neq \mathrm{conc}(\rho)$.
A5	$(p \to [\rho]_i\,p) \wedge (\neg p \to [\rho]_i\,\neg p)$	with $p \in \mathcal{P}$.
A6	$(\langle\rho\rangle_i\,\varphi \wedge P_i\,\langle\rho\rangle_i\,\psi) \to \langle\rho\rangle_i\,(\varphi \wedge P_i\,\psi)$	
A7	$I_i\,\gamma \to \gamma$	
Rules		
MP	Given φ and $\varphi \to \psi$, prove ψ	
E-Gen	Given φ, prove $K_i\,\varphi$	
I-Gen	Given φ, prove $[\rho]_i\,\varphi$	

Table 1. Axioms and rules for EI_K.

[2] Facts of the internal language can be directly interpreted in epistemic inference models, but rules cannot. Formally, we will assume a translation from $\mathcal{I}$ into $\mathcal{EI}$ that maps a fact into itself and a rule into an implication whose antecedent is the (finite) conjunction of the rule's premises and whose consequent is the rule's conclusion.

[3] Formulas of the form $I_i\,\Gamma$ are abbreviations of $\bigwedge_{\gamma\in\Gamma} I_i\,\gamma$, for a finite $\Gamma \subseteq \mathcal{I}$.

Theorem 3.6 (Soundness) *The logic* EI_K *is sound with respect to the class* $\mathbf{EI}_K$*.*

Proof. *For soundness, we just need to prove that axioms of* EI_K *are valid in* $\mathbf{EI}_K$*, and that its rules preserve validity. We omit the details here.* QED

Strong completeness is equivalent to satisfiability of consistent set of formulas, as mentioned in Proposition 4.12 of [6].

Theorem 3.7 (Completeness) *The logic* EI_K *is strongly complete with respect to the class* $\mathbf{EI}_K$*.*

Proof. *We define the canonical model* M^{EI_K} *for the logic* EI_K*. With the the Lindenbaum's Lemma, the Existence Lemma and the Truth Lemma, we show that every* EI_K*-consistent set of formulas is satisfiable in* M^{EI_K}*. Finally, we show that* M^{EI_K} *is indeed a model in* $\mathbf{EI}_K$*. See section A.1 for details.* QED

4. External interaction: explicit observations

So far, our language can express the agent's *internal* dynamics, but it cannot express *external* ones. We can express how inference steps modify the explicit information, but we cannot express how both explicit and implicit one are affected by external observations. Here we add the other fundamental source of information; in this section, we extend the language to express updates. For easiness of reading and writing, we remove subindexes referring to agents.

Updates are usually represented as operations that modify the semantic model. In *Public Announcement Logic* (PAL), for example, an announcement is defined by an operation that removes the worlds where the announced formula does not hold, restricting the epistemic relation to those that are not deleted.

In our semantic model, we have a finer representation of the agent's information. We have explicit information (her information sets) but we also have implicit one (what she can add to her information set via inference). Then, we can extend PAL by defining different kinds of model operations, affecting explicit and implicit information in different forms, and therefore expressing different ways the agent processes the new information. Here, we present one of the possible definitions, what we have called *explicit observations.*

Definition 4.1 (Explicit observation) Let $M = (W, \sim, D_\rho, V, Y)$ be an epistemic inference model, and let γ be a formula of the internal language. The epistemic inference model $M_{+\gamma!} = (W', \sim', D'_\rho, V', Y')$ is given by

- $W' := \{ w \in W \mid M, w \models \gamma \}$
- $\sim' := \{ (w, u) \in W' \times W' \mid w \sim u \}$
- $D'_\rho := \{ (w, u) \in W' \times W' \mid D_\rho wu \}$
- $V'(w) := V(w) \quad \text{for } w \in W'$
- $Y'(w) := Y(w) + \gamma \quad \text{for } w \in W'$

◁

Our explicit observation operation behave as the standard public announcement with respect to worlds, valuation and relations. With respect to the information set functions, we have chosen a simple definition: once a formula is announced, it will become part of the agent's explicit information. The choice is also a good one, since the operation is closed for models in $\mathbf{EI}_K$.

Proposition 4.2 *If* M *is a model in* $\mathbf{EI}_K$*, so it is* $M_{+\gamma!}$*.*

Proof. *See section A.2.* QED

The new language $\mathcal{EEI}$ extends $\mathcal{EI}$ by closing it under explicit observations. Take a formula γ in the internal language; if φ is a formula in $\mathcal{EEI}$, so it is $[+\gamma!]\,\varphi$. The semantics for formulas already in $\mathcal{EI}$ is defined as before (definition 3.3). For explicit observation formulas, we have the following.

Definition 4.3 Let M be a model in $\mathbf{EI}_K$, and let $w \in W$ be a world in it. Then:

$$M, w \models [+\gamma!]\,\varphi \quad \textit{iff} \quad M, w \models \gamma \text{ implies } M_{+\gamma!}, w \models \varphi$$

◁

Our second result is a syntactic characterization of the formulas in $\mathcal{EEI}$ that are valid in models in $\mathbf{EI}_K$. By proposition 4.2, the explicit observation operation is closed for models in $\mathbf{EI}_K$, so we can rely on the logic EI_K: all we have to do is give a set of reduction axioms for formulas of the form $[+\gamma!]\,\varphi$. The standard reduction axioms for atomic propositions, negations, disjunctions and epistemic formulas work for $\mathcal{EEI}$ too; we just have to add axioms indicating how information set formulas and inference formulas are affected.

Theorem 4.4 *The logic* EEI_K*, built from axioms and rules of* EI_K *(see table 1) plus axioms and rules in table 4.4, is sound and strongly complete for the class* $\mathbf{EI}_K$*.*

Proof. *Soundness comes from the validity of the new axioms and the validity-preserving property of the new rule. Strong completeness comes from the fact that, by a repetitive application of such axioms, any explicit observation formula can be reduced to a formula in* $\mathcal{EI}$*, for which* EI_K *is strongly complete with respect to* $\mathbf{EI}_K$*.* QED

Axioms		
EO-1	$[+\gamma!]\, p \ \leftrightarrow\ (\gamma \rightarrow p)$	
EO-2	$[+\gamma!]\, \neg\varphi \ \leftrightarrow\ (\gamma \rightarrow \neg[+\gamma!]\, \varphi)$	
EO-3	$[+\gamma!]\, (\varphi \vee \psi) \ \leftrightarrow\ ([+\gamma!]\, \varphi \vee [+\gamma!]\, \psi)$	
EO-4	$[+\gamma!]\, K\, \varphi \ \leftrightarrow\ (\gamma \rightarrow K\, [+\gamma!]\, \varphi)$	
EO-5	$[+\gamma!]\, I\, \gamma \ \leftrightarrow\ \top$	
EO-6	$[+\gamma!]\, I\, \delta \ \leftrightarrow\ (\gamma \rightarrow I\, \delta)$	for $\delta \neq \gamma$
EO-7	$[+\gamma!]\, [\rho]\, \varphi \ \leftrightarrow\ (\gamma \rightarrow [\rho]\, [+\gamma!]\, \varphi)$	
Rules		
EO-Gen	Given φ, prove $[+\gamma!]\, \varphi$	

Table 2. Axioms and rules for explicit observations.

The language $\mathcal{EEI}$ can express uncertainty (as classic epistemic logic does), inference (as the modal approaches of [7, 8, 13, 12]) and update (as PAL). Moreover, it can express its combinations. With it, we are able to talk about the merging of *internal* dynamics, expressing the way the agent *"unpacks"* her implicit information, with external ones, expressing how her interaction with her environment modifies what she is informed about.

We have provided semantics for the language; semantics that reflect the nature of each process. Inferences are represented as relations between information sets. This reflects the idea that, with enough initial explicit information, the agent may get all the implicit information by the adequate rule applications. Update, on the other hand, is defined as a model operation. It is a process that not only provides explicit information, but also modifies implicit one. This reflects the idea that updates yields information that might have not been available to the agent before.

Among the semantic models, we distinguish the class $\mathbf{EI}_K$, which contains those where the agent's information is in fact knowledge. We give a syntactic characterization of the valid formulas in $\mathbf{EI}_K$ by means of the sound and complete logic EEI_K.

5. Comparison with other works

The present work is a combination of three main ideas: the representation of explicit information as set of formulas, relations between such sets to represent inferences and model operations to represent updates. The first two have been used in some other works; we present a brief comparison between some of them and our approach.

5.1. Fagin-Halpern's logics of awareness

Fagin and Halpern presented in [9] what they called *logic of general awareness* ($\mathfrak{L}_A$). Given a set of agents, formulas of the language are given by a set of atomic propositions $\mathcal{P}$ closed under negation, conjunction and the modal operators A_i and L_i (for an agent i). Formulas of the form $A_i\varphi$ are read as *"the agent i is aware of φ"*, and formulas of the form $L_i\varphi$ are read as *"the agent i implicitly believes that φ"*. The operator B_i, which expresses explicit beliefs, is defined as $B_i\varphi := A_i\varphi \wedge L_i\varphi$.

A *Kripke structure for general awareness* is defined as a tuple $M = (W, \mathfrak{A}_i, \mathfrak{L}_i, V)$, where $W \neq \emptyset$ is the set of possible worlds, $\mathfrak{A}_i : W \rightarrow \wp(\mathfrak{L}_A)$ is a function that assigns a set of formulas of $\mathfrak{L}_A$ to the agent i in each world (her awareness set), the relation $\mathfrak{L}_i \subseteq (W \times W)$ is a serial, transitive and Euclidean relation over W for each agent i ($\mathfrak{L}_A$ deals with beliefs rather than knowledge) and $V : \mathcal{P} \rightarrow \wp(W)$ is a valuation function.

Given a Kripke structure for general awareness $M = (W, \mathfrak{A}_i, \mathfrak{L}_i, V)$, semantics for atomic propositions, negations and conjunctions are given in the standard way. For formulas of the form $A_i\, \varphi$ and $L_i\, \varphi$, we have

$$\begin{array}{lll} M, w \models A_i\varphi & \textit{iff} & \varphi \in \mathfrak{A}_i(w) \\ M, w \models L_i\varphi & \textit{iff} & \text{for all } u \in W, \\ & & \mathfrak{L}_i wu \text{ implies } M, u \models \varphi \end{array}$$

It follows that $M, w \models B_i\varphi$ *iff* $\varphi \in A_i(w)$ and, for all $u \in W$, $\mathfrak{L}_i wu$ implies $M, u \models \varphi$.

Given the similarities between the functions $\mathfrak{A}_i$ and Y_i and between the relations $\mathfrak{L}_i$ and $\sim_i$, formulas $A_i\varphi$ and $L_i\varphi$ in $\mathfrak{L}_A$ behaves exactly like $I_i\, \varphi$ and $K_i\, \varphi$ in $\mathcal{EEI}$. The difference in the approaches is in the dynamic part.

For the internal dynamics (inference), the language $\mathfrak{L}_A$ does not express changes in the agent's awareness sets. Later in the same paper, Fagin and Halpern explore the incorporation of time to the language by adding a deterministic serial binary relation $\mathfrak{T}$ over W to represent steps in time. Still, they do not indicate what the process(es) that change the awareness sets is (are).

In our approach, pairs in the inference relation $D_{(\rho,i)}$ have a specific interpretation: they indicate *steps in the agent's reasoning process*. Because of this, we have a particular definition of how they should behave (properties **P1**, **P2**, and **P3**). Moreover, external dynamics (observations), which are not considered $\mathfrak{L}_A$, are represented in a different way, as model operations.

There is another conceptual difference. In $\mathfrak{L}_A$, elements of the awareness sets are just formulas; in $\mathcal{EI}$, elements of the information sets are not only formulas (what we have called *facts*) but also *rules*. The information of the agent consists not only on facts, but also on rules that allow her to infer new facts. It is not that the agent knows that after a rule application her information set will change; it is that she knows the *process* that leads the change. We interpret a rule as an object that can be part of the agent's information, and whose presence is needed for the agent to be able to apply it.

5.2. Duc's dynamic epistemic logic

In [7] and [8], Ho Ngoc Duc proposes a dynamic epistemic logic to reason about agents that are neither logically omniscient nor logically ignorant.

The syntax of the language is very similar to the inference part of our language. There is an internal language, the classic propositional one (PL), to express agent's knowledge. There is also another language to talk about how this knowledge evolves. Formally, At denotes the set of formulas of the form $K\gamma$, for γ in PL. The language $\mathcal{L}_{BDE}$ contains At and is closed under negation, conjunction and the modal operator $\langle F \rangle$. Formulas of the form $K\gamma$ are read as *"γ is known"*; formulas of the form $\langle F \rangle\varphi$ are read as *"φ is true after some course of thought"*.

A model M is a tuple (W, R, Y), where $W \neq \emptyset$ is the set of *possible worlds*, $R \subseteq (W \times W)$ is a transitive binary relation and $Y : W \to \wp(At)$ associates a set of formulas of At to each possible world. A *BDE*-model is a model M such that: (1) for all $w \in W$, if $K\gamma \in Y(w)$ and Rwu, then $K\gamma \in Y(u)$; (2) for all $w \in W$, if $K\gamma$ and $K(\gamma \to \delta)$ are in $Y(w)$, then $K\delta$ is in $Y(u)$ for some u such that Rwu; (3) if γ is a propositional tautology, then for all $w \in W$ there is a world u such that Rwu and $K\gamma \in Y(u)$. Such restrictions guarantees that the set of formulas will grow as the agent reasons, and that her knowledge will be closed under modus ponens and will contain all tautologies at some point in the future.

Given a *BDE*-model, the semantics for negation and conjunctions are standard. The semantics of atomic and reasoning-steps formulas are given by:

$$
\begin{array}{lll}
M, w \models K\gamma & \textit{iff} & K\gamma \in Y(w) \\
M, w \models \langle F \rangle\varphi & \textit{iff} & \text{there is } u \in W \text{ such that} \\
 & & Rwu \text{ and } M, u \models \varphi
\end{array}
$$

Note that the language does not indicate what a *"course of though"* is; again, our framework is more precise. Also, it does not consider sentences about the world. Finally, the language is restricted to express what the agent can infer through some *"course of though"*, but it does not express external dynamics, as explicit observations in $\mathcal{EEI}$ do.

6. Further work

In order to give a finer representation of the inference process, we have chosen to represent information as set of formulas. This is also a solution for the famous *logical omniscience* problem, since sets of formulas do not need to satisfy *a priori* any particular property, like being closed under some consequence relation. Among other approaches for the problem, there is the non-classical worlds approach for epistemic logic. The idea is to add worlds in which the usual rules of logic do not hold. The knowledge of the agents is affected since non-classical worlds may be considered possible. It would be interesting to look at this approach as an alternative for representing the agent's explicit information, and see what the differences are.

Our framework do not represent in a completely faithful way the intuitive idea of the application of a rule. It is possible to have a world in which a rule can be applied, and not to have a world that results from its application. We can focus on models on which, if a rule is applicable, then *there is a world* that results from its application. This forces us to change the defined explicit observation operation since, in general, the resulting model will not have the required property: the added formula can make applicable a rule that was not applicable before. The immediate solution is to create all needed worlds, but this iterative process complicates the operation, and the existence of reduction axioms is not so clear anymore.

As mentioned in the text, properties **P4** and **P5** characterize models in which the information the agent has is in fact knowledge, that is, the epistemic relation is an equivalence one and formulas in all information sets are true at the correspondent world. It would be interesting to be able to talk about not only knowledge but also *beliefs*. Some recent works ([17, 4, 5] among others) combine these two notions, giving us a nice way of studying these two propositional attitudes together.

Property **P1** defines not only the situation when a rule can be applied (whenever a rule a rule and all its premises are in the agent's information set), but also what results from the application (the given information set extended by the conclusion of the rule). The property indeed restricts our models to those that use rules in a *deductive* way, that is, to those that represent just *deductive* inference. There are other interesting inference processes, like *abduction* or *belief revision*; they are not deductive, but they are important and widely used, with particular relevance on incomplete information situations. Within the proposed framework, we can represent different inference processes, and we can study how all of them work together.

For the external dynamics, we mentioned that this finer representation of knowledge allows us to define different kinds of observations. Since we represent both explicit and implicit information, we can define different model operations, allowing us to explore the different ways an agent process new information.

In the context of agent diversity ([14, 15]), a finer representation of the inference process allows us to make a distinction between agents with different reasoning abilities. The rules an agent has in her information set may be very different from those in the information set of another, and they will not be able to perform the same inference steps. Moreover, some of them may be able to perform several inference steps at once instead of a single one. The idea works also for external dynamics: agents may have different observational power. It will be interesting to explore how agents that differs in their reasoning and observational abilities interact with each other.

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A. Technical appendix

A.1. Proof of completeness

As mentioned, the key observation is that a logic Λ is strongly complete with respect to a class of structures if and only if every Λ-consistent set of formulas is satisfiable on some structure of the given class (Proposition 4.12 of [6]). Using the the canonical model technique, we show that every EI_K-consistent set of formulas is satisfiable in a model in $\mathbf{EI}_K$. Proofs of Lindenbaum's Lemma, Existence Lemmas and Truth Lemma are standard.

Lemma A.1 (Lindenbaum's Lemma) *For any EI_K-consistent set of formulas Σ, there is a maximal EI_K-consistent set Σ^+ such that $\Sigma \subseteq \Sigma^+$.*

Definition A.2 (Canonical model) The canonical model of the logic EI_K is the epistemic inference model $M^{\mathsf{EI}_K} = (W^{\mathsf{EI}_K}, \sim_i^{\mathsf{EI}_K}, D_{(\rho,i)}^{\mathsf{EI}_K}, V^{\mathsf{EI}_K}, Y_i^{\mathsf{EI}_K})$, where:

- W^{EI_K} is the set of all maximal EI_K-consistent set of formulas.
- $w \sim_i^{\mathsf{EI}_K} u$ *iff* for all φ in $\mathcal{EI}$, $\varphi \in u$ implies $P_i\, \varphi \in w$ (equivalently, $w \sim_i^{\mathsf{EI}_K} u$ *iff* for all φ in $\mathcal{EI}$, $K_i\, \varphi \in w$ implies $\varphi \in u$).
- $w D_{(\rho,i)}^{\mathsf{EI}_K} w'$ *iff* for all φ in $\mathcal{EI}$, $\varphi \in w'$ implies $\langle \rho \rangle_i\, \varphi \in w$ (equivalently, $w D_{(\rho,i)}^{\mathsf{EI}_K} w'$ *iff* for all φ in $\mathcal{EI}$, $[\rho]_i\, \varphi \in w$ implies $\varphi \in w'$).
- $V^{\mathsf{EI}_K}(w) := \{\, p \in \mathcal{P} \mid p \in w \,\}$.
- $Y_i^{\mathsf{EI}_K}(w) := \{\, \gamma \in \mathcal{I} \mid I_i\, \gamma \in w \,\}$.

$\lhd$

Lemma A.3 (Existence Lemmas) *For any world $w \in W^{\mathsf{EI}_K}$, if $P_i\, \varphi \in w$, then there is a world $u \in W^{\mathsf{EI}_K}$ such that $w \sim_i^{\mathsf{EI}_K} u$ and $\varphi \in u$. For any world $w \in W^{\mathsf{EI}_K}$, if $\langle \rho \rangle_i\, \varphi \in w$, then there is a world $w' \in W^{\mathsf{EI}_K}$ such that $D_{(\rho,i)}^{\mathsf{EI}_K} ww'$ and $\varphi \in w'$.*

Lemma A.4 (Truth Lemma) *For all $w \in W^{\mathsf{EI}_K}$, we have $M^{\mathsf{EI}_K}, w \models \varphi$ iff $\varphi \in w$.*

By the mentioned Proposition of [6], all we have to show is that every EI_K-consistent set of formulas is satisfiable, so take any such set Σ. By Lindenbaum's Lemma, we can extend it to a maximal EI_K-consistent set of formulas Σ^+; by the Truth Lemma, we have $M^{\mathsf{EI}_K}, \Sigma^+ \models \Sigma$, so Σ is satisfiable in the canonical model of EI_K at Σ^+. Now we have to show that the canonical model M^{EI_K} is indeed a model in $\mathbf{EI}_K$.

Axioms **T**, **4** and **B** are canonical for reflexivity, transitivity and symmetry, respectively, so $\sim_i^{\mathsf{EI}_K}$ is an equivalence relation and property **P4** is fulfilled. It remains to show that M^{EI_K} satisfy **P1**, **P2**, **P3** and **P5**. We have removed the agent's subindexes for easiness of writing and reading.

Remember that any maximal EI_K-consistent set Φ is closed under modus ponens, that is, if φ and $\varphi \to \psi$ are in Φ, so it is ψ.

P1 Suppose $D_\rho^{\mathsf{EI}_K} ww'$; we want to show that $(\mathrm{prem}(\rho) + \rho) \subseteq Y^{\mathsf{EI}_K}(w)$ and that $Y^{\mathsf{EI}_K}(w') = Y^{\mathsf{EI}_K}(w) + \mathrm{conc}(\rho)$.

For the first part, $D_\rho^{\mathsf{EI}_K} ww'$ implies $M^{\mathsf{EI}_K}, w \models \langle \rho \rangle \top$, so $\langle \rho \rangle \top \in w$. By axiom **A2** and modus ponens closure, we have $I\,(\mathrm{prem}(\rho) + \rho) \in w$. Then, $\mathrm{prem}(\rho)$ and ρ are in $Y^{\mathsf{EI}_K}(w)$.

For the second part, we will show both inclusions, i.e., we will show that $Y^{\mathsf{EI}_K}(w) + \mathrm{conc}(\rho) \subseteq Y^{\mathsf{EI}_K}(w')$ and $Y^{\mathsf{EI}_K}(w') \subseteq Y^{\mathsf{EI}_K}(w) + \mathrm{conc}(\rho)$.

- Take any $\gamma \in Y^{\mathsf{EI}_K}(w)$; then, $I\, \gamma \in w$. By axiom **A3** and the modus ponens closure,

$[\rho]\, I\,\gamma \in w$. Since $D_\rho^{\mathsf{EI}_K}\, ww'$, we have $I\,\gamma \in w'$ and then $\gamma \in Y^{\mathsf{EI}_K}(w')$.

It remains to show that $\mathrm{conc}(\rho) \in Y^{\mathsf{EI}_K}(w')$. Since axiom **A1** is in w and $D_\rho^{\mathsf{EI}_K}\, ww'$, we have $I\ \mathrm{conc}(\rho) \in w'$ and therefore $\mathrm{conc}(\rho) \in Y^{\mathsf{EI}_K}(w')$.

- Take any $\gamma \in (Y^{\mathsf{EI}_K}(w') - \mathrm{conc}(\rho))$; then, $I\,\gamma \in w'$. Since $D_\rho^{\mathsf{EI}_K}\, ww'$, we have $\langle\rho\rangle\, I\,\gamma \in w$ and, by axiom **A4**, we have $I\,\gamma \in w$; then, $\gamma \in Y^{\mathsf{EI}_K}(w)$. Hence, $Y^{\mathsf{EI}_K}(w') - \mathrm{conc}(\rho) \subseteq Y^{\mathsf{EI}_K}(w)$, and therefore $Y^{\mathsf{EI}_K}(w') \subseteq Y^{\mathsf{EI}_K}(w) + \mathrm{conc}(\rho)$.

P2 Suppose $D_\rho^{\mathsf{EI}_K}\, ww'$; we want to show that w and w' satisfy the same propositional letters. Note that we have **A5** in w, and then both $p \to [\rho]\, p$ and $\neg p \to [\rho]\, \neg p$ are in w for every $p \in \mathcal{P}$, since it is a maximal consistent set.

If $M^{\mathsf{EI}_K}, w \models p$ then, by definition of V^{EI_K}, we have $p \in w$. But $(p \to [\rho]\, p) \in w$ and, by the modus ponens closure, $[\rho]\, p \in w$. Then, since $D_\rho^{\mathsf{EI}_K}\, ww'$, we have $p \in w'$, so $M^{\mathsf{EI}_K}, w' \models p$.

If $M^{\mathsf{EI}_K}, w \not\models p$, then $M^{\mathsf{EI}_K}, w \models \neg p$; by definition of V^{EI_K}, we have $\neg p \in w$. But $(\neg p \to [\rho]\, \neg p) \in w$, so the modus ponens closure implies $[\rho]\, \neg p \in w$. Then, since $D_\rho^{\mathsf{EI}_K}\, ww'$, we have $\neg p \in w'$, so $M^{\mathsf{EI}_K}, w' \models \neg p$, i.e., $M^{\mathsf{EI}_K}, w' \not\models p$.

P3 Note that axiom **A6** is a Sahlqvist formula (a very simple Sahlqvist formula indeed; see section 3.6 of [6] for details). Its first-order local correspondent is the formula

$$(\forall w')(\forall u)(\forall u')\big((D_\rho\, ww' \wedge w \sim u \wedge D_\rho\, uu') \\ \to (D_\rho\, ww' \wedge u \sim u')\big)$$

which is equivalent to our desired property

$$\chi(w) := (\forall w')(\forall u)(\forall u') \\ ((D_\rho\, ww' \wedge w \sim u \wedge D_\rho\, uu') \to u \sim u')$$

By theorem 4.42 of [6], we know that **A6** is canonical for $\chi(w)$, i.e., the canonical frame for any normal modal logic containing **A6** has the property $\chi(w)$. In particular, M^{EI_K} has the property.

P5 We want to show that $\gamma \in Y^{\mathsf{EI}_K}(w)$ implies $M^{\mathsf{EI}_K}, w \models \gamma$. Suppose $\gamma \in Y^{\mathsf{EI}_K}(w)$; by definition of Y^{EI_K}, we have $I\,\gamma \in w$; by axiom **A7** and the modus ponens closure, $\gamma \in w$; by the Truth Lemma, $M^{\mathsf{EI}_K}, w \models \gamma$.

A.2. Proof of Proposition 4.2

We will show that $M_{+\gamma!} = (W', \sim', D'_\rho, V', Y')$ satisfy **P1-P5**.

P1 Suppose $D'_\rho\, wu$; we want to show that $(\mathrm{prem}(\rho) + \rho) \subseteq Y'(w)$ and that $Y'(u) = Y'(w) + \mathrm{conc}(\rho)$. If $D'_\rho wu$, then $w, u \in W'$ and $D_\rho wu$. Since M satisfy **P1**, we have $(\mathrm{prem}(\rho) + \rho) \subseteq Y(w)$ and $Y(u) = Y(w) + \mathrm{conc}(\rho)$. By definition of Y' and the fact that $w, u \in W'$, we have $(\mathrm{prem}(\rho) + \rho) \subseteq Y'(w)$ and $Y'(u) = Y'(w) + \mathrm{conc}(\rho)$.

P2 Suppose $D'_\rho wu$; we want to show that w, u satisfy the same propositional letters in M. Since $D'_\rho wu$, w and u are in W' and $D_\rho wu$. By property **P2** of M, we know that w and u satisfy the same propositional letters in M; by definition of V', w and u satisfy the same propositional letters in $M_{+\gamma!}$.

P3 Suppose $w_1 \sim' u_1$ and $D'_\rho\, w_1 w_2, D'_\rho\, u_1 u_2$ for some rule ρ; we want to show that $w_2 \sim' u_2$. By $w_1 \sim' u_1$, $D'_\rho\, w_1 w_2$ and $D'_\rho\, u_1 u_2$, we have $w_1 \sim u_1$, $D_\rho\, w_1 w_2$ and $D_\rho\, u_1 u_2$, with $w_1, w_2, u_1, u_2 \in W'$. By **P3** of M, $w_2 \sim u_2$; by definition of $\sim'$, we get $w_2 \sim' u_2$.

P4 It follows from the definition that if $\sim$ is an equivalence relation, so it is $\sim'$.

P5 Suppose $\delta \in Y'(w)$; we want to show that $M_{+\gamma!}, w \models \delta$. If $\delta \in Y'(w)$, we have either $\delta \in Y(w)$ or else $\delta = \gamma$. In the first case, we get $M, w \models \delta$ by **P5** of M; in the second case, we get $M, w \models \delta$ by definition of W' and the fact that $w \in W'$. But δ is just a propositional formula, and by definition, the valuations for w in M and $M_{+\gamma!}$ are the same. Then, $M_{+\gamma!}, w \models \delta$.

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