Strategies made explicit in Dynamic Game Logic

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Abstract

We propose an explicit logic of strategies ($SDGL$) in the Dynamic Game Logic (DGL) framework and provide a complete axiomatization for this logic. Some discussions are put forward regarding $SDGL$ and DGL, raising an interesting issue about their combination.

1. Introduction: cudos for strategies

Many events that happen in our daily life can be thought of as games. In fact, besides the 'games' in the literal sense, our day-to-day dialogues, interactions, legal procedures, social and political actions, biological phenomena - all these can be viewed as games together with their goals and strategies. The theory of games has its various applications in the areas of economics, logic, computer science as well as linguistics. Games play a very important role in modelling intelligent interaction. In Rubinstein's words, 'I view game theory as an analysis of concepts used in social reasoning when dealing with situations of conflict' [16].

As evident from the existing literature, much of game theory deals with strategic equilibriums. Various equilibrium theories have been developed till date both for zero-sum as well as non zero-sum games, starting from the initial concept of Nash [12], which have their implications in the studies of the society. They help in providing a 'plan of action' to the agents participating in the state of affairs, which could be articulated as 'games', when faced with strategic decision making in situations of conflict.

Over the past few decades a lot of work has been done in the epistemic foundations of game theory, studying the formal logics of knowledge and belief. The formal systems expressing players' knowledge and beliefs about themselves as well as their competitors were looked at in much details - a tremendous amount of work is still going on. But a very related and relevant issue - players' strategies/plan of actions to play the game, which they base on their epistemic states almost have rarely been looked upon, until very recently. To mention a few, [15] proposes a logic of strategies in games over finite graphs, whereas [17] makes strategies explicit in Alternating-time Temporal Logic. The incorporation of 'strategies' within the logical language would very well aid in the currently popular ventures into social choice mechanism designs.

Strategies of the players playing the game form a basic ingredient of game theory, whether looked upon from the winning point of view or from the best-response one. A lot of other issues like the rationality of the players, their goals and preferences are also very important issues, but they are outside the scope of this work, though we plan to incorporate them in the future.

Our main goal in this work is to incorporate explicit notions of strategies in the framework of Dynamic Game Logic (DGL) [13]. Not unlike other logics talking about game and coalition structures [1, 14], DGL suffers from '$\exists$-sickness' : the detailed level of game structures getting suppressed by existential quantifiers of "having a strategy" [7]. We intend to provide a logic ($SDGL$) that makes the game structures explicit to a great extent.

In general, strategies are partial transition relations and hence dynamic modal logic provides a good framework to talk about them, as mentioned in [4, 5]. But the main challenge here is to combine the strategy calculus together with the game calculus. As one can easily guess, the constructs of Propositional Dynamic Logic [11] play an important role in achieving this amalgamation. In this regard, we should mention that, a lot of discussions and proposals have already been put forward by van Benthem [9, 8]. This effort can be looked upon as a follow-up of one of these proposals.

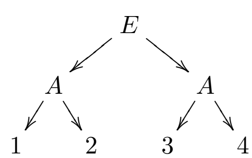
After providing a brief overview of DGL in the next section, we propose a logic for strategizing DGL ($SDGL$) in section 3 with a complete axiomatization. Section 4 provides some discussions over the two logics DGL and $SDGL$, with several pointers towards future work mentioned in the last one.

2. Dynamic Game Logics : an overview

We now give a brief review of DGL, the dynamic game logic of two-person sequential games in this section, which was first proposed in [13], and further developed by [14], [6], [10] and others. DGL talks about 'generic' games which can be played starting from any state s on the 'game boards' and the semantics is based on the 'forcing relations' describing the powers each player has to end in a set of final states, starting from a single initial state.

> $s\rho^i_G X$: player i has a *strategy* for playing game **G** from state s onwards, whose resulting final states are always in the set X, whatever the other players choose to do.

To exemplify, let us move onto real extensive games for once. Consider the game tree:



In this game, player *E* has two strategies, forcing the sets of end states $\{1, 2\}$, $\{3, 4\}$, while player *A* has four strategies, forcing one of the sets $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$.

These forcing relations satisfy the following two simple set-theoretic conditions [3]:

(C1) Monotonicity: If $s\rho^i_G X$ and $X \subseteq X'$, then $s\rho^i_G X'$.

(C2) Consistency: If $s\rho^E_G Y$ and $s\rho^A_G Z$, then Y, Z overlap.

In the semantics of DGL as proposed in [13, 14], another extra condition is assumed :

(C3) Determinacy: If it is not the case that $s\rho^E_G X$, then, $s\rho^A_G S$-X, and the same for A vis-a-vis E.

Both [13] and [14] talks about determined games. This simplifies things a lot, but fails to express the roles of the players in the games. The dynamic logic for non-determined games was studied extensively in [10] which also introduced the notion of parallel games in the syntax. For the present work the concurrent game construct has not been dealt with. The *iteration* operation for repeated play as present in [13] has also not been considered here. We only consider the following constructs which form new games: *choice* ($G \cup G'$), *dual* (G^d), *and sequential composition* $(G; G')$. The readers could easily guess the intuitive meanings of these constructs. For the sake of continuation to the next section, in what follows, the DGL for non-determined games has been briefly discussed. To start with, it should be noted here that the players' powers have a recursive structure in the complex games:

Fact 2.1 *Forcing relations for players in complex sequential two-person games satisfy the following equivalences:*

$$\begin{array}{ccc}
s\rho^E_{G\cup G'} X & \textit{iff} & s\rho^E_G X \textit{ or } s\rho^E_{G'} X \\
s\rho^A_{G\cup G'} X & \textit{iff} & s\rho^A_G X \textit{ and } s\rho^A_{G'} X \\
s\rho^E_{G^d} X & \textit{iff} & s\rho^A_G X \\
s\rho^A_{G^d} X & \textit{iff} & s\rho^E_G X \\
s\rho^i_{G;G'} X & \textit{iff} & \exists Z : s\rho^i_G Z \textit{ and for all } z \in Z,\ z\rho^i_{G'} X.
\end{array}$$

The basic models that play the role of game boards are defined as follows:

Definition 2.2 A game model is a structure $\mathcal{M} = (S, \{\rho^i_g \mid g \in \Gamma\}, V)$, where S is a set of states, V is a valuation assigning truth values to atomic propositions in states, and for each $g \in \Gamma$, $\rho^i_g \subseteq$ S $\times$ $\mathcal{P}$(S). We assume that for each g, the relations are upward closed under supersets (the earlier Monotonicity), while also, the earlier Consistency condition holds for the forcing relations of the players A, E. ◁

The language of DGL (without game iteration) is defined as follows:

Definition 2.3 Given a set of atomic games Γ and a set of atomic propositions Φ, game terms γ and formulas ϕ are defined inductively:

$$\gamma := g \mid \phi? \mid \gamma;\gamma \mid \gamma \cup \gamma \mid \gamma^d$$
$$\phi := \bot \mid p \mid \neg\phi \mid \phi \vee \phi \mid \langle \gamma, i\rangle\phi,$$

where $p \in \Phi$, $g \in \Gamma$ and $i \in \{A, E\}$.

◁

The truth definition for formulas ϕ in a model $\mathcal{M}$ at a state s is standard, except for the modality $\langle \gamma, i\rangle\phi$, which is interpreted as follows:

> $\mathcal{M}, s \models \langle \gamma, i\rangle\phi$ iff there exists X : $s\rho^i_\gamma X$ and $\forall x \in X : \mathcal{M}, x \models \phi$.

The complete axiomatization of this logic has been proposed and proved in [10] :

Theorem 2.4 *DGL is complete and its validities are axiomatized by the following axioms:*

a) all propositional tautologies and inference rules

b) if $\vdash \phi \to \psi$ *then* $\vdash \langle g, i\rangle\phi \to \langle g, i\rangle\psi$

c) $\langle g, E\rangle\phi \to \neg\langle g, A\rangle\neg\phi$

d) reduction axioms:

$$\langle \alpha\cup\beta, E\rangle\phi \leftrightarrow \langle\alpha, E\rangle\phi \vee \langle\beta, E\rangle\phi$$
$$\langle \alpha\cup\beta, A\rangle\phi \leftrightarrow \langle\alpha, A\rangle\phi \wedge \langle\beta, A\rangle\phi$$
$$\langle \gamma^d, E\rangle\phi \leftrightarrow \langle\gamma, A\rangle\phi$$
$$\langle \gamma^d, A\rangle\phi \leftrightarrow \langle\gamma, E\rangle\phi$$
$$\langle \alpha;\beta, i\rangle\phi \leftrightarrow \langle\alpha, i\rangle\langle\beta, i\rangle\phi$$
$$\langle \delta?, E\rangle\phi \leftrightarrow (\delta\wedge\phi)$$
$$\langle \delta?, A\rangle\phi \leftrightarrow (\neg\delta\wedge\phi).$$

This logic is also decidable. As can be noticed, the truth definition of the modal game formulas of the form $\langle\gamma, i\rangle\phi$ is given in terms of existence of strategies, without going into their structures. In what follows, the strategy structures have been explicitly dealt together with the game structures.

3. Strategizing DGL

3.1. A logic for strategies

Mentioning strategies explicitly in the dynamic game logic framework prompt us to divert from the usual DGL semantics that takes into consideration 'generic' games on game boards. The whole point is to bring *strategies* within the logical language which till now have their place in giving meaning to the game as well as coalition modalities [14]. Adding explicit strategy terms to DGL, the language of Strategized DGL ($SDGL$) is defined by,

Definition 3.1 Given a set of atomic games Γ, a set of atomic strategies Σ, a *finite* set of atomic actions Π and a set of atomic propositions Φ, game terms γ, strategy terms σ, action terms π and formulas ϕ are defined inductively in the following way:

$$\gamma := g \mid \phi? \mid \gamma^d \mid \gamma;\gamma \mid \gamma\cup\gamma$$
$$\sigma := s \mid \sigma\cup\sigma \mid \sigma;\sigma$$
$$\pi := b \mid \pi\cup\pi \mid \pi^*$$
$$\phi := \bot \mid p \mid \neg\phi \mid \phi\vee\phi \mid [\pi]\phi \mid \langle\pi\rangle\phi \mid \langle\sigma, i, \gamma\rangle\phi$$

where $p\in\Phi$, $s\in\Sigma$, $g\in\Gamma$, $b\in\Pi$, and $i\in\{A, E\}$. ◁

Regarding the intuitive understanding of the strategy terms, '$\cup$' corresponds to the choice of strategies, and ';' to the composition of strategies. It should be mentioned here that, the way the semantics is given later, it would have been enough to use just one combination operation of the strategy terms. The use of both of them aids in understanding the intuition behind their usage.

Moving away from the 'generic' game structures, the models take the form of extensive game trees with a few additional actions. Before going into all these, we need a parent model which is given as follows.

Definition 3.2 A model is a structure $\mathcal{M}$ = ⟨ S, $\{R_\pi : \pi$'s are actions$\}$, *ref*, **L**, **R**, V ⟩, where S is a set of states and V is a valuation assigning truth values to atomic propositions in states. For each π, R_π is a binary relation on S. *ref*, **L**, **R** are all reflexive relations over S, with ⟨ S, $\{R_\pi : \pi$'s are actions$\}$⟩ forming a regular action frame. ◁

In this model, atomic and composite games from a specified 'start'-state are defined in the following. It should be mentioned that all these game structures are taken to be finite, defined over finite subsets of S.

Definition 3.3 Game($\mathcal{M}$, s, γ) is a structure defined recursively as follows:

(i) For atomic games g, Game($\mathcal{M}$, s, g) is a structure given as follows: $\langle W\subseteq S, s\in W, \{R_b\downarrow_W: b\in\Pi\}, V = V^{\mathcal{M}}\downarrow_W, P : W\to\{E, A, end\}\rangle$.

(ii) For test games $\phi?$, Game($\mathcal{M}$, s, $\phi?$) is a structure given as follows: $\langle\{s\}, s, ref\downarrow_{\{s\}}, V = V^{\mathcal{M}}\downarrow_{\{s\}}, P : \{s\}\to\{end\}\rangle$.

(iii) Given Game($\mathcal{M}$, s, γ), Game($\mathcal{M}$, s, γ^d) is the structure, $\langle W\subseteq S, s\in W, \{R_b\downarrow_W: b\in\Pi\}, V = V^{\mathcal{M}}\downarrow_W, P : W\to\{E, A, end\}\rangle$, where all the constituents of the structure are the same as the corresponding ones in Game($\mathcal{M}$, s, γ), except for P_{γ^d}, which satisfies the property: P_{γ^d}(w) = E/A, whenever P_γ(w) = A/E, respectively.

(iv) Given Game($\mathcal{M}$, s, γ) and Game($\mathcal{M}$, s, γ'), Game($\mathcal{M}$, s, $\gamma\cup\gamma'$) is the structure given by, $\langle W\subseteq S, s\in W, \{R_b\downarrow_W: b\in\Pi\}, \mathbf{L}\downarrow_{\{s,s\}}, \mathbf{R}\downarrow_{\{s,s\}}, V = V^{\mathcal{M}}\downarrow_W, P : W\to\{E, A, end\}\rangle$, where $W = W_\gamma \uplus W_{\gamma'}$, and P extends both P_γ and $P_{\gamma'}$.

(v) Given Game($\mathcal{M}$, s_1, γ) and Game($\mathcal{M}$, s_2, γ'), Game($\mathcal{M}$, s, $\gamma;\gamma'$) is defined if for each $t\in P_\gamma^{-1}(end)$, Game($\mathcal{M}$, t, γ') can be defined. Suppose we have Game($\mathcal{M}$, t_1, γ'), …, Game($\mathcal{M}$, t_n, γ'). In that case, Game($\mathcal{M}$, s, $\gamma;\gamma'$) is the structure $\langle W\subseteq S, s\in W, \{R_b\downarrow_W: b\in\Pi\}, V = V^{\mathcal{M}}\downarrow_W, P : W\to\{E, A, end\}\rangle$, where $W = W_\gamma\cup W^1_{\gamma'}\cup\ldots\cup W^n_{\gamma'}$; $s = s_1$; P extends P_γ, P^1_γ, …, P^n_γ, with the restriction that for $w\in P_\gamma^{-1}(end)\cap W$, $P(w) = P_{\gamma'}(s_2)$. ◁

Because of some technical reasons regarding satisfiability, *choice* games can only be defined for the games with the same initial state, which is not really a big issue. The sequential composition game could also be defined under certain restrictions as mentioned above. It is now time to define strategies of the players in a game, which again has a recursive definition. Note that we will only talk about full strategies here and the definition is given likewise.

Definition 3.4 Given Game($\mathcal{M}$, s, γ), a strategy for a player i, given by the relation $\mathcal{R}_i^\gamma$ is defined by,

(i) For Game($\mathcal{M}$, s, g), $\mathcal{R}_E^g[\mathcal{R}_A^g] \subseteq \bigcup\{R_b \downarrow_{W_g} : b \in \Pi\}$ satisfying the following conditions:

(a) $s \in Dom(\mathcal{R}_E^g)[Dom(\mathcal{R}_A^g)]$, and $Ran(\mathcal{R}_E^g)$ $[Ran(\mathcal{R}_A^g)] \cap P_g^{-1}(end) \neq \emptyset$

(b) For each $t \in P_g^{-1}(E, A) - \{s\}$, $t \in Dom(\mathcal{R}_E^g)$ $[Dom(\mathcal{R}_A^g)]$ iff $t \in Ran(\mathcal{R}_E^g)[Ran(\mathcal{R}_A^g)]$.

(c) For each $s \in P^{-1}(E)[P^{-1}(A)], \exists$ unique s' such that $(s, s') \in \mathcal{R}_E^g[\mathcal{R}_A^g]$.

(d) For each $s \in P^{-1}(A)[P^{-1}(E)]$, $(s, s') \in \bigcup\{R_b \downarrow_{W_g} : b \in \Pi\}$ implies $(s, s') \in \mathcal{R}_E^g[\mathcal{R}_A^g]$.

(e) Nothing else is in $\mathcal{R}_E^g[\mathcal{R}_A^g]$.

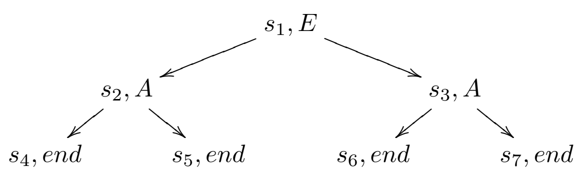
(ii) For Game($\mathcal{M}$, s, ϕ?), $\mathcal{R}_i^{\phi?}$ = *ref* $\downarrow_{\{s\}}$.

(iii) For Game($\mathcal{M}$, s, γ^d), $\mathcal{R}_E^{\gamma^d} = \mathcal{R}_A^\gamma$, and $\mathcal{R}_A^{\gamma^d} = \mathcal{R}_E^\gamma$.

(iv) For Game($\mathcal{M}$, s, $\gamma \cup \gamma'$),
$\mathcal{R}_E^{\gamma\cup\gamma'}$ = $\mathbf{L} \downarrow_{\{s,s\}} \cup \mathcal{R}_E^\gamma$ or, $\mathbf{R} \downarrow_{\{s,s\}} \cup \mathcal{R}_E^{\gamma'}$,
and, $\mathcal{R}_A^{\gamma\cup\gamma'}$ = $\mathbf{L} \downarrow_{\{s,s\}} \cup \mathbf{R} \downarrow_{\{s,s\}} \cup \mathcal{R}_A^\gamma \cup \mathcal{R}_A^{\gamma'}$.

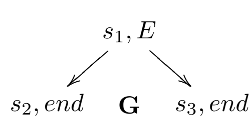
(v) For Game($\mathcal{M}$, s, $\gamma; \gamma'$), $\mathcal{R}_i^{\gamma;\gamma'}$ = $\mathcal{R}_i^\gamma \cup \mathcal{R}_{i_{j_1}}^{\gamma'} \cup \ldots \cup \mathcal{R}_{i_{j_l}}^{\gamma'}$, where the indices correspond to the number of times the 'end'-state is reached in $\mathcal{R}_i^\gamma$. $\triangleleft$

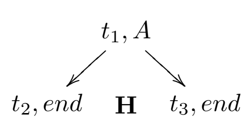
For an example of the players' strategies, consider the simple extensive game tree:



The strategies of E are $\{(s_1, s_2), (s_2, s_4), (s_2, s_5)\}$, and $\{(s_1, s_3), (s_3, s_6), (s_3, s_7)\}$, whereas the strategies for A are $\{(s_1, s_2), (s_1, s_3), (s_2, s_4), (s_3, s_6)\}$, $\{(s_1, s_2), (s_1, s_3), (s_2, s_4), (s_3, s_7)\}$, $\{(s_1, s_2), (s_1, s_3),$ $(s_2, s_5), (s_3, s_6)\}$, and $\{(s_1, s_2), (s_1, s_3), (s_2, s_5), (s_3, s_7)\}$.

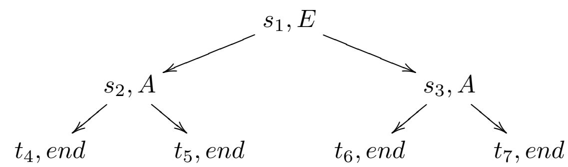
If we consider the *choice* operations of two such games, the strategies of the players could be easily computed. For the *sequential composition*, consider the following two games:





The strategies of E in G are $\{(s_1, s_2)\}$, and $\{(s_1, s_3)\}$, and in H is $\{(t_1, t_2), (t_1, t_3)\}$, and similarly, that for A in G is $\{(t_1, t_2), (t_1, t_3)\}$, and in H are $\{(t_1, t_2)\}$, and $\{(t_1, t_3)\}$.

Suppose, the model is such that $G; H$ could be defined and it is as follows:



The readers can notice that it is just the game given as example earlier, and hence could easily verify that the strategies of the players in this complex game conform with the definition given to compute the strategies of the *sequential composition* games, from the simpler ones.

Before going into the truth-definitions of formulas, let us mention a few words about interpreting the strategy terms of the language. The strategy terms are always interpreted corresponding to some game structure Game($\mathcal{M}$, s, γ) and player i. Let $\mathbb{R}_i^\gamma$ denote the set of all strategies for player i in Game($\mathcal{M}$, s, γ).

Definition 3.5 Given Game($\mathcal{M}$, s, γ) and player i, a strategy function $\mathcal{F}_i^\gamma$ is a partial function from the set of all strategy terms to $\mathbb{R}_i^\gamma$, satisfying the following conditions.

(i) For $s \in \Sigma$, $\mathcal{F}_i^\gamma(s)$ is defined, only when γ is an atomic or a test game.

(ii) For the choice game $\alpha \cup \beta$, $\mathcal{F}_i^{\alpha\cup\beta}$ is given by,

$$\mathcal{F}_E^{\alpha\cup\beta}(\sigma \cup \tau) = \mathbf{L} \downarrow_{W_{\alpha\cup\beta}} \cup \mathcal{R}_E^\alpha \quad \text{iff} \quad \mathcal{F}_E^\alpha(\sigma) = \mathcal{R}_E^\alpha,$$

$$\mathcal{F}_E^{\alpha\cup\beta}(\sigma \cup \tau) = \mathbf{R} \downarrow_{W_{\alpha\cup\beta}} \cup \mathcal{R}_E^\beta \quad \text{iff} \quad \mathcal{F}_E^\beta(\tau) = \mathcal{R}_E^\beta,$$

$$\mathcal{F}_A^{\alpha\cup\beta}(\sigma \cup \tau) = \mathcal{R}_A^{\alpha\cup\beta} \quad \text{iff} \quad \mathcal{F}_A^\alpha(\sigma) = \mathcal{R}_A^\alpha \text{ and, } \mathcal{F}_A^\beta(\tau) = \mathcal{R}_A^\beta.$$

(iii) $\mathcal{F}_E^{\gamma^d}(\sigma) = \mathcal{R}_E^{\gamma^d}$ iff $\mathcal{F}_A^\gamma(\sigma) = \mathcal{R}_A^\gamma$, and,
$\mathcal{F}_A^{\gamma^d}(\sigma) = \mathcal{R}_E^{\gamma^d}$ iff $\mathcal{F}_A^\gamma(\sigma) = \mathcal{R}_A^\gamma$.

(iv) For the composition game $\alpha; \beta$, $\mathcal{F}_i^{\alpha;\beta}$ satisfies,

$$\mathcal{F}_i^{\alpha;\beta}(\tau; \eta) = \mathcal{R}_i^{\alpha;\beta} \quad \text{iff} \quad \mathcal{F}_i^\alpha(\tau) = \mathcal{R}_i^\alpha \text{ and, } \mathcal{F}_i^\beta(\eta) = \mathcal{R}_i^\beta.$$

$\triangleleft$

Note that the way these partial functions are given, it takes care of the cases of mismatched syntax (like, $\langle\sigma \cup \tau, E, \alpha; \beta\rangle\phi$), which does not have any corresponding structure in the model. For the semantics of our language, we define the truth of a formula ϕ in $\mathcal{M}$ at a state s in the obvious manner, with the action modalities defined in the usual PDL-style and the following key clause for the game-strategy modality:

$\mathcal{M}, s \models \langle\sigma, i, \gamma\rangle\phi$ iff for all $s' \in Ran(\mathcal{F}_i^\gamma(\sigma)) \cap P^{-1}(end)$ in Game$(\mathcal{M}, s, \gamma)$, $\mathcal{M}, s' \models \phi$.

Here are some validities of this logic.

- $\langle\sigma, i, \gamma\rangle\phi \rightarrow \langle\sigma, i, \gamma\rangle(\phi \vee \psi)$
- $\langle\sigma, i, \gamma\rangle(\phi \wedge \psi) \leftrightarrow \langle\sigma, i, \gamma\rangle\phi \wedge \langle\sigma, i, \gamma\rangle\psi$

3.2. Axioms and completeness

We now provide a complete axiomatization of $SDGL$.

Theorem 3.6 *$SDGL$ is complete and its validities are axiomatized by*

a) all propositional tautologies and inference rules

b) generalization rule for the action modalities

c) axioms for the action constructs:

$[\pi](\phi \rightarrow \psi) \rightarrow ([\pi]\phi \rightarrow [\pi]\psi)$

$\langle\pi\rangle\phi \leftrightarrow \neg[\pi]\neg\phi$

$\langle\pi_1 \cup \pi_2\rangle\phi \leftrightarrow \langle\pi_1\rangle\phi \vee \langle\pi_2\rangle\phi$

$\langle\pi^*\rangle\phi \leftrightarrow (\phi \vee \langle\pi\rangle\langle\pi^*\rangle\phi)$

$[\pi^*](\phi \rightarrow [\pi]\phi) \rightarrow (\phi \rightarrow [\pi^*]\phi)$

d) $\langle s, i, g\rangle\phi \rightarrow \langle b_1 \cup \ldots \cup b_n\rangle\langle(b_1 \cup \ldots \cup b_n)^*\rangle\phi$*, where* $\Pi = \{b_1, \ldots, b_n\}$

e) $\langle\sigma, i, \gamma\rangle(\phi \rightarrow \psi) \rightarrow (\langle\sigma, i, \gamma\rangle\phi \rightarrow \langle\sigma, i, \gamma\rangle\psi)$

f) if $\vdash \phi \rightarrow \psi$ *then* $\vdash \langle\sigma, i, \gamma\rangle\phi \rightarrow \langle\sigma, i, \gamma\rangle\psi$

g) reduction axioms:

$\langle\sigma \cup \tau, E, \alpha \cup \beta\rangle\phi \leftrightarrow \langle\sigma, E, \alpha\rangle\phi \vee \langle\tau, E, \beta\rangle\phi$

$\langle\sigma \cup \tau, A, \alpha \cup \beta\rangle\phi \leftrightarrow \langle\sigma, A, \alpha\rangle\phi \wedge \langle\tau, A, \beta\rangle\phi$

$\langle\sigma, E, \gamma^d\rangle\phi \leftrightarrow \langle\sigma, A, \gamma\rangle\phi$

$\langle\sigma, A, \gamma^d\rangle\phi \leftrightarrow \langle\sigma, E, \gamma\rangle\phi$

$\langle\tau;\eta, i, \alpha;\beta\rangle\phi \leftrightarrow \langle\tau, i, \alpha\rangle\langle\eta, i, \beta\rangle\phi$

$\langle\sigma, E, \delta?\rangle\phi \leftrightarrow (\delta \wedge \phi)$

$\langle\sigma, A, \delta?\rangle\phi \leftrightarrow (\neg\delta \wedge \phi)$

h) strategy rules:

for each $X \subseteq \Pi$*, the rule below:*

if $\vdash \phi \rightarrow \langle(\cup X)\rangle\langle(\cup X)^*\rangle\psi$ *then* $\vdash \phi \rightarrow \langle s, i, g\rangle\psi$.

Proof. Soundness of some of the interesting reduction axioms and the strategy rules for the game-strategy modality are shown below. The readers can easily verify the validity of the rest.

1. $\langle\sigma \cup \tau, E, \alpha \cup \beta\rangle\phi \leftrightarrow \langle\sigma, E, \alpha\rangle\phi \vee \langle\tau, E, \beta\rangle\phi$

Suppose $\mathcal{M}, s \models \langle\sigma \cup \tau, E, \alpha \cup \beta\rangle\phi$. Then, for all $s' \in Ran(\mathcal{F}_E^{\alpha\cup\beta}(\sigma \cup \tau)) \cap P_{\alpha\cup\beta}^{-1}(end)$ in Game$(\mathcal{M}, s, \alpha \cup \beta)$, $\mathcal{M}, s' \models \phi$. Now, $\mathcal{F}_E^{\alpha\cup\beta}(\sigma \cup \tau) = \mathbf{L} \downarrow_{W_{\alpha\cup\beta}} \cup\mathcal{R}_E^\alpha$ or, $\mathbf{R} \downarrow_{W_{\alpha\cup\beta}} \cup\mathcal{R}_E^\beta$. W.l.o.g. suppose that $\mathcal{F}_E^{\alpha\cup\beta}(\sigma \cup \tau) = \mathbf{L} \downarrow_{W_{\alpha\cup\beta}} \cup\mathcal{R}_E^\alpha$. Then, for all $s' \in Ran(\mathbf{L} \downarrow_{W_{\alpha\cup\beta}} \cup\mathcal{R}_E^\alpha) \cap P_{\alpha\cup\beta}^{-1}(end)$ in Game$(\mathcal{M}, s, \alpha \cup \beta)$, $\mathcal{M}, s' \models \phi$. By definition of strategies in $\cup$ games, this implies that, for all $s' \in Ran(\mathcal{R}_E^\alpha) \cap P_\alpha^{-1}(end)$ in Game$(\mathcal{M}, s, \alpha)$, $\mathcal{M}, s' \models \phi$. Hence, for all $s' \in Ran(\mathcal{F}_E^\alpha(\sigma)) \cap P_\alpha^{-1}(end)$ in Game$(\mathcal{M}, s, \alpha)$, $\mathcal{M}, s' \models \phi$.So, we have that, $\mathcal{M}, s \models \langle\sigma, E, \alpha\rangle\phi$. Similarly, if $\mathcal{F}_E^{\alpha\cup\beta}(\sigma \cup \tau) = \mathbf{R} \downarrow_{W_{\alpha\cup\beta}} \cup\mathcal{R}_E^\beta$, one can show that, $\mathcal{M}, s \models \langle\tau, E, \alpha\rangle\phi$. So, $\mathcal{M}, s \models \langle\sigma, E, \alpha\rangle\phi$ or $\mathcal{M}, s \models \langle\tau, E, \alpha\rangle\phi$. Hence, $\mathcal{M}, s \models \langle\sigma, E, \alpha\rangle\phi \vee \langle\tau, E, \beta\rangle\phi$.

For the converse, suppose that $\mathcal{M}, s \models \langle\sigma, E, \alpha\rangle\phi$. Then, for all $s' \in Ran(\mathcal{F}_E^\alpha(\sigma)) \cap P_\alpha^{-1}(end)$ in Game$(\mathcal{M}, s, \alpha)$, $\mathcal{M}, s' \models \phi$. So, for all $s' \in Ran(\mathcal{R}_E^\alpha) \cap P_\alpha^{-1}(end)$ in Game$(\mathcal{M}, s, \alpha)$, $\mathcal{M}, s' \models \phi$, which implies that, for all $s' \in Ran(\mathbf{L} \downarrow_{W_{\alpha\cup\beta}} \cup\mathcal{R}_E^\alpha) \cap P_{\alpha\cup\beta}^{-1}(end)$ in Game$(\mathcal{M}, s, \alpha \cup \beta)$, $\mathcal{M}, s' \models \phi$. Hence, reasoning as earlier we have that, $\mathcal{M}, s \models \langle\sigma \cup \tau, E, \alpha \cup \beta\rangle\phi$. The proof for the other disjunct can be dealt with in a similar manner.

2. $\langle\sigma \cup \tau, A, \alpha \cup \beta\rangle\phi \leftrightarrow \langle\sigma, A, \alpha\rangle\phi \wedge \langle\tau, A, \beta\rangle\phi$

Suppose $\mathcal{M}, s \models \langle\sigma \cup \tau, A, \alpha \cup \beta\rangle\phi$. Then, for all $s' \in Ran(\mathcal{F}_A^{\alpha\cup\beta}(\sigma \cup \tau)) \cap P_{\alpha\cup\beta}^{-1}(end)$ in Game$(\mathcal{M}, s, \alpha \cup \beta)$, $\mathcal{M}, s' \models \phi$. Now, $\mathcal{F}_A^{\alpha\cup\beta}(\sigma \cup \tau) = \mathcal{R}_A^{\alpha\cup\beta}$, i.e. $\mathbf{L} \downarrow_{\{s,s\}} \cup\mathbf{R} \downarrow_{\{s,s\}} \cup\mathcal{R}_A^\gamma \cup \mathcal{R}_A^{\gamma'}$. It follows that for all $s' \in Ran(\mathcal{R}_A^\alpha) \cap P_\alpha^{-1}(end)$ in Game$(\mathcal{M}, s, \alpha)$, $\mathcal{M}, s' \models \phi$, and for all $s' \in Ran(\mathcal{R}_A^\beta) \cap P_\beta^{-1}(end)$ in Game$(\mathcal{M}, s, \beta)$, $\mathcal{M}, s' \models \phi$. Then, from the definitions it follows that $\mathcal{M}, s \models \langle\sigma, A, \alpha\rangle\phi \wedge \langle\tau, A, \beta\rangle\phi$. The converse can be proved by retracing the steps backwards.

3. $\langle\sigma, E, \gamma^d\rangle\phi \leftrightarrow \langle\sigma, A, \gamma\rangle\phi$

Suppose $\mathcal{M}, s \models \langle\sigma, E, \gamma^d\rangle\phi$. Then, for all $s' \in Ran(\mathcal{F}_E^{\gamma^d}(\sigma)) \cap P_{\gamma^d}^{-1}(end)$ in Game$(\mathcal{M}, s, \gamma^d)$, $\mathcal{M}, s' \models \phi$. By definition of strategies in the dual game, this implies that, for all $s' \in Ran(\mathcal{F}_A^\gamma(\sigma)) \cap P_\gamma^{-1}(end)$ in Game$(\mathcal{M}, s, \gamma)$, $\mathcal{M}, s' \models \phi$ and so, $\mathcal{M}, s \models \langle\sigma, A, \gamma\rangle\phi$. For the converse proof, retrace back.

4. $\langle\tau;\eta, i, \alpha;\beta\rangle\phi \leftrightarrow \langle\tau, i, \alpha\rangle\langle\eta, i, \beta\rangle\phi$

Suppose $\mathcal{M}, s \models \langle\tau;\eta, i, \alpha;\beta\rangle\phi$. Then, for all

$s' \in Ran(\mathcal{F}_i^{\alpha;\beta}(\sigma;\tau)) \cap P_{\alpha;\beta}^{-1}(end)$ in Game$(\mathcal{M}, s, \alpha;\beta)$, $\mathcal{M}, s' \models \phi$. Hence, for all $s' \in Ran(\mathcal{R}_i^{\alpha} \cup \mathcal{R}_{i_{j_1}}^{\beta} \cup \ldots \cup \mathcal{R}_{i_{j_l}}^{\beta}) \cap P_{\alpha;\beta}^{-1}(end)$ in Game$(\mathcal{M}, s, \alpha;\beta)$, $\mathcal{M}, s' \models \phi$. Then, for k = 1, …, l, for all $t \in Ran(\mathcal{R}_{i_{j_k}}^{\beta}) \cap P_{\beta_{j_k}}^{-1}(end)$ in Game$(\mathcal{M}, t_{j_k}, \beta)$, $\mathcal{M}, t \models \phi$, and hence for all $t' \in Ran(\mathcal{R}_i^{\alpha}) \cap P_{\alpha}^{-1}(end)$, in Game$(\mathcal{M}, s, \alpha)$, $\mathcal{M}, t' \models \langle \eta, i, \beta \rangle \phi$. So, $\mathcal{M}, s \models \langle \tau, i, \alpha \rangle \langle \eta, i, \beta \rangle \phi$.

For the converse part, suppose $\mathcal{M}, s \models \langle \tau, i, \alpha \rangle \langle \eta, i, \beta \rangle \phi$. Then, for all $t' \in Ran(\mathcal{R}_i^{\alpha}) \cap P_{\alpha}^{-1}(end)$, in Game$(\mathcal{M}, s, \alpha)$, $\mathcal{M}, t' \models \langle \eta, i, \beta \rangle \phi$, where $\mathcal{F}_i^{\alpha}(\tau) = \mathcal{R}_i^{\alpha}$. This can be possible, only when, for each $t \in P_{\gamma}^{-1}(end)$, Game($\mathcal{M}$, t, γ') can be defined. Hence, Game$(\mathcal{M}, s, \alpha;\beta)$ is defined, and for all $s' \in Ran(\mathcal{R}_i^{\alpha} \cup \mathcal{R}_{i_{j_1}}^{\beta} \cup \ldots \cup \mathcal{R}_{i_{j_l}}^{\beta}) \cap P_{\alpha;\beta}^{-1}(end)$ in Game$(\mathcal{M}, s, \alpha;\beta)$, $\mathcal{M}, s' \models \phi$. So, $\mathcal{M}, s \models \langle \tau;\eta, i, \alpha;\beta \rangle \phi$.

The validity of the strategy rules follows from the fact that, if there is a path from some state s to a state satisfying some formula ϕ, then the Game$(\mathcal{M}, s, g)$ and a corresponding strategy relation $\mathcal{R}_i^g$ can be defined in such a way that $\langle s, i, g \rangle \phi$ holds at s.

The completeness of the axiom system is proved by showing that every consistent formula is satisfiable. Let α be a consistent formula. Let $Cl(\alpha)$ denote the subformula closure of α, satisfying the FL-closure conditions for the action modalities with the following extra conditions:

(i) If $\langle \sigma \cup \tau, E, \alpha \cup \beta \rangle \phi \in Cl(\alpha)$, then $\langle \sigma, E, \alpha \rangle \phi \vee \langle \tau, E, \beta \rangle \phi \in Cl(\alpha)$.
(ii) If $\langle \sigma \cup \tau, A, \alpha \cup \beta \rangle \phi \in Cl(\alpha)$, then $\langle \sigma, A, \alpha \rangle \phi \wedge \langle \tau, A, \beta \rangle \phi \in Cl(\alpha)$.
(iii) If $\langle \tau;\eta, i, \alpha;\beta \rangle \phi \in Cl(\alpha)$, then $\langle \tau, i, \alpha \rangle \langle \eta, i, \beta \rangle \phi \in Cl(\alpha)$.
(iv) $Cl(\alpha)$ is closed under single negations.

Any maximal consistent subset of $Cl(\alpha)$ is said to be an atom. Let $\mathcal{A}$ denote the set of all such atoms. For $T \in \mathcal{A}$, let $\widehat{T}$ denote the conjunction of all the formulas present in T. For $C, D \in \mathcal{A}$, define $C\mathcal{R}_{\pi}D$ if $\widehat{C} \wedge \langle \pi \rangle \widehat{D}$ is consistent. The *regular* canonical model $\mathcal{C}$ is defined to be the tuple $\langle \mathcal{A}, \{\mathcal{R}_{\pi}$: π's are actions$\}, ref, \mathcal{L}, \mathcal{R}, \mathcal{V} \rangle$, where, *ref*, $\mathcal{L}, \mathcal{R}$ are reflexive relations on $\mathcal{A}$, and $\mathcal{V}(p) = \{T \in \mathcal{A} : p \in T\}$, and $\mathcal{R}_{\pi}$'s satisfy the *regularity* conditions. The existence lemma for the modalities $\langle \pi \rangle$, can be proved in the usual way, and we have that $\mathcal{C}, A \models \phi$ iff $\phi \in A$, for each $\phi \in Cl(\alpha)$, and each $A \in \mathcal{C}$ where ϕ is either an atomic or a boolean or an action modal formula.

It remains to be shown that $\mathcal{C}, A \models \langle \sigma, i, \gamma \rangle \phi$ iff $\langle \sigma, i, \gamma \rangle \phi \in A$. Because of the reduction axioms, it suffices to show that for each $\langle s, i, g \rangle \phi \in Cl(\alpha)$, and each $A \in \mathcal{C}$, $\mathcal{C}, A \models \langle s, i, g \rangle \phi$ iff $\langle s, i, g \rangle \phi \in A$. In other words, we have to show that $\langle s, i, g \rangle \phi \in A$ iff in Game$(\mathcal{C}, A, g)$, $Ran(\mathcal{F}_i^g(s)) \cap P^{-1}(end)$ is the set of all atoms T, such that $\phi \in T$.

Suppose $\langle s, i, g \rangle \phi \in A$. Then because of axiom (d), Game$(\mathcal{C}, A, g)$, and $\mathcal{F}_i^g$ can be defined in such a way, that the implication holds. The converse follows from the fact that if $< b_{i_1} > \ldots < b_{i_m} > \phi$ is consistent, then so is $< b_{i_1} > \ldots < b_{i_m} > \phi \wedge \langle s, i, g \rangle \phi$, which holds because of the strategy rules. QED

4. DGL and $SDGL$ - a comparison

As mentioned earlier, DGL talks about *generic* games played on game boards, and the meaning of the game modalities is given by existence of strategies. $SDGL$ brings out these strategies to the fore. Strategy combinations for playing composite games are talked about in this framework which brings out the extensional nature of strategies, though according to certain views, strategies are inherently intensional. As mentioned by van Benthem [4, 5], strategies of the players in the game tree can be talked about using the program constructs of the dynamic modal logic. Some proposals for combining strategies to achieve a certain goal are also made there.

The task was to combine the strategy constructs together with the game constructs. $SDGL$ proposes a way to do it. As evident from the previous section, one has to resort to the PDL-style action constructs. To make strategies explicit, one can no longer talk about *generic* games. Extensive game trees come into the treatise - games are defined as tree structures, and strategies are defined as subtrees.

In the tradition of DGL semantics, the so-called *forcing* relations satisfy the conditions of upward-monotonicity and consistency (determinacy also, in case of Parikh's and Pauly's DGL). The sets of states forced by these relations have an inherent 'disjunctive' interpretation. A 'conjunctive' interpretation of these sets which is needed when *parallel* game constructs are introduced, has been taken in [10]. It is interesting to note that, the way *strategies* are defined as relations between states, it corroborates with the 'conjunctive' interpretation of the set of 'end'-states reached. Hence, this language rather suggests 'downward monotonicity' at this conjunctive level.

It is clear that there are some sentences which could be expressed in $SDGL$, but not in DGL. But it is also the case that there are certain statements that can be expressed in DGL, but not in $SDGL$: for example, 'player i does not have any strategy in the game g to achieve ϕ' can be expressed in DGL as $\neg\langle g, i\rangle\phi$. Under these circumstances it would be ideal to have a logic that could express both. This gives rise to the following issue:

Question What would be the complete axiomatization of a logic that has both Parikh's original game modalities as well as the game-strategy modalities presented in the earlier section of this paper?

In fact, for the set of strategy relations $\mathbb{R}_i^\gamma$ for player i in Game($\mathcal{M}, s, \gamma$), one can easily define ρ_γ^i (cf.§2), as follows:

$s\rho_\gamma^i X$ iff $X = Ran(\mathcal{R}_i^\gamma) \cap P_\gamma^{-1}(end)$, for some $\mathcal{R}_i^\gamma \in \mathbb{R}_i^\gamma$.

It remains to be seen what conditions have to be imposed on ρ_γ^i to maintain compatibility. This is precisely the same issue as finding joint logics of proofs and provability in arithmetic, on which a lot of effort has been made in the recent past. For a detailed overview, one can have a look at [2]. The most natural analogy that one can think of having both such existential criterion, as well as the witnesses conforming to it could be found in first order logic - $\exists x\phi$ together with term substitutions like $\phi[\sigma/x]$.

5. Conclusions and intentions

This paper proposes a logic which makes strategies explicit in the dynamic game logic framework. The need for the dynamic modal logic syntax for achieving such targets becomes apparent. An interesting issue of getting a joint logic of complex game modalities together with game-strategy modalities emerges. Some possible areas for future investigations are given below.

Explicit strategies for other logics Several other languages talking about game structures and coalition structures like Alternating-time temporal logic and Coalition logic could be investigated so as to add an explicit notion of strategies, which merely occur as an existential notion in the semantics of these logics. This could very well aid in the social choice mechanism designs.

Adding knowledge and preference notions To come closer to the real game scenario which are played by the rational players, one has to incorporate the knowledge/belief as well as preference modalities in the existing framework, i.e. epistemic versions of these game logics with explicit strategies need to be explored.

Games with imperfect information It is evident that the uniform strategies in the imperfect information games do not conform with the compositional analysis that has been done here. That study is inherently different taking into account the knowledge level of the players, which provides a very interesting challenge.

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