Reasoning with Probabilities

July 31, 2009

Plan for the Course

✓ Introduction and Background

✓ Probabilistic Epistemic Logics

✓: Dynamic Probabilistic Epistemic Logics

✓: Reasoning with Probabilities

Day 5: Conclusions and General Issues

Compactness

A set of formulas is satisfiable if there is a model (or pointed model) for which every formula in the set is true.

Definition (Finite satisfiability)

A set of formulas is satisfiable if every finite subset of formulas is satisfiable

Definition (Compact logic)

A logic is compact if every finitely satisfiable set of formulas is satisfiable

Related to topological compactness of a set

- Topology: every open cover has a finite subcover
- Logic: every unsatisfiable set of formulas has an unsatisfiable finite subset

Non-compactness of probability logic

Probability logic is not compact:

- $\{\neg(P(\varphi) \geq r)\} \cup \{P(\varphi) \geq s \mid r > s\}$ is finitely satisfiable, but unsatisfiable due to the Archimedean property.

Let $(s_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{Q} \cap (-\infty, r)$ for which $\lim_{n \to \infty} s_n = r$.

$$\chi_1 = \neg(P(\varphi) \geq r)$$

$$\chi_{n+1} = \chi_n \wedge P(\varphi) \geq s_n$$

Then $\bigcap [\![\psi \vee \chi_n]\!] = [\![\psi]\!]$ and

- $\{\neg(P(\psi) \geq a)\} \cup \{P(\psi \vee \chi_n) \geq a \mid n \in \mathbb{N}\}$ is finitely satisfiable, but unsatisfiable due to countable additivity (and hence continuity from above).

Beliefs as Probabilities

Why should beliefs satisfy the Kolomogrov axioms?

Dutch book arguments.

Dutch Book and subjective probabilities

- Most of the probabilities in this course are subjective probabilities: probabilities agents assign to the likelihood of certain events.
- Subjective probabilities are often viewed in terms of an agent's willingness to bet.
- The Syncronic Dutch Book literature provides justification for the laws of probability using betting games.
- The Diachronic Dutch Book literature provides justification for Bayesian updating as a means for changing subjective probabilities using betting games.

Strategies for Synchronic Dutch Book

Let $(\Omega, \mathcal{A})$ be a finite measurable space. Consider three players α, β, and η

- α's strategy ("a system of beliefs") is a function $\mu : \mathcal{A} \to \mathbb{R}$
  View $\mu(A)$ as a price for a unit wager for event A
- β's strategy ("a system of bets") is a function $\nu : \mathcal{A} \to \mathbb{R}$
  View $\nu(A)$ as being the quantity β buys of unit wagers for event A
- η's strategy ("the actual outcome") is an $\omega \in \Omega$.

Payoffs

Fix a strategy profile (μ, ν, ω).

- α's payoff is

$$\sum_{\{A \in \mathcal{A} \mid \omega \notin A\}} \mu(A)\nu(A) + \sum_{\{A \in \mathcal{A} \mid \omega \in A\}} (\mu(A) - 1)\nu(A)$$

- β's payoff is

$$\sum_{\{A \in \mathcal{A} \mid \omega \notin A\}} -\mu(A)\nu(A) + \sum_{\{A \in \mathcal{A} \mid \omega \in A\}} (1 - \mu(A))\nu(A)$$

- η's payoff is 0 regardless of the strategies played

Synchronic Dutch Book

Definition (Dutch Book)

β's strategy is a Dutch Book with respect to α's strategy if regardless of η's strategy, β will receive a positive payoff.

Theorem (Diachronic Dutch Book Theorem)

If α's strategy μ is not a probability measure, then β has a strategy that is a Dutch book with respect to α's strategy.

Theorem (Converse Diachronic Dutch Book Theorem)

If α's strategy μ is a probability measure, then β has no strategy that is a Dutch book with respect to α's strategy.

Proof of Dutch Book Theorem

Possible violations of the laws of probability:

- $\mu(A) < 0$ for some A
- $\mu(S) > 1$
- $\mu(A \cap B) + \mu(A \cap \overline{B}) > \mu(A)$
- $\mu(A \cap B) + \mu(A \cap \overline{B}) < \mu(A)$

Proof continued

If $\mu(A) < 0$ for some A,

- $\nu(B) = 0$ for all $B \neq A$
- $\nu(A) = a$ for any positive number a (β buys a quantity of a unit wagers)
- This guarantees β at least $a|\mu(A)|$ (and at most $a|\mu(A)| + a$).

If $\mu(S) > 1$ for some A,

- $\nu(A) = 0$ for all $A \neq S$
- $\nu(S) = -a$ for any positive number a (β sells a quantity of a unit wagers)
- this guarantees β at least $a\mu(S)$ (and at most $a\mu(S) + a$).

Proof continued

If $\mu(A \cap B) + \mu(A \cap \overline{B}) > \mu(A)$,

- $\nu(A \cap B) = -1$
- $\nu(A \cap \overline{B}) = -1$
- $\nu(A) = 1$

Then

- $\omega \in A \cap B$ implies β's payoff is $\mu(A \cap B) - 1 + \mu(A \cap \overline{B}) - \mu(A) + 1 > 0$
- $\omega \in A \cap \overline{B}$ implies β's payoff is $\mu(A \cap B) + \mu(A \cap \overline{B}) - 1 - \mu(A) + 1 > 0$
- $\omega \in \overline{A}$ implies β's payoff is $\mu(A \cap B) + \mu(A \cap \overline{B}) - 1 - \mu(A) + 1 > 0$

Proof continued

If $\mu(A \cap B) + \mu(A \cap \overline{B}) < \mu(A)$,

- $\nu(A \cap B) = 1$
- $\nu(A \cap \overline{B}) = 1$
- $\nu(A) = -1$

The proof is the same as for $\mu(A \cap B) + \mu(A \cap \overline{B}) > \mu(A)$, but with every sign reversed.

Proof of converse Dutch Book Theorem

Let $\mathcal{B} = \{B_1, \ldots, B_n\}$ be the finest partition of Ω (finite) for which each $B_i \in \mathcal{A}$.

Fix a probability measure μ for α. Given ν, let ν' be given by

$$\nu'(A) = \begin{cases} \sum_{\{A' | A \subseteq A'\}} \nu(A') & A \in \mathcal{B} \\ 0 & \text{otherwise} \end{cases}$$

If $\omega \in B_\omega \in \mathcal{B}$, then β's payoff when ν is played:

$$\begin{aligned} & -\sum_{A \in \mathcal{A}} \mu(A)\nu(A) + \sum_{\{A \in \mathcal{A} | \omega \in A\}} \nu(A) \\ = & -\sum_{A \in \mathcal{A}} \sum_{\{B \in \mathcal{B} | B \subseteq A\}} \mu(B)\nu(A) + \sum_{\{A \in \mathcal{A} | B_\omega \subseteq A\}} \nu(A) \\ = & -\sum_{B \in \mathcal{B}} \sum_{\{A \in \mathcal{A} | B \subseteq A\}} \mu(B)\nu(A) + \sum_{\{A \in \mathcal{A} | B_\omega \subseteq A\}} \nu(A) \\ = & -\sum_{B \in \mathcal{B}} \mu(B)\nu'(B) + \nu'(B) \end{aligned}$$

which is β's payoff when ν' is played.

Proof continued

If $\nu'(B_i) \geq 0$ for all i, let $\nu'(B_M) = \max\{\nu'(B_i)\}$ and $\nu'(B_m) = \min\{\nu'(B_i)\}$. Then if $\omega \in B_M$, β's payoff is

$$\begin{aligned}
&\nu'(B_M) - \sum_{B \in \mathcal{B}} \mu(B)\nu'(B) \\
&\geq \nu'(B_M) - \sum_{B \in \mathcal{B}} \mu(B)\nu'(B_M) \\
&= \nu'(B_M) - \nu'(B_M) \sum_{B \in \mathcal{B}} \mu(B) = 0
\end{aligned}$$

and if $\omega \in B_m$, β's payoff

$$\begin{aligned}
&\nu'(B_m) - \sum_{B \in \mathcal{B}} \mu(B)\nu'(B) \\
&\leq \nu'(B_m) - \sum_{B \in \mathcal{B}} \mu(B)\nu'(B_m) = 0
\end{aligned}$$

If $\nu'(B_i) \leq 0$ for all i, use the same reasoning as the case where $\nu'(B_i) \geq 0$.

Proof continued

If $\nu'(B_i) > 0$ for some i and $\nu'(B_i) < 0$ for some i, let

- $\nu'(B_M) = \max\{\nu'(B_i) > 0\}$
- $\nu'(B_N) = \max\{-\nu'(B_i) \mid \nu'(B_i) < 0\}$.

Using the same reasoning as for the cases with $\nu'(B_i) \geq 0$ for all B_i or $\nu'(B_i) \leq 0$ for all B_i,

- if $\omega \in B_M$, β's payoff is at least 0
- if $\omega \in B_N$, β's payoff is at most 0.

Strategies for Diachronic Dutch Book

Let

- $(\Omega, \mathcal{A})$ be a finite measurable space,
- $D_1, \ldots, D_n \in \mathcal{A}$ partition Ω,
- $\mathcal{A}_i = \{A \cap D_i : A \in \mathcal{A}\}$ for each i.

Consider three players α, β, and η

- α's strategy ("a system of beliefs") is a probability measure $\mu : \mathcal{A} \to \mathbb{R}$ for which $\mu(D_i) \neq 0$ for $1 \leq i \leq n$, together with probability measures $\{\mu_i : \mathcal{A}_1 \to \mathbb{R}\}_{i=1}^n$
- β's strategy ("a system of bets") is a function $\nu : \mathcal{A} \to \mathbb{R}$ together with functions $\{\nu_i : \mathcal{A}_1 \to \mathbb{R}\}_{i=1}^n$,
- η's strategy ("the actual outcome") is an $\omega \in \Omega$.

Payoffs

Fix a strategy profile $(\{\mu, \mu_1, \ldots, \mu_n\}, \{\nu, \nu_1, \ldots, \nu_n\}, \omega)$. Define the function $\pi : \mathcal{P}(\Omega) \to \mathbb{R}$ by

$$\pi(\mathcal{X}, \mu, \nu) = \sum_{\{A \in \mathcal{X} \mid \omega \notin A\}} \mu(A)\nu(A) + \sum_{\{A \in \mathcal{X} \mid \omega \in A\}} (\mu(A) - 1)\nu(A)$$

- α's payoff is $\pi(\mathcal{A}, \mu, \nu) + \sum_{\{i \mid \omega \in S_i,\}} \pi(\mathcal{A}_i, \mu_i, \nu_i)$
- β's payoff is $-\pi(\mathcal{A}, \mu, \nu) - \sum_{\{i \mid \omega \in S_i\}} \pi(\mathcal{A}_i, \mu_i, \nu_i)$
- η's payoff is 0 regardless of the strategies played

Diachronic Dutch Book

Definition (Dutch Book)

β's strategy is a Dutch Book with respect to α's strategy if regardless of η's strategy, β will receive a positive payoff.

Theorem (Diachronic Dutch Book Theorem)

If there is an i and an $A \in \mathcal{A}$ such that $\mu_i(A \cap D_i) \neq \mu(A \cap D_i)/\mu(D_i)$, then β has a strategy that is a Dutch book with respect to α's strategy.

Theorem (Converse Diachronic Dutch Book Theorem)

If there for all i and $A \in \mathcal{A}$, it is the case that $\mu_i(A \cap D_i) = \mu(A \cap D_i)/\mu(D_i)$, then β has no strategy that is a Dutch book with respect to α's strategy.

Proof of Diachronic Dutch Book Theorem

Suppose $\mu_i(A \cap D_i) > \mu(A \cap D_i)/\mu(D_i)$. Let

- $\nu(D_i) = -\mu_i(A \cap D_i)$
- $\nu(A \cap D_i) = 1$
- $\nu_i(A \cap D_i) = -1$

Then

- $\omega \notin D_i$ implies β's payoff is $\mu(D_i)\mu_i(A \cap D_i) - \mu(A \cap D_i) > \mu(D_i)\mu(A \cap D_i)/\mu(D_i) - \mu(A \cap D_i) = 0$
- $\omega \in \overline{A} \cap D_i$ implies β's payoff is $\mu(D_i)\mu_i(A \cap D_i) - \mu_i(A \cap D_i) - \mu(A \cap D_i) + \mu_i(A \cap D_i) = \mu(D_i)\mu_i(A \cap D_i) - \mu(A \cap D_i) > 0$
- $\omega \in A \cap D_i$ implies β's payoff is the same as with $\omega \in \overline{A} \cap D_i$. Two new stakes must be payed (1 and -1).

Proof of Diachronic Dutch Book Theorem cont'd

Suppose $\mu_i(A \cap D_i) < \mu(A \cap D_i)/\mu(D_i)$. Let

- $\nu(D_i) = \mu_i(A \cap D_i)$
- $\nu(A \cap D_i) = -1$
- $\nu_i(A \cap D_i) = 1$

Then every term in the previous slide is negated, and the inequalities are not reversed.

Proof of Converse Dutch Book Theorem

Suppose $\mu, \mu_1, \ldots, \mu_n$ is α's strategy for which μ and each μ_i is a probability measure, and $\mu_i(A \cap D_i) = \mu(A \cap D_i)/D_i$ for each i and A.

Given any strategy $\nu, \nu_1, \ldots, \nu_n$ for β, let $\nu_{i,D_i}(A) = 0$ for each $A \in \mathcal{A}$, and for each $B \in \mathcal{A}_i$ and $A \in \mathcal{A}$, let

$$\nu_{i,B}(A) = \begin{cases} \nu_i(B) & A = B \\ -\mu(B)\nu_i(B)/\mu(D_i) & A = D_i \\ 0 & \text{otherwise} \end{cases}$$

Let $\nu_i'(A) = 0$ for each i and A, and

$$\nu'(A) = \nu(A) + \sum_{i=1}^{n} \sum_{B \in \mathcal{A}_i} \nu_{i,B}(A)$$

Then the payoffs are the same if we replace β's strategy with $\nu', \nu_1', \ldots, \nu_n'$. As $\nu_i' = 0$, we can appeal to the converse synchronic Dutch Book theorem.

Cable Guy Paradox

The Cable Guy is coming. You have to be home in order for him to install your new cable service, but to your chagrin he cannot tell you exactly when he will come. He will definitely come between 8 AM and 4 PM tomorrow, but you have no more information. I offer to keep you company while you wait. To make it more interesting, we decide to bet on the Cable Guy's arrival time. We subdivide teh relevant part of the day into two 4 hour long intervals: morning $(8, 12]$ and afternoon $(12, 4)$. You nominate an interval in which you will bet. If he arrives in your interval, I pay you 10 EUR. Otherwise (he arrives in my interval) you pay me 10 EUR.

Which interval should you bet on?

A. Hájek. *The Cable Guy Paradox*. Analysis, **65** (2005).

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Avoid Certain Frustration Principle

Suppose you now have a choice between two options. You should not choose one of these options if you are certain that a rational future self of yours will prefer that you had chosen the other one – unless both options have this property.

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Avoid Self-Undermining Choices Principle

Whenever you have a choice between two options, you should not make a self-undermining choice if you can avoid doing so.

The Two-Envelope Puzzle

There are two envelopes with money in them. The sum of money in one of the envelopes is twice as large as the other sum. Each of the envelopes is equally likely to hold the larger sum. You are assigned at random one of the envelopes and may take the money inside. However, before you open your envelope you are offered the possibility of switching the envelopes and taking the money inside the other one. Should you switch?

D. Samet, I. Samet and D. Shmeidler. *One Observation behind Two-Envelope Puzzles*. .

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The Sleeping Beauty Puzzle

Some researchers are going to put you to sleep. During the two days that your sleep will last, they will brießy wake you up either once or twice, depending on the toss of a fair coin (heads: once; tails: twice). After each waking, they will put you back to sleep with a drug that makes you forget that waking. When you are first awakened, to what degree ought you believe that the outcome of the coin toss is heads?

See, for example,

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Conclusions

- Discussed a number of different logical systems incorporating both *knowledge* (hard information) and probabilities (beliefs, soft information).

  These results have been generalized (coalgebraic framework)

R. Goldblatt. *Deduction systems for coalgebras over measurable spaces.* Journal of Logic and Computation (2008).

- Both logical frameworks (modal logic) and probabilistic frameworks (type spaces) have been used to reason about beliefs in game theoretic situations. How to compare the different types of analyses? The logical frameworks presented here can be a "bridge" between the two different types of analyses.

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Thank you.