Reasoning with Probabilities

Eric Pacuit Joshua Sack

July 29, 2009

Plan for the Course

✓ Introduction and Background

✓ Probabilistic Epistemic Logics

Day 3: Dynamic Probabilistic Epistemic Logics

Day 4: Reasoning with Probabilities

Day 5: Conclusions and General Issues

Plan for Today

- Background: Dynamic Epistemic Logic
- Dynamic Epistemic Probabilistic Logic
- Dynamics with measure spaces

Question: How should we update our probabilistic epistemic models in the presence of new information?

$P_i(\phi \mid A)$: "the probability of ϕ given (true) information A."

Background: DEL

Let Φ be set of proposition letters and Ag a set of agents. An epistemic model is a tuple $M = (W, \sim, \|\cdot\|)$, where

- W is a set of possible worlds
- $\sim$ is a collection of relations $\sim_i \subseteq W \times W$ for each $i \in Agt$.
- $\|\cdot\| : \Phi \to \mathcal{P}(W)$.

- $M, w \models p$ iff $w \in \|p\|$
- $M, w \models \neg\phi$ iff $M, w \not\models \phi$
- $M, w \models \phi \wedge \psi$ iff $M, w \models \phi$ and $M, w \models \psi$
- $M, w \models K_i\phi$ iff for all $v \in W$ if $w \sim_i v$ then $M, v \models \phi$

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Background: DEL

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$$\mathcal{M} \otimes E$$

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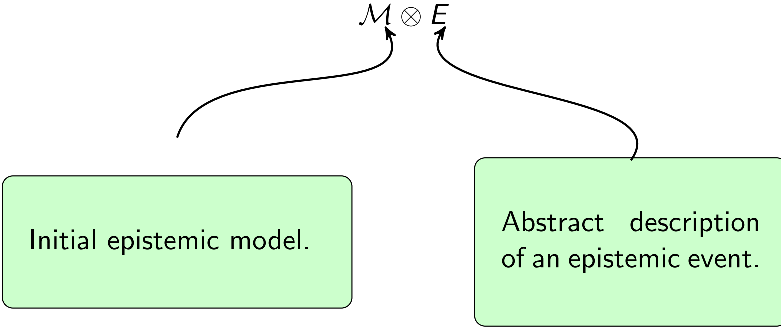

Abstract Description of the Event

Ann looks at the card while Bob is looking away.

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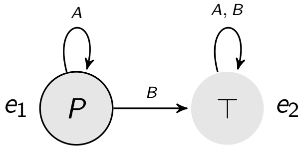

Product Update

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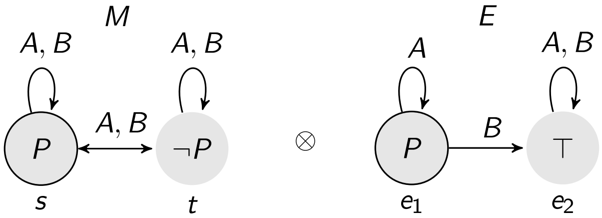

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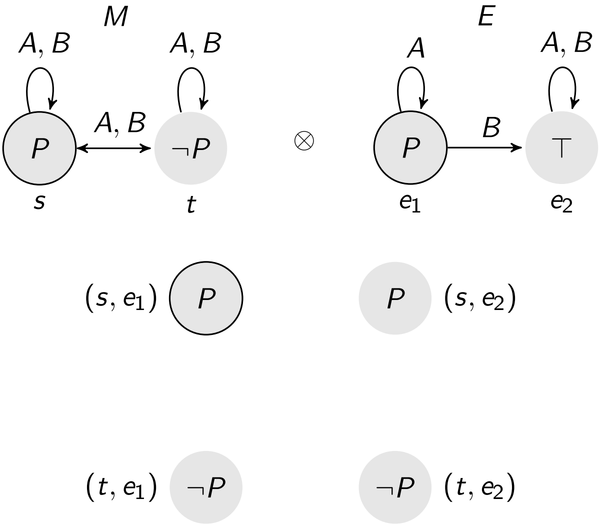

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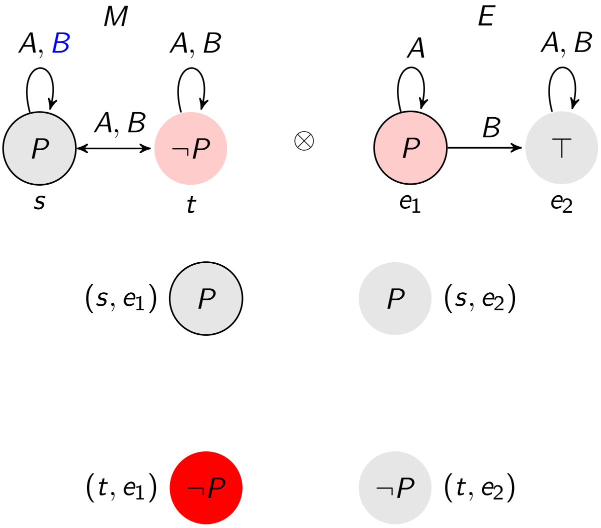

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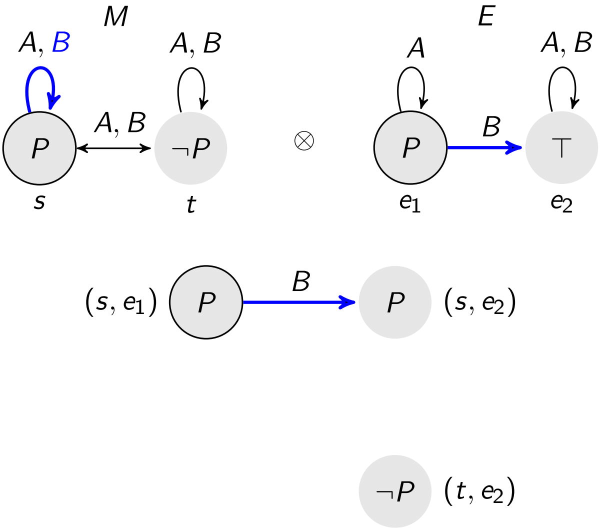

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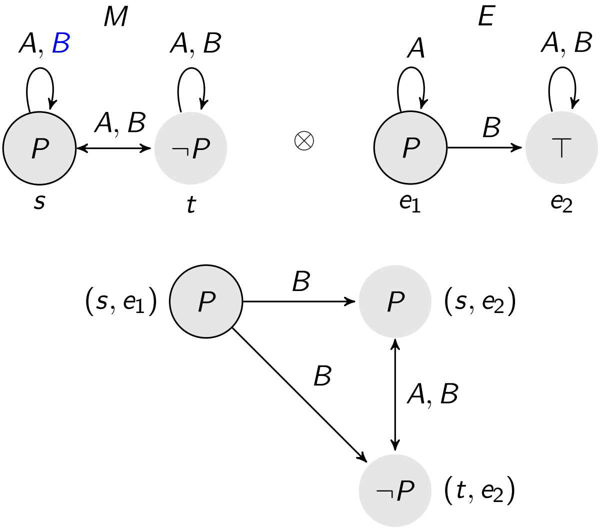

Product Update Details

Let $\mathbb{M} = \langle W, R, V \rangle$ be a Kripke model.

An event model is a tuple $\mathbb{A} = \langle A, S, Pre \rangle$, where $S \subseteq A \times A$ and $Pre : \mathcal{L} \to \mathcal{P}(A)$.

The update model $\mathbb{M} \otimes \mathbb{A} = \langle W', R', V' \rangle$ where

- $W' = \{(w, a) \mid w \models Pre(a)\}$
- $(w, a)R'(w', a')$ iff wRw' and aSa'
- $(w, a) \in V(p)$ iff $w \in V(p)$

$\mathcal{M}, w \models [A, a]\phi$ iff $\mathcal{M}, w \models Pre(a)$ implies $\mathcal{M} \otimes A, (w, a) \models \phi$.

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Literarture

A. Baltag and L. Moss. *Logics for Epistemic Programs*. 2004.

W. van der Hoek, H. van Ditmarsch and B. Kooi. *Dynamic Episetmic Logic*. 2007.

Example: Public Announcement Logic

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\begin{array}{rcl}
[\psi]p & \leftrightarrow & (\psi \rightarrow p) \\
[\psi]\neg\phi & \leftrightarrow & (\psi \rightarrow \neg[\psi]\phi) \\
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Theorem Every formula of Public Announcement Logic is equivalent to a formula of Epistemic Logic.

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The situation is more complicated with common knowledge.

J. van Benthem, J. van Eijk, B. Kooi. *Logics of Communication and Change*. 2006.

J. van Benthem, J. Gerbrandy and B. Kooi. *Dynamic update with probabilities*. Manuscript (2009).

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1. W is a *finite* set of possible worlds
2. $\sim$ is a collection of relations $\sim_i \subseteq W \times W$ for each $i \in Agt$.
3. $P : Ag \to (W \to (W \to [0,1]))$ assigns a probability funciton over W to each agent $i \in Ag$ and each state $w \in W$. (write $P_i(s)(t)$ for the probability assigned to t by i at state s).
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Truth

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$M, w \models K_i\phi$ iff for all $v \in W$ if $w \sim_i v$ then $M, v \models \phi$

$M, w \models P_i(\phi) = k$ iff $\sum_{t \,:\, M,t \models \phi} P_i(s)(t) \;=\; k$

Conditions

If $P_i(s)(t) > 0$ then $P_i(s) = P_i(t)$

$P_i(s)$ assigns positive probabilities only to states that are in the $\sim_i$-equivalence class.

$$P_i(\phi) = k \rightarrow K_i(P_i(\phi) = k)$$

Monty Hall Puzzle

Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car, behind the others, goats. You pick a door, say number 1, and the host, who knows what's behind the doors, opens another door, say number 3, which has a goat. He says to you, "Do you want to pick door number 2?" Is it to your advantage to switch your choice of doors?

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2. *Occurrence probabilities for events*: from the update model A, representing the agents' views on what sort of process produces new information

3. *Observational probability*: reflecting the agents' uncertainty as to which event is currently being observed.

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Probabilistic Update Models

$A = (E, \sim, \Phi, pre, P)$ where:

- E is a non-empty finite set of events
- $\sim$ is a set of equivalence relations $\sim_i$ on E for each $i \in Ag$
- Φ is a set of pairwise inconsistent sentences called **preconditions**
- *pre* assigns to each preconditions $\phi \in \Phi$ a probability distribution over E (write $pre(\phi, e)$)
- For each i, P_i assigns to each event e a probability distribution over E.

Probabilistic product update rule

$M \otimes A = (S', \sim', P', \|\cdot\|')$

- $S' = \{(s,e) \mid s \in S, e \in E \text{ and } pre(s,e) > 0\}$
  ($pre(s,e) = pre(\phi, e)$ where $\phi \in \Phi$ is the element of Φ true at s (if none exists, set $pre(s,e) = 0$).
- $(s,e) \sim'_i (s',e')$ iff $s \sim_i s'$ and $e \sim_i e'$
- $P'_i((s,e),(s',e')) :=$

$$\frac{P_i(s)(s') \cdot pre(s',e') \cdot P_i(e)(e')}{\sum_{s'' \in S, e'' \in E} P_i(s)(s'') \cdot pre(s'',e'') \cdot P_i(e)(e'')}$$

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1. prior probability
2. occurrence probability
3. observational probability

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1. prior probability
2. occurrence probability
3. observational probability
4. Normalize

$$P_i'((s,e),(s',e')) \;:=\; \frac{P_i(s)(s') \cdot pre(s',e') \cdot P_i(e)(e')}{\sum_{s''\in S, e''\in E} P_i(s)(s'') \cdot pre(s'',e'') \cdot P_i(e)(e'')}$$

Example

Suppose you are reading about some horrible disease on a website, and start wondering whether you have it. The chances of having the disease are very slight, 1 in 100,000. The website states that one of the symptoms of this disease is that a certain gland is swollen. If you have the disease the chance that this gland is swollen is 97%, while if you do not have the disease, the chance is 0 that it is swollen. You immediately examine the gland. The problem is that it is hard to determine if it is swollen or not. It is the first time you actually examine the gland and — not being a physician — you do not know what its size ought to be. You are uncertain, but you think the chances are 50% that the gland is swollen. What chances should you assign to having the disease?

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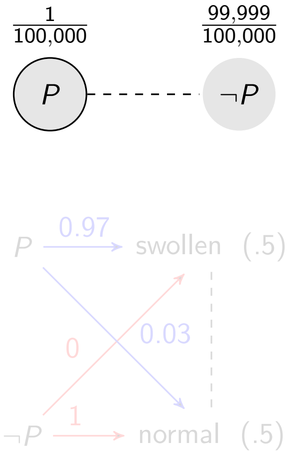
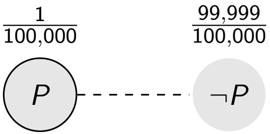

$\frac{1}{100{,}000}$
$\frac{99{,}999}{100{,}000}$
P
$\neg P$
P
0.97
swollen (.5)
0.03
0
1
$\neg P$
normal (.5)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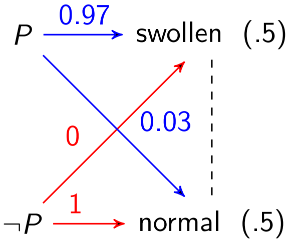
P
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swollen (.5)
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Theorem There are reduction axioms for the dynamic probabilistic update rule (using the Halpern et al.)

J. van Benthem, J. Gerbrandy and B. Kooi. *Dynamic update with probabilities*. Manuscript (2009).

Involving updates with σ-algebras

A first step to involving σ-algabras in dynamic epistemic probabilistic logic: involve

- Prior probabilities
- observation probabilities
- not occurrence probabilities
- non-trivial σ-algebras (σ-algebras that are not the powerset of the sample space).

Probabilistic update model

Let Φ be a set of proposition letters and Agt a set of agents.
Probabilistic update model: $U = (E, R, \text{pre}, \mathbf{P})$, where

- $(E, R, \mathbf{P})$ is a finite probabilistic epistemic model
- pre is a function mapping E to a function from probabilistic epistemic models to subsets of their carrier sets.

Update Products

Updating occurs in two stages

- Unrestricted product: cartesian product of states and standard measure product of probability spaces
- Relativization: The set of states in the relativized model are determined by the precondition function.

Unrestricted product

Definition (Unrestricted Product)

The unrestricted product between a probabilistic epistemic model M and an update model U is $M \otimes_N U$ with the following components:

1. $X^{\otimes} = X \times E$
2. $R^{\otimes}$ is a collection of relations $R_i^{\otimes}$, such that $(x, e)R_i^{\otimes}(z, f)$ iff $xR_i^M z$ and $eR_i^U f$
3. $\|p\|^{\otimes} = \|p\| \times E$
4. $\mathbf{P}^{\otimes}$ consists of a collection of triples $(S_{i,(x,e)}, \mathcal{A}_{i,(x,e)}, \mu_{i,(x,e)})$
   - $(S_{i,(x,e)}, \mathcal{A}_{i,(x,e)})$ is the product measurable space between $(S_{i,x}, \mathcal{A}_{i,x})$ and $(S_{i,e}, \mathcal{A}_{i,e})$.
   - $\mu_{i,(x,e)}$ is the product measure of $\mu_{i,x}$ and $\mu_{i,e}$.

Relativizing probability space

Given a probability space $P = (S, \mathcal{A}, \mu)$, P relativized to Y is $(S^Y, \mathcal{A}^Y, \mu^Y)$:

- if $\mu(Y) = 0$, then let $S^Y = Y$, $\mathcal{A}^Y = \{\emptyset, Y\}$, and μ be the only probability measure defined on $\mathcal{A}^Y$
- if $\mu(Y) \neq 0$, then let $\widehat{\mu} : \mathcal{P}(Y) \to [0, 1]$ be the outer measure defined by

$$\widehat{\mu_Y}(B) = \frac{\mu^*(B)}{\mu^*(Y)}$$

  for each $B \subseteq Y$. Then let
  1. $S^Y = S \cap Y$
  2. $\mathcal{A}^Y = \mathcal{A}(\widehat{\mu^Y}) \cap \{A \cap Y : A \in \mathcal{A}\}$, (where $\mathcal{A}(\widehat{\mu^Y})$ is the set of $\widehat{\mu^Y}$-measurable sets).
  3. μ^Y is the restriction of $\widehat{\mu^Y}$ to $\mathcal{A}^Y$

Relativization

Definition (Relativization)

The relativization of a probabilistic epistemic model M to $Y \subseteq X$ is given by $M \otimes_R Y$ with the following components:

1. $X^Y = Y$
2. R^Y is a collection of relations R_i^Y, such that $xR_i^Y z$ iff $xR_i^M z$ and $x, z \in Y$
3. $\|p\|^Y = \|p\|^M \cap Y$
4. $\mathbf{P}^Y$ is the collection $(S_{i,x}^Y, \mathcal{A}_{i,x}^Y, \mu_{i,x}^Y)$ of relativized probability spaces.

Formal definition of update product

Definition (Update Product)

Let

- $U = (E, R, \mathbf{P}, \text{pre})$ be an update model
- $M = (X, R, \|\cdot\|, \mathbf{P})$ be a probabilistic model.
- $Y = \{(x, e) : x \in \text{pre}(e)(M)\}$.

The update product between M and U is written $M \otimes U$ and is defined as $(M \otimes_N U) \otimes_R Y$.

This is the approach taken in

- J. Sack (2008) Extending probabilistic dynamic epistemic logic, *Synthese: Knowledge, Rationality and Action*, 169, pp. 241–257.

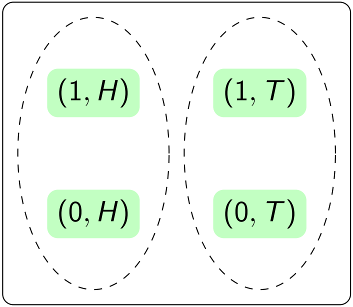
Fagin, Halpern, and Tuttle example (repeated)

Recall the following example:

Suppose there are two agents i and k.

1. k is first given a bit 0 or 1. k learns he has this bit, i is aware that k received a bit, but i does not know what bit k received.
2. k flips a fair coin and looks at the result. i sees k look at the result, but does not what the result is.
3. k performs action s if the coin agrees with the bit (given that heads agrees with 1 and tails agrees with 0), and performs action d otherwise.

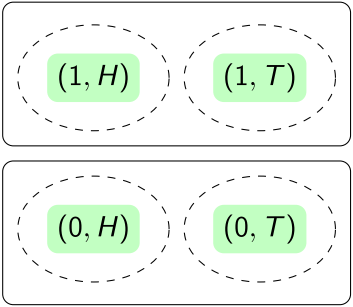
Here is a possibility for i's probability spaces. The sample space enclosed in a box, and the σ-algebra equivalence classes are enclosed in the dotted ovals.



$\mathbf{M}_1$

The sample space is the same as the set of states i considers possible. Individual states cannot be measurable (otherwise 0 or 1 must be assigned a probability).

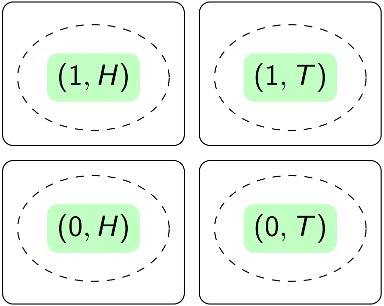
Another possibility has a sample space containing only the states with the correct bit (but recall that i considers all states possible and both sample spaces possible).



$\mathbf{M}_2$

Without assigning probability to the bit, i can now assign a probability to the actions s and d.

Here i is uncertain among 4 probability spaces.



$\mathbf{M}_3$

Modeling a sequence of events

It is suggested that each of these models may reasonably represent i's probability spaces at a certain stage in the sequence of events (but to make better sense of the transition, we add a little more in parentheses that was not in the original statement of the example):

- M_1 with the time before the bit is given to k (suppose i does not yet know that k will perform action s or d).
- M_2 with the time after the bit is given to k, (after k tells i he will do either s or d depending on the coin toss,) but before the coin is flipped.
- M_3 with the time after the coin is tossed, (after k spontaneously offers i a bet about what action he will take,) but before k performs his action.

What update models should be used?

From M_1 to M_2, there are two events:

1. a semi-private announcement of the bit to k
2. a public announcement that k plans to do either action s or d.

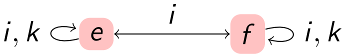
From M_2 to M_3, there are two events:

1. a semi-private announcement to k of the result of the coin toss
2. a public announcement regarding k's bet offer

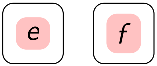
We first consider going from M_1 to M_2 using just one update model, and similarly from M_2 to M_3 with just one update model. We then consider going from M_1 to M_2 using a sequence of two update models, and similarly from M_2 to M_3.

Semi-private announcement

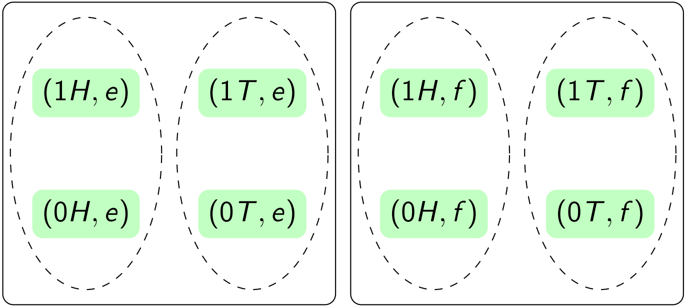
The relational structure of a semi-private announcement is given by



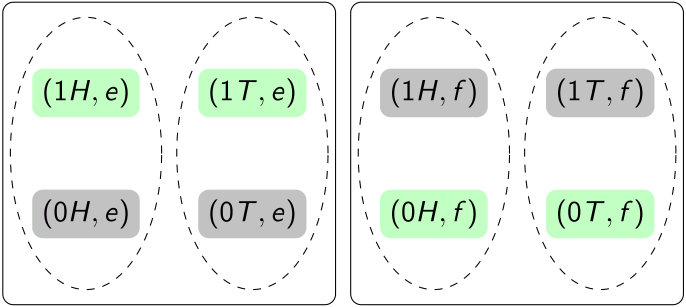
i and k's probability spaces:

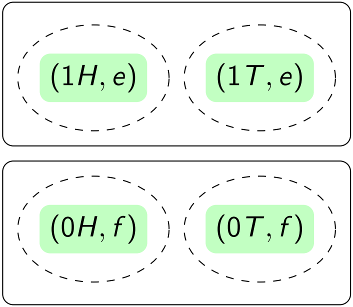


M_1 to M_2

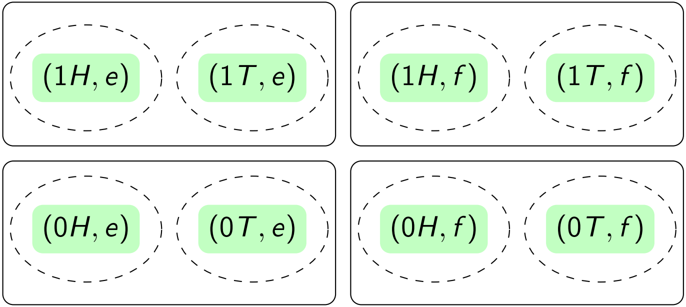


M_1 to M_2

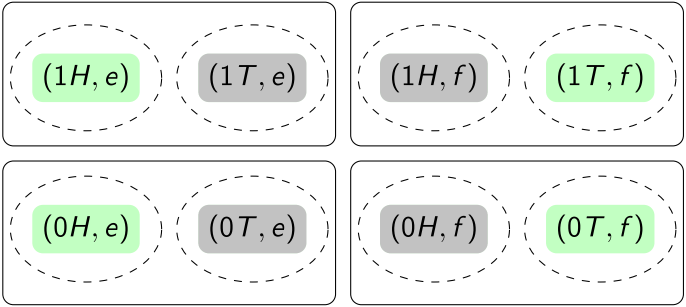


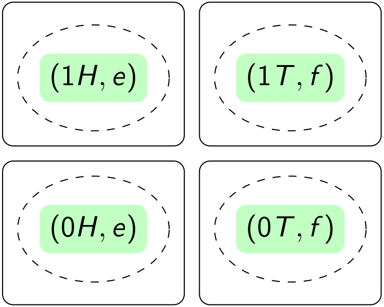

$(1H, e)$
$(1T, e)$
$(0H, f)$
$(0T, f)$

M_2 to M_3



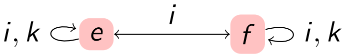
M_2 to M_3



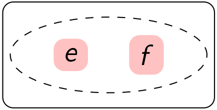

$(1H, e)$
$(1T, f)$
$(0H, e)$
$(0T, f)$

From M_1 to M_2 first stage: semi-private announcement

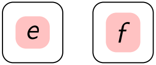
relational structure:



i's probability space:



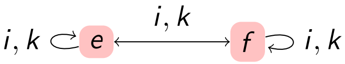
k's probability spaces:



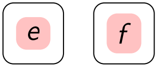
$pre(e)$ includes states with 1, and $pre(f)$ includes states with 0.

From M_1 to M_2 second stage: public announcement

relational structure:



This is the public announcement "the precondition of e or the precondition of f" as long as no state satisfies both preconditions.
i and k's probability spaces:



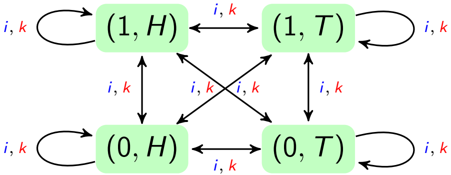
$pre(e)$ includes states with 1, and $pre(f)$ includes states with 0.

From M_2 to M_3

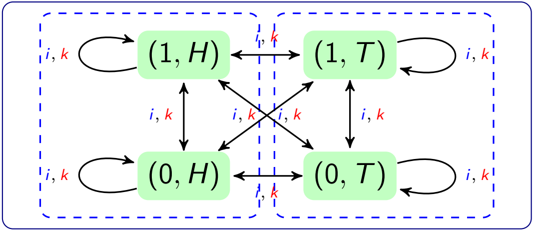
The semi-private and public announcement action models are the same in all components except for the precondition function *pre*.

- Instead of 1, the precondition of e is H
- Instead of 0, the precondition of f is T.

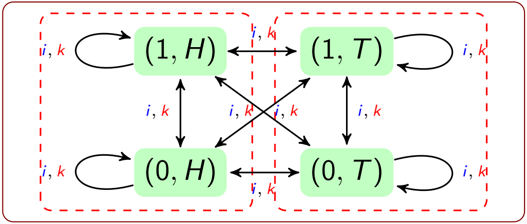
M_1



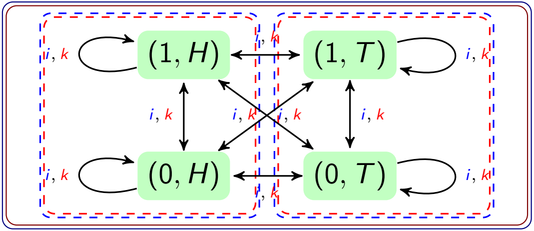
M_1



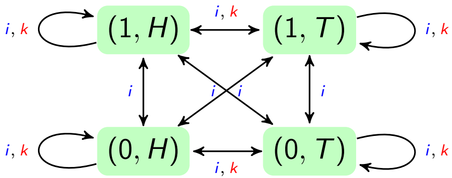
M_1



M_1

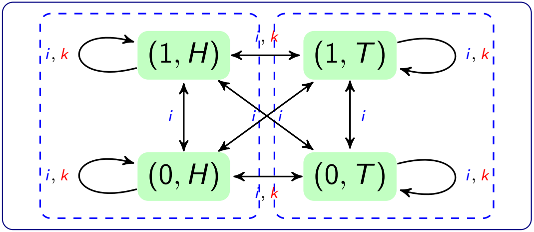


after 1st semi-private announcement



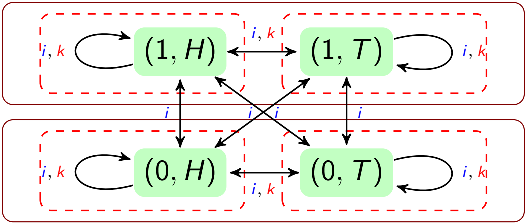
Bit is semi-privately announced to k

after 1st semi-private announcement



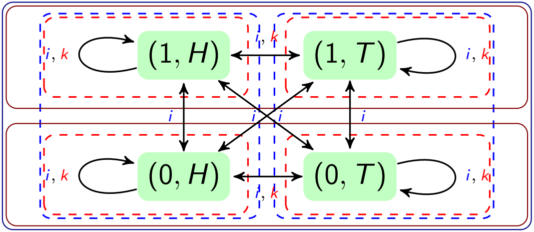
Bit is semi-privately announced to k

after 1st semi-private announcement



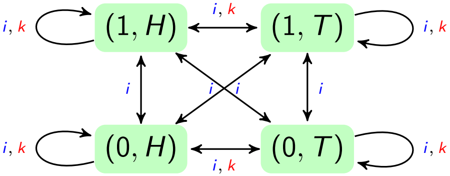
Bit is semi-privately announced to k

after 1st semi-private announcement



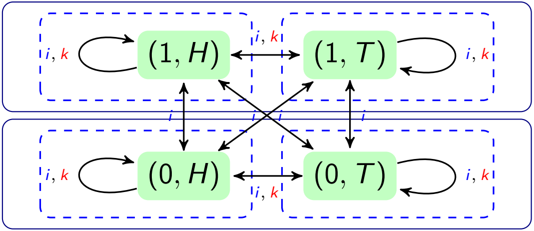
Bit is semi-privately announced to k

after first public announcement (M_2)



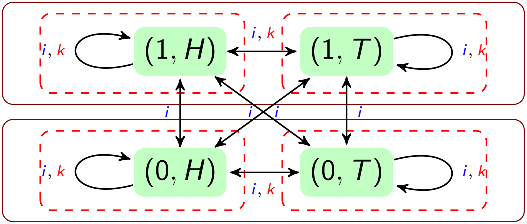
k's intention to do either s or d is announced.

after first public announcement (M_2)



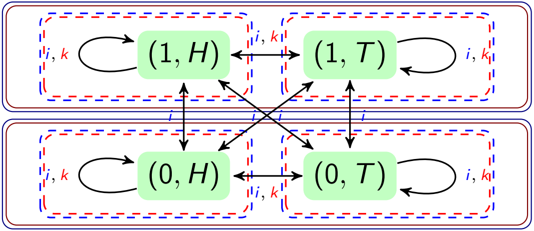
k's intention to do either s or d is announced.

after first public announcement (M_2)



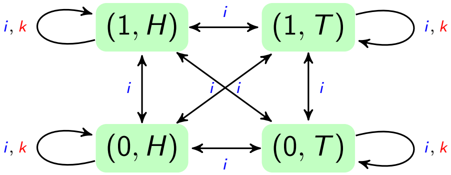
k's intention to do either s or d is announced.

after first public announcement (M_2)



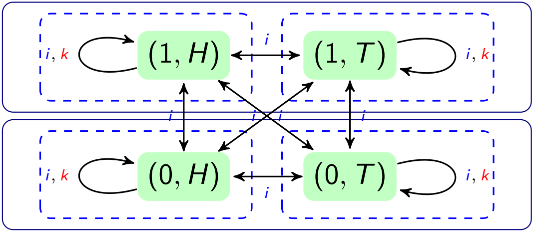
k's intention to do either s or d is announced.

after 2nd semi-private announcement



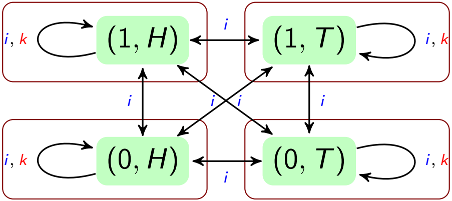
The result of the coin flip is semi-privately announced to k.

after 2nd semi-private announcement



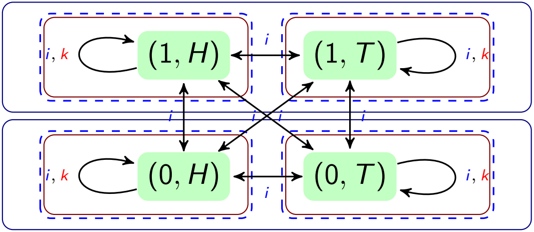
The result of the coin flip is semi-privately announced to k.

after 2nd semi-private announcement



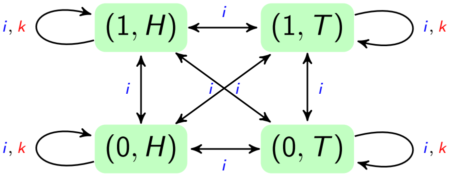
The result of the coin flip is semi-privately announced to k.

after 2nd semi-private announcement



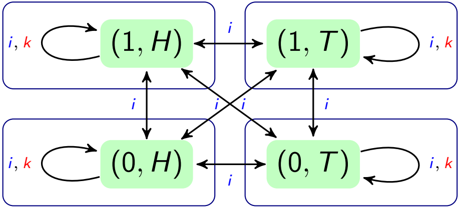
The result of the coin flip is semi-privately announced to k.

M_3



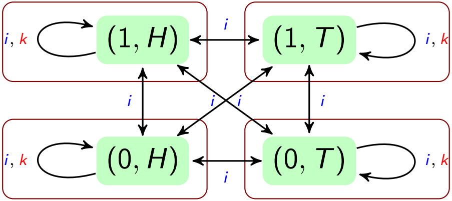
k's offer of a bet is announced.

M_3



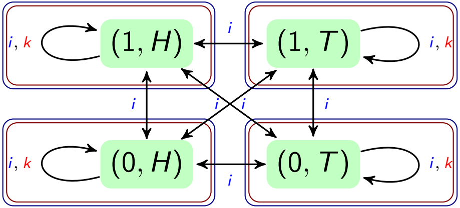
k's offer of a bet is announced.

M_3



k's offer of a bet is announced.

M_3



k's offer of a bet is announced.

Variations of update product

- The σ-algebra of the update product is capped by the σ-algebras of the prior probability space and the update frame's probability space. Would it be reasonable to define the sigma-algebra as the largest for which a probability can be defined?
- Can we guarantee that outer measures need not be involved in the updating process? This of course may depend on the specific language used.
- The case of updating by sets of measure 0 posses a technical hurdle. Such updates are not the focus of update logics (but are in belief revision), thus definitions are chosen to maximize technical convenience. Other variations may be considered.