Logics of Rational Agency

Lecture 1

Eric Pacuit

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Tilburg Univeristy
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July 27, 2009
<table>
<thead>
<tr>
<th>Lecture 1:</th>
<th>Introduction, Motivation and Background</th>
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Plan for the Course

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Lecture 2: Basic Ingredients for a Logic of Rational Agency

Lecture 3: Logics of Rational Agency and Social Interaction, Part I

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Course Website

http://ai.stanford.edu/~epacuit/classes/esslli/log-ratagency.html

Reading Material

✓ Pointers to literature on the website

Concerning Modal Logic

✓ Modal Logic by P. Blackburn, M. de Rijke and Y. Venema.

✓ Modal Logic for Open Minds by Johan van Benthem (published soon)
Plan for the Course

We are interested in reasoning about rational agents interacting in *social* situations.
Plan for the Course

We are interested in reasoning about rational agents interacting in social situations.

- Philosophy (social philosophy, epistemology)
- Game Theory
- Social Choice Theory
- AI (multiagent systems)
We are interested in reasoning about rational agents interacting in social situations.

What is a rational agent?

▶ maximize expected utility (instrumentally rational)
▶ react to observations
▶ revise beliefs when learning a surprising piece of information
▶ understand higher-order information
▶ plans for the future
▶ ????
We are interested in reasoning about rational agents interacting in social situations.

There is a jungle of formal systems!

- logics of informational attitudes (knowledge, beliefs, certainty)
- logics of action & agency
- temporal logics/dynamic logics
- logics of motivational attitudes (preferences, intentions)

(Not to mention various game-theoretic/social choice models and logical languages for reasoning about them)
We are interested in reasoning about rational agents interacting in social situations.

There is a jungle of formal systems!

- How do we compare different logical systems studying the same phenomena?
- How complex is it to reason about rational agents?
- (How) should we merge the various logical systems?
- What do the logical frameworks contribute to the discussion on rational agency?

(Not to mention various game-theoretic/social choice models and logical languages for reasoning about them)
Plan for the Course

We are interested in reasoning about rational agents interacting in social situations.

- playing a card game
- having a conversation
- executing a social procedure
- ...

Goal: incorporate/extend existing game-theoretic/social choice analyses
Formally, a game is described by its strategy sets and payoff functions.
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Adjusted Winner

**Adjusted winner** \((AW)\) is an algorithm for dividing \(n\) divisible goods among two people (invented by Steven Brams and Alan Taylor).

For more information see

- *Fair Division: From cake-cutting to dispute resolution* by Brams and Taylor, 1998
- *The Win-Win Solution* by Brams and Taylor, 2000
- [www.nyu.edu/projects/adjustedwinner](http://www.nyu.edu/projects/adjustedwinner)
Suppose Ann and Bob are dividing three goods: A, B, and C.

Step 1. Both Ann and Bob divide 100 points among the three goods.

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<tr>
<th>Item</th>
<th>Ann</th>
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<tr>
<td>A</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>B</td>
<td>65</td>
<td>46</td>
</tr>
<tr>
<td>C</td>
<td>30</td>
<td>50</td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
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Adjusted Winner: Example

Suppose Ann and Bob are dividing three goods: $A$, $B$, and $C$. 
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Example: Adjusted Winner
Adjusted Winner: Example

Suppose Ann and Bob are dividing three goods: $A$, $B$, and $C$.

**Step 2.** The agent who assigns the most points receives the item.
Suppose Ann and Bob are dividing three goods: A, B, and C.

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<td>$C$</td>
<td>0</td>
<td>50</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>70</strong></td>
<td><strong>50</strong></td>
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Adjusted Winner: Example

Suppose Ann and Bob are dividing three goods: $A$, $B$, and $C$.

**Step 3.** Equitability adjustment:
Suppose Ann and Bob are dividing three goods: $A$, $B$, and $C$.

**Step 3.** Equitability adjustment:
Notice that $\frac{65}{46} \geq \frac{5}{4} \geq 1 \geq \frac{30}{50}$

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**Step 3.** Equitability adjustment:

Give $A$ to Bob (the item whose ratio is closest to 1)
Adjusted Winner: Example

Suppose Ann and Bob are dividing three goods: \( A, B, \) and \( C \).

**Step 3.** Equitability adjustment:

Give \( A \) to Bob (the item whose ratio is closest to 1)

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Suppose Ann and Bob are dividing three goods: $A$, $B$, and $C$.

**Step 3.** Equitability adjustment:

Still not equal, so give (some of) $B$ to Bob: $65p = 100 - 46p$.

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Example: Adjusted Winner

Adjusted Winner: Example

Suppose Ann and Bob are dividing three goods: $A$, $B$, and $C$.

**Step 3.** Equitability adjustment:

yielding $p = \frac{100}{111} = 0.9009$

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<tr>
<td>$A$</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>$B$</td>
<td>58.559</td>
<td>4.559</td>
</tr>
<tr>
<td>$C$</td>
<td>0</td>
<td>50</td>
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**Total** | 58.559 | 58.559 |
Adjusted Winner: Formal Definition

Suppose that $G_1, \ldots, G_n$ is a fixed set of goods.
Adjusted Winner: Formal Definition

Suppose that \( G_1, \ldots, G_n \) is a fixed set of goods.

A **valuation** of these goods is a vector of natural numbers \( \langle a_1, \ldots, a_n \rangle \) whose sum is 100.

Let \( \alpha, \alpha', \alpha'', \ldots \) denote possible valuations for Ann and \( \beta, \beta', \beta'', \ldots \) denote possible valuations for Bob.
Adjusted Winner: Formal Definition

Suppose that $G_1, \ldots, G_n$ is a fixed set of goods.
Adjusted Winner: Formal Definition

Suppose that $G_1, \ldots, G_n$ is a fixed set of goods.

An allocation is a vector of $n$ real numbers where each component is between 0 and 1 (inclusive). An allocation $\sigma = \langle s_1, \ldots, s_n \rangle$ is interpreted as follows.

For each $i = 1, \ldots, n$, $s_i$ is the proportion of $G_i$ given to Ann.

Thus if there are three goods, then $\langle 1, 0.5, 0 \rangle$ means, “Give all of item 1 and half of item 2 to Ann and all of item 3 and half of item 2 to Bob.”
Example: Adjusted Winner

Fairness

- **Proportional** if both Ann and Bob receive at least 50% of their valuation: \( \sum_{i=1}^{n} s_i a_i \geq 50 \) and \( \sum_{i=1}^{n} (1 - s_i) b_i \geq 50 \)
Example: Adjusted Winner

Fairness

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- **Envy-Free** if no party is willing to give up its allocation in exchange for the other player’s allocation:
  \( \sum_{i=1}^{n} s_i a_i \geq \sum_{i=1}^{n} (1 - s_i) a_i \) and \( \sum_{i=1}^{n} (1 - s_i) b_i \geq \sum_{i=1}^{n} s_i b_i \)
Example: Adjusted Winner

Fairness

- **Proportional** if both Ann and Bob receive at least 50% of their valuation: \( \sum_{i=1}^{n} s_ia_i \geq 50 \) and \( \sum_{i=1}^{n} (1 - s_i)b_i \geq 50 \)

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  \[ \sum_{i=1}^{n} s_ia_i \geq \sum_{i=1}^{n} (1 - s_i)a_i \quad \text{and} \quad \sum_{i=1}^{n} (1 - s_i)b_i \geq \sum_{i=1}^{n} s_ib_i \]

- **Equitable** if both players receive the same total number of points: \( \sum_{i=1}^{n} s_ia_i = \sum_{i=1}^{n} (1 - s_i)b_i \)
Example: Adjusted Winner

Fairness

- **Proportional** if both Ann and Bob receive at least 50% of their valuation: \( \sum_{i=1}^{n} s_i a_i \geq 50 \) and \( \sum_{i=1}^{n} (1 - s_i) b_i \geq 50 \)

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- **Equitable** if both players receive the same total number of points: \( \sum_{i=1}^{n} s_i a_i = \sum_{i=1}^{n} (1 - s_i) b_i \)

- **Efficient** if there is no other allocation that is strictly better for one party without being worse for another party: for each allocation \( \sigma' = \langle s'_1, \ldots, s'_n \rangle \) if \( \sum_{i=1}^{n} a_i s'_i > \sum_{i=1}^{n} a_i s_i \), then \( \sum_{i=1}^{n} (1 - s'_i) b_i < \sum_{i=1}^{n} (1 - s_i) b_i \). (Similarly for Bob)
Easy Observations

- For two-party disputes, proportionality and envy-freeness are equivalent.

- \( AW \) only produces equitable allocations (equitability is essentially built in to the procedure).

- \( AW \) produces allocations \( \sigma \) that in which at most one good is split.
Adjusted Winner is Fair

**Theorem (Brams and Taylor)** \( AW \) produces allocations that are efficient, equitable and envy-free (with respect to the announced valuations)
Some Questions

▶ Can we make use of geometric intuitions?

▶ Is $AW$ a “continuous” function?

▶ It seems that the more the agents’ utilities differ, the more points $AW$ gives to each agent.

▶ The agents’ utility functions are assumed to be linear, what about non-linear utility functions?

▶ Can an agent benefit by making use of information about the other agent’s valuation?
Some Questions

▶ Can we make use of geometric intuitions? Yes!

▶ Is $AW$ a “continuous” function? Yes and No

▶ It seems that the more the agents’ utilities differ, the more points $AW$ gives to each agent. Yes, we can prove this.

▶ The agents’ utility functions are assumed to be linear, what about non-linear utility functions? The nonlinear situation may be interesting.

▶ Can an agent benefit by making use of information about the other agent’s valuation? Yes, but in most cases it is not a “safe” strategy.

Strategizing

Can the agents improve their allocation by misrepresenting their preferences?
Strategizing

Can the agents improve their allocation by misrepresenting their preferences?

Yes
Strategizing

Can the agents improve their allocation by misrepresenting their preferences?

Yes

However, while honesty may not always be the best policy it is the only safe one, i.e., the only one which will guarantee 50%.
Strategizing

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<tr>
<td>Matisse</td>
<td>75</td>
<td>25</td>
</tr>
<tr>
<td>Picasso</td>
<td>25</td>
<td>75</td>
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Ann will get the Matisse and Bob will get the Picasso and each gets 75 of his or her points.
Suppose Ann knows Bob’s preferences, but Bob does not know Ann’s.

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<td>$M$</td>
<td>75</td>
<td>25</td>
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<td>$P$</td>
<td>25</td>
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So Ann will get $M$ plus a portion of $P$.

According to Ann’s announced allocation, she receives 50 points.

According to Ann’s actual allocation, she receives $75 + 0.33 \times 25 = 83.33$ points.
Strategizing: A Theorem

Theorem (Brams and Taylor) Assume there are two goods, $G_1$ and $G_2$, all true and announced values are restricted to integers, and suppose Bob’s announced valuation of $G_1$ is $x$, where $x \geq 50$. Assume Ann’s true valuation of $G_1$ is $b$. Then her optimal announced valuation of $G_1$ is:

$$
\begin{cases}
  x + 1 & \text{if } b > x \\
  x & \text{if } b = x \\
  x - 1 & \text{if } b < x
\end{cases}
$$
Strategizing: Example

Suppose both players know each other’s preferences but neither knows that the other knows their own preference.

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<td>$M$</td>
<td>26</td>
<td>74</td>
</tr>
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<td>$P$</td>
<td>74</td>
<td>26</td>
</tr>
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Each will get 74 of his or her announced points, but each one is really getting only 25 of his or her true points.
Strategizing: Example

Suppose both players know each other’s preferences. Moreover, Ann knows that Bob knows her preference and Bob doesn’t know that Ann knows.

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What happens as the level of knowledge increases?
What about a logical analysis?

- Which aspects of social situations should we focus on?
  - Knowledge, Beliefs, Group Knowledge, Preferences, Desires, Ability, Actions, Intentions, Goals, Obligations, etc.
- One grand system, or many smaller systems that loosely “fit” together?
- Combining systems is hard! (conceptually and technically)
- Logics of rational agents in social situations. vs. Logics about rational agents in social situations.
- Normative vs. Descriptive
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- Logics *of* rational agents in social situations.

  vs.

- Logics *about* rational agents in social situations.
What about a logical analysis?

- Which aspects of social situations should we focus on? Knowledge, Beliefs, Group Knowledge, Preferences, Desires, Ability, Actions, Intentions, Goals, Obligations, etc.

- One grand system, or many smaller systems that loosely “fit” together?

- *Combining* systems is hard! (conceptually and technically)

- Logics *of* rational agents in social situations.
  vs.
  Logics *about* rational agents in social situations.

- Normative vs. Descriptive
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Time for some details.
What is a modal?

A modal qualifies the truth of a judgement.
What is a modal?

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John _______ happy.
What is a modal?

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- is necessarily
- is possibly
What is a modal?

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- can do something to ensure that he is
- ...
The Basic Modal Language

A wff of Modal Logic is defined *inductively*:

1. Any atomic propositional variable is a wff
2. If $P$ and $Q$ are wff, then so are $\neg P$, $P \land Q$, $P \lor Q$ and $P \rightarrow Q$
3. If $P$ is a wff, then so is $\Box P$ and $\Diamond P$
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Boolean Logic
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Unary operator
The Basic Modal Language

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2. If $P$ and $Q$ are wff, then so are $\neg P$, $P \land Q$, $P \lor Q$ and $P \rightarrow Q$
3. If $P$ is a wff, then so is $\square P$ and $\diamond P$

Eg., $\square(P \rightarrow \diamond Q) \lor \square \diamond \neg R$
Propositional Modal Logic

Modal Formulas

\neg (\Box A \rightarrow B)

\neg \Box (A \rightarrow B)

(\neg \Box A \rightarrow B)
Modal Formulas

\[ \neg (\Box A \rightarrow B) \]

\[ \neg \Box (A \rightarrow B) \]

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Modal Formulas

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Propositional Modal Logic

Modal Formulas

\neg (\Box A \rightarrow B)

\neg (A \rightarrow B)

(\neg \Box A \rightarrow B)

\neg

\rightarrow

\rightarrow

\rightarrow

\Box

\rightarrow

\Box

\rightarrow

\Box

\rightarrow

A


B

A


B

A


B

A


B

A
One Language, Many Interpretations

**Alethic**

\( \Box A \): A is necessary

\( \Diamond A \): A is possible

**Deontic**

\( \Box A \): A is obligatory

\( \Diamond A \): A is permitted

\((OA, PA)\)
One Language, Many Interpretations

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- □A: A is necessary
- ◊A: A is possible

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(\(OA, PA\))
One Language, Many Interpretations

**Alethic**

□A: A is necessary

◊A: A is possible

**Deontic**

□A: A is obligatory

◊A: A is permitted

(OA, PA)

**Epistemic**

□A: Ann knows ϕ

◊A: it is consistent with Ann’s information that ϕ

(KA, LA)
Valid?

\[ \square P \leftrightarrow \neg \Diamond \neg P \]
Valid?

$$\Box P \leftrightarrow \neg \Diamond \neg P$$  

*P is necessary/obligatory iff \(\neg P\) is not possible/permitted*
Valid?

\[ \square P \leftrightarrow \neg \Diamond \neg P \]

*P is necessary/obligatory iff \( \neg P \) is not possible/permitted*

\[ \square P \rightarrow P \]
Valid?

\[ \square P \leftrightarrow \neg \lozenge \neg P \]

*P is necessary/obligatory iff \( \neg P \) is not possible/permitted*

\[ \square P \rightarrow P \]

*If \( P \) is necessary/known/obligatory then \( P \) is true*
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Valid?

\[ \Box P \leftrightarrow \neg \Diamond \neg P \]

*P is necessary/obligatory iff \( \neg P \) is not possible/permitted*

\[ \Box P \rightarrow P \]

*If \( P \) is necessary/known/obligatory then \( P \) is true*

\[ \Box P \rightarrow \Box \Box P \]

*If Ann knows \( P \) then Ann knows that she knows \( P \)*
Valid?

\(\square P \leftrightarrow \neg\Diamond \neg P\)

*P is necessary/obligatory iff \(\neg P\) is not possible/permitted*

\(\square P \rightarrow P\)

*If P is necessary/known/obligatory then P is true*

\(\square P \rightarrow \square\square P\)

*If Ann knows P then Ann knows that she knows P*

\(\Diamond P \rightarrow \square\Diamond P\),
Valid?

□P ↔ ¬◊¬P

*P is necessary/obligatory iff ¬P is not possible/permitted*

□P → P

*If P is necessary/known/obligatory then P is true*

□P → □□P

*If Ann knows P then Ann knows that she knows P*

◊P → □◊P, □◊P → ◊□P, etc.
Can we give a natural *semantics* for the basic modal language?
The main idea:

▶ "It is sunny outside" is currently true, but it is not necessary (for example, if we were currently in Ohio).

▶ We say $P$ is necessary provided $P$ is true in all (relevant) situations (states, worlds, possibilities).

▶ A Kripke structure is
1. A set of states, or worlds (each world specifies the truth value of all propositional variables)
2. A relation on the set of states (specifying the "relevant situations")
Kripke Structures

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  2. A **relation** on the set of states (specifying the “relevant situations”)
A Kripke Structure

1. Set of states

$w_1$ $w_2$ $w_3$ $w_4$ $w_5$
A Kripke Structure

1. Set of states (propositional valuations)

- $w_1$: $A$
- $w_2$: $B$
- $w_3$: $B$
- $w_4$: $B, C$
- $w_5$: $A, B$
A Kripke Structure

1. Set of states (propositional valuations)
2. Accessibility relation
A Kripke Structure

1. Set of states (propositional valuations)
2. Accessibility relation

denoted $w_3 R w_5$
Truth of Modal Formulas

We interpret formulas at states in a Kripke structure: \( w \models P \) means \( P \) is true at state \( w \).
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We write $wRv$ is $v$ is accessible from state $w$. 
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1. \( \Box P \) is true at state \( w \) iff \( P \) is true in all accessible worlds.
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1. \( \square P \) is true at state \( w \) iff \( P \) is true in all accessible worlds.
   \( w \models \square P \) iff for all \( v \), if \( wRv \) then \( v \models P \)
Truth of Modal Formulas

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We write \( wRv \) is \( v \) is accessible from state \( w \).

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   \[ w \models \Box P \text{ iff for all } v, \text{ if } wRv \text{ then } v \models P \]

2. \( \Diamond P \) is true at state \( w \) iff \( P \) is true at some accessible world.
Truth of Modal Formulas

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\[
\models w \models \Box P \iff \text{for all } v, \text{ if } wRv \text{ then } v \models P
\]

2. \( \Diamond P \) is true at state \( w \) iff \( P \) is true at some accessible world.
\[
\models w \models \Diamond P \iff \text{there exists } v \text{ such that } wRv \text{ and } v \models P.
\]
Example
Example

\[ w_4 \models B \land C \]
Example

\[ w_3 \models \Box B \]
Example

$w_3 \models □B$
Example

Propositional Modal Logic

\[ w_3 \models \lozenge C \]
Example

$w_3 \models \Diamond C$
Example

\[ w_3 \not\models \Box C \]
Propositional Modal Logic

Example

\[ w_3 \not\models \Box C \]
Example

Propositional Modal Logic

\[ w_1 \models \lozenge \Box B \]

\[ A \xrightarrow{w_1} B \xrightarrow{w_2} B, C \xrightarrow{w_4} A, B \xrightarrow{w_5} B \xrightarrow{w_3} B \]
Propositional Modal Logic

Example

\[ w_1 \models 
\]
Propositional Modal Logic

Example

\[ w_1 \models \diamond \Box B \]
Example

$w_1 \models A$

$w_2 \models B$

$w_3 \models B$

$w_4 \models B, C$

$w_5 \models □ C$

$w_5 \models A, B$
Example

\[ w_5 \models □(B \land \neg B) \]
Example

\[ w_5 \models \neg \lozenge B \]
Propositional Modal Logic

\[ w_1 \models \Box B \land B? \]
\[ w_1 \models \Diamond \Diamond B? \]
\[ w_1 \models \Diamond \Diamond \Diamond B? \]
\[ w_1 \models \Box \Box B? \]
\[ w_1 \models \Box \Diamond C? \]
\[ w_1 \models \Diamond \Diamond C? \]
Propositional Modal Logic

$w_1 \not\models \Box B \land B$

$w_1 \models \Diamond \Diamond B$?

$w_1 \models \Diamond \Diamond \Diamond B$?

$w_1 \models \Box \Box B$?

$w_1 \models \Box \Diamond C$?

$w_1 \models \Diamond \Diamond C$?
Propositional Modal Logic

$w_1 \not\models \Box B \land B$

$w_1 \models \Diamond \Diamond B$

$w_1 \models \Diamond \Diamond \Diamond B$

$w_1 \models \Box \Box B$?

$w_1 \models \Box \Diamond C$?

$w_1 \models \Diamond \Diamond C$?
Propositional Modal Logic

$\lnot w_1 \models \Box B \land B$

$w_1 \models \Diamond \Diamond B$

$w_1 \models \Diamond \Diamond \Diamond B$

$w_1 \models \Box \Box B$

$w_1 \models \Box \Diamond C$?

$w_1 \models \Diamond \Diamond C$?
Propositional Modal Logic

$w_1 \not\models \Box B \land B$

$w_1 \models \Diamond \Diamond B$

$w_1 \models \Diamond \Diamond \Diamond B$

$w_1 \not\models \Box \Box B$

$w_1 \models \Box \Diamond C?$

$w_1 \models \Diamond \Diamond C?$

Eric Pacuit: LORI, Lecture 1
Propositional Modal Logic

$w_1 \models \Diamond \Diamond \Box B \land B$

$w_1 \models \Diamond \Diamond \Diamond \Diamond B$

$w_1 \not\models \Box \Box \Box B$

$w_1 \not\models \Box \Diamond C$

$w_1 \models \Diamond \Diamond C$?
Propositional Modal Logic

\[ A \wedge B \]

\[ A \Rightarrow B \]

\[ B \triangleleft C \]

\[ B \triangleleft C \]

\[ B \triangleleft C \]

\[ w_1 \not\models \Box B \land B \]

\[ w_1 \models \Diamond \Diamond \Diamond B \]

\[ w_1 \models \Box \Box \Box \Box B \]

\[ w_1 \not\models \Box \Diamond C \]

\[ w_1 \models \Diamond \Diamond C \]
Propositional Modal Logic

\[ w_1 \not\models \Box B \land B \]
\[ w_1 \models \Diamond \Diamond B \]
\[ w_1 \models \Diamond \Diamond \Diamond B \]
\[ w_1 \not\models \Box \Box B \]
\[ w_1 \not\models \Box \Diamond C \]
\[ w_1 \models \Diamond \Diamond C \]
Propositional Modal Logic

\[ \begin{align*}
  w_1 & \not\models □B \land B \\
  w_1 & \models ♦♦B \\
  w_1 & \models ♦♦♦B \\
  w_1 & \not\models □□B \\
  w_1 & \not\models □♦C \\
  w_1 & \models ♦♦C 
\end{align*} \]
Some Facts

- □P ∨ ¬□P is always true (i.e., true at any state in any Kripke structure), but what about □P ∨ □¬P?
Propositional Modal Logic

Some Facts

- □P ∨ ¬□P is always true (i.e., true at any state in any Kripke structure), but what about □P ∨ □¬P?

- □P ∧ □Q → □(P ∧ Q) is true at any state in any Kripke structure.
Some Facts

- □P ∨ ¬□P is always true (i.e., true at any state in any Kripke structure), but what about □P ∨ □¬P?

- □P ∧ □Q → □(P ∧ Q) is true at any state in any Kripke structure. What about □(P ∨ Q) → □P ∨ □Q?
Some Facts

- □P ∨ ¬□P is always true (i.e., true at any state in any Kripke structure), but what about □P ∨ □¬P?

- □P ∧ □Q → □(P ∧ Q) is true at any state in any Kripke structure. What about □(P ∨ Q) → □P ∨ □Q?

- □P ↔ ¬◊¬P is true at any state in any Kripke structure.
But, we are not always interested in all Kripke structures.
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For example, consider the epistemic interpretation: A state $v$ is accessible from $w$ ($wRv$) provided “given the agents information, $w$ and $v$ are indistinguishable”.
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Eg., for each state $w$, $w$ is accessible from itself ($R$ is a reflexive relation).
But, we are not always interested in all Kripke structures.

For example, consider the epistemic interpretation: A state $v$ is accessible from $w$ ($wRv$) provided “given the agents information, $w$ and $v$ are indistinguishable”. What are natural properties?

Eg., for each state $w$, $w$ is accessible from itself ($R$ is a reflexive relation).

**Some Facts**

- $\square P \rightarrow P$ is true at any state in any Kripke structure where each state is accessible from itself.
- $\square P \rightarrow \diamond P$ is true at any state in any Kripke structure where each state has at least one accessible world.
Something to think about....

Which pair of states cannot be distinguished by a modal formula? What about a first order formula?
**Slogan 1:** Modal languages are simple yet expressive languages for talking about relational structures.

**Slogan 2:** Modal languages provide an internal, local perspective on relational structures.

\[ \Box(\Box \bot \lor \Diamond \Box \bot) \]
Propositional Modal Logic

\[ \Box(\Box \bot \lor \Diamond \Box \bot) \]
\[ \Box (\Box \bot \lor \Diamond \Box \bot) \]
Propositional Modal Logic

\[ \Box (\Box \bot \lor \Diamond \Box \bot) \]
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\[ \Box (\Box \bot \lor \Diamond \Box \bot) \]
Propositional Modal Logic

\[ \Box (\Box \bot \lor \Diamond \Box \bot) \]
\[\Box (\Box \bot \lor \Diamond \Box \bot)\]
Propositional Modal Logic

\[\Box(\Box \bot \lor \Diamond \Box \bot)\]
□(□⊥ ∨ ♦□⊥)

Eric Pacuit: LORI, Lecture 1
A Kripke frame is a pair $\langle W, R \rangle$ where $R \subseteq W \times W$. 

Notation
A Kripke frame is a pair \( \langle W, R \rangle \) where \( R \subseteq W \times W \).

Let \( F = \langle W, R \rangle \) be a Kripke frame and \( M = \langle W, R, V \rangle \) a model based on \( M \).
Notation

A Kripke frame is a pair $\langle W, R \rangle$ where $R \subseteq W \times W$.

Let $\mathcal{F} = \langle W, R \rangle$ be a Kripke frame and $\mathcal{M} = \langle W, R, V \rangle$ a model based on $\mathcal{M}$.

$\phi$ is satisfiable in $\mathcal{M}$ if there exists $w \in W$ such that $\mathcal{M}, w \models \phi$
Notation

A Kripke frame is a pair \( \langle W, R \rangle \) where \( R \subseteq W \times W \).

Let \( F = \langle W, R \rangle \) be a Kripke frame and \( M = \langle W, R, V \rangle \) a model based on \( M \).

\( \varphi \) is satisfiable in \( M \) if there exists \( w \in W \) such that \( M, w \models \varphi \)

\( \varphi \) is valid in \( M \) (\( M \models \varphi \)) if \( \forall w \in W, \ M, w \models \varphi \)
A Kripke frame is a pair \( \langle W, R \rangle \) where \( R \subseteq W \times W \).

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\( \varphi \) is valid in \( M \) (\( M \models \varphi \)) if \( \forall w \in W, M, w \models \varphi \).

\( \varphi \) is valid on a frame \( F \) (\( F \models \varphi \)) if for all models \( M \) based on \( F \), \( M \models \varphi \).
Definable Properties

A modal formula $\varphi$ defines a class of frames $K$ provided

$$\mathcal{F} \in K \text{ iff } \mathcal{F} \models \varphi$$

- $\Box \varphi \rightarrow \Box \Box \varphi$ defines the class of transitive frames.
- $\varphi \leftrightarrow \Box \varphi$ defines the class of frames consisting of isolated reflexive points ($\forall x \in W, \ x R y \rightarrow x = y$).
- $\Box (\Box \varphi \rightarrow \varphi)$ defines the class of secondary-reflexive frames ($\forall w, v \in W, \text{ if } w R v \text{ then } v R v$).
Definable Properties

A modal formula \( \varphi \) defines a class of frames \( K \) provided

\[
F \in K \text{ iff } F \models \varphi
\]

✓ \( \Box \varphi \rightarrow \Box \Box \varphi \) defines the class of transitive frames.

▷ \( \varphi \leftrightarrow \Box \varphi \) defines the class of frames consisting of isolated reflexive points (\( \forall x \in W, \; xRy \rightarrow x = y \)).

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Definable Properties

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Definable Properties

A modal formula $\varphi$ defines a class of frames $K$ provided

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✓ $\Box (\Box \varphi \rightarrow \varphi)$ defines the class of secondary-reflexive frames ($\forall w, v \in W, \text{ if } wRv \text{ then } vRv$).

Some modal formulas correspond to genuine second-order properties: L"ob $(\Box (\Box \varphi \rightarrow \varphi) \rightarrow \Box \varphi)$, McKinsey $(\Box \Diamond \varphi \rightarrow \Diamond \Box \varphi)$
Definable Properties

A modal formula $\varphi$ defines a class of frames $K$ provided

$$F \in K \text{ iff } F \models \varphi$$

✓ $\Box \varphi \to \Box \Box \varphi$ defines the class of transitive frames.

✓ $\varphi \leftrightarrow \Box \varphi$ defines the class of frames consisting of isolated reflexive points ($\forall x \in W, \ xRy \to x = y$).

✓ $\Box(\Box \varphi \to \varphi)$ defines the class of secondary-reflexive frames ($\forall w, v \in W, \text{ if } wRv \text{ then } vRv$).

The Sahlqvist Theorem gives an algorithm for finding a first-order correspondant for certain modal formulas.
Slogan 3: Modal logics are not isolated formal systems.
The Standard Translation

\[ st_x : \mathcal{L} \rightarrow \mathcal{L}_1 \]
The Standard Translation

\[ st_x : \mathcal{L} \rightarrow \mathcal{L}_1 \]

First-order language with an appropriate signature

Lemma

For each \( w \in W \), \( M, w \models \phi \) iff \( M \models st_x(\phi)[x/w] \).
Propositional Modal Logic

The Standard Translation

\[ st_x : \mathcal{L} \rightarrow \mathcal{L}_1 \]

\[
\begin{align*}
  st_x(p) & = Px \\
  st_x(\neg \varphi) & = \neg st_x(\varphi) \\
  st_x(\varphi \land \psi) & = st_x(\varphi) \land st_x(\psi)
\end{align*}
\]
The Standard Translation

\[
st_x : \mathcal{L} \rightarrow \mathcal{L}_1
\]

\[
\begin{align*}
st_x(p) &= Px \\
st_x(\neg \varphi) &= \neg st_x(\varphi) \\
st_x(\varphi \land \psi) &= st_x(\varphi) \land st_x(\psi) \\
st_x(\Box \varphi) &= \forall y(xRy \rightarrow st_y(\varphi))
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Propositional Modal Logic

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\end{align*}
\]

Fact: Modal logic falls in the two-variable fragment of \( \mathcal{L}_1 \).
Propositional Modal Logic

The Standard Translation

\[ st_x : \mathcal{L} \rightarrow \mathcal{L}_1 \]

\[
\begin{align*}
    st_x(p) &= Px \\
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\end{align*}
\]

Lemma For each \( w \in \mathcal{W} \), \( M, w \models \varphi \) iff \( M \models st_x(\varphi)[x/w] \).
What can we say with modal logic? What about in comparison with first-order logic?
Definition Let $M_1 = \langle W_1, R_1, V_1 \rangle$ and $M_2 = \langle W_2, R_2, V_2 \rangle$. The **disjoint union** is the structure $M_1 \oplus M_2 = \langle W, R, V \rangle$ where

- $W = W_1 \cup W_2$
- $R = R_1 \cup R_2$
- for all $p \in \text{At}$, $V(p) = V_1(p) \cup V_2(p)$
**Disjoint Union**

**Definition** Let $\mathcal{M}_1 = \langle W_1, R_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, R_2, V_2 \rangle$. The **disjoint union** is the structure $\mathcal{M}_1 \uplus \mathcal{M}_2 = \langle W, R, V \rangle$ where

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- $R = R_1 \cup R_2$
- for all $p \in \text{At}$, $V(p) = V_1(p) \cup V_2(p)$

**Lemma** For each collection of Kripke structures $\{\mathcal{M}_i \mid i \in I\}$, for each $w \in W_i$, $\mathcal{M}_i, w \models \varphi$ iff $\bigcup_{i \in I} \mathcal{M}_i, w \models \varphi$
Disjoint Union

**Definition** Let $\mathbb{M}_1 = \langle W_1, R_1, V_1 \rangle$ and $\mathbb{M}_2 = \langle W_2, R_2, V_2 \rangle$. The disjoint union is the structure $\mathbb{M}_1 \uplus \mathbb{M}_2 = \langle W, R, V \rangle$ where

- $W = W_1 \cup W_2$
- $R = R_1 \cup R_2$
- for all $p \in At$, $V(p) = V_1(p) \cup V_2(p)$

**Fact** The universal modality is not definable in the basic modal language.
Definition \( \mathcal{M}' = \langle W', R', V' \rangle \) is a generated submodel of \( \mathcal{M} = \langle W, R, V \rangle \) provided:

- \( W' \subseteq W \) is R-closed:
  
  for each \( w' \in W \) and \( v \in W \), if \( wRv \) then \( v \in W' \).

- \( R' = R \cap W' \times W' \)

- for all \( p \in \text{At} \), \( V'(p) = V(p) \cap W' \)
Generated Submodel

**Definition** $\mathcal{M}' = \langle W', R', V' \rangle$ is a generated submodel of $\mathcal{M} = \langle W, R, V \rangle$ provided

- $W' \subseteq W$ is $R$-closed:
  - for each $w' \in W$ and $v \in W$, if $wRv$ then $v \in W'$.
- $R' = R \cap W' \times W'$
- for all $p \in \text{At}$, $V'(p) = V(p) \cap W'$

**Lemma** If $\mathcal{M}'$ is a generated submodel of $\mathcal{M}$ then for each $w \in W'$, $\mathcal{M}', w \models \varphi$ iff $\mathcal{M}, w \models \varphi$
Generated Submodel

**Definition** \(M' = \langle W', R', V' \rangle\) is a generated submodel of \(M = \langle W, R, V \rangle\) provided

- \(W' \subseteq W\) is \(R\)-closed:
  - for each \(w' \in W\) and \(v \in W\), if \(wRv\) then \(v \in W'\).
- \(R' = R \cap W' \times W'\)
- for all \(p \in \text{At}\), \(V'(p) = V(p) \cap W'\)

**Fact** The backwards looking modality is not definable in the basic modal language.
Bounded Morphism

**Definition** A bounded morphism between models $M = \langle W, R, V \rangle$ and $M' = \langle W', R', V' \rangle$ is a function $f$ with domain $W$ and range $W'$ such that:

- **Atomic harmony:** for each $p \in \text{At}$, $w \in V(p)$ iff $f(w) \in V'(p)$
- **Morphism:** if $wRv$ then $f(w)Rf(v)$
- **Zag:** if $f(w)R'v'$ then $\exists v \in W$ such that $f(v) = v'$ and $wRv$

Fact The universal modality is not definable in the basic modal language.
Bounded Morphism

**Definition** A **bounded morphism** between models $\mathbb{M} = \langle W, R, V \rangle$ and $\mathbb{M}' = \langle W', R', V' \rangle$ is a function $f$ with domain $W$ and range $W'$ such that:

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**Lemma** If $\mathbb{M}'$ is a bounded morphic image of $\mathbb{M}$ then for each $w \in W$, $\mathbb{M}, w \models \varphi$ iff $\mathbb{M}', f(w) \models \varphi$
Propositional Modal Logic

Bounded Morphism

**Definition** A bounded morphism between models $M = \langle W, R, V \rangle$ and $M' = \langle W', R', V' \rangle$ is a function $f$ with domain $W$ and range $W'$ such that:

**Atomic harmony:** for each $p \in \text{At}$, $w \in V(p)$ iff $f(w) \in V'(p)$

**Morphism:** if $wRv$ then $f(w)Rf(v)$

**Zag:** if $f(w)R'v'$ then $\exists v \in W$ such that $f(v) = v'$ and $wRv$

**Fact** Counting modalities are not definable in the basic modal language (eg., $\lozenge_1 \varphi$ iff $\varphi$ is true in more than 1 accessible world).
A bisimulation between $\mathcal{M} = \langle W, R, V \rangle$ and $\mathcal{M}' = \langle W', R', V' \rangle$ is a non-empty binary relation $Z \subseteq W \times W'$ such that whenever $wZw'$:

Atomic harmony: for each $p \in \text{At}$, $w \in V(p)$ iff $w' \in V'(p)$

Zig: if $wRv$, then $\exists v' \in W'$ such that $vZv'$ and $w'R'v'$

Zag: if $w'R'v'$ then $\exists v \in W$ such that $vZv'$ and $wRv$
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We write $\mathcal{M}, w \leftrightarrow \mathcal{M}', w'$ if there is a $Z$ such that $wZw'$. 
A **bisimulation** between $M = \langle W, R, V \rangle$ and $M' = \langle W', R', V' \rangle$ is a non-empty binary relation $Z \subseteq W \times W'$ such that whenever $wZw'$:

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**Zag:** if $w'R'v'$ then $\exists v \in W$ such that $vZv'$ and $wRv$

We write $M, w \leftrightarrow M', w'$ iff $\forall \varphi \in \mathcal{L}$, $M, w \models \varphi$ iff $M', w' \models \varphi$.
A **bisimulation** between $\mathcal{M} = \langle W, R, V \rangle$ and $\mathcal{M'} = \langle W', R', V' \rangle$ is a non-empty binary relation $Z \subseteq W \times W'$ such that whenever $wZw'$:

**Atomic harmony:** for each $p \in \text{At}$, $w \in V(p)$ iff $w' \in V'(p)$

**Zig:** if $wRv$, then $\exists v' \in W'$ such that $vZv'$ and $w'R'v'$

**Zag:** if $w'R'v'$ then $\exists v \in W$ such that $vZv'$ and $wRv$

**Lemma** If $\mathcal{M}, w \leftrightarrow \mathcal{M'}, w'$ then $\mathcal{M}, w \leftrightarrow \mathcal{M'}, w'$.
Bisimulation

A **bisimulation** between $\mathcal{M} = \langle W, R, V \rangle$ and $\mathcal{M}' = \langle W', R', V' \rangle$ is a non-empty binary relation $Z \subseteq W \times W'$ such that whenever $wZw'$:

**Atomic harmony:** for each $p \in \text{At}$, $w \in V(p)$ iff $w' \in V'(p)$

**Zig:** if $wRv$, then $\exists v' \in W'$ such that $vZv'$ and $w'R'v'$

**Zag:** if $w'R'v'$ then $\exists v \in W$ such that $vZv'$ and $wRv$

**Lemma** On finite frames, if $\mathcal{M}, w \leftrightarrow \mathcal{M}', w'$ then $\mathcal{M}, w \leftrightarrow \mathcal{M}', w'$. 
The Van Benthem Characterization Theorem

Modal logic is the bisimulation invariant fragment of first-order logic.
The Van Benthem Characterization Theorem

For any first-order formula \( \varphi(x) \), TFAE:

1. \( \varphi(x) \) is invariant for bisimulation
2. \( \varphi(x) \) is equivalent to the standard translation of a basic modal formula.
Logics of Rational Agency
Basic Ingredients

- What are the basic building blocks? (the nature of time (continuous or discrete/branching or linear), how (primitive) events or actions are represented, how causal relationships are represented and what constitutes a state of affairs.)

- Single agent vs. many agents.

- What are the primitive operators?
  - Informational attitudes
  - Motivational attitudes
  - Normative attitudes

- Static vs. dynamic
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Basic Ingredients

- What are the basic building blocks? (the nature of time (continuous or discrete/branching or linear), how (primitive) events or actions are represented, how causal relationships are represented and what constitutes a state of affairs.)

- Single agent vs. many agents.

- What are the primitive operators?
  - Informational attitudes
  - Motivational attitudes
  - Normative attitudes

- Static vs. dynamic
✓ informational attitudes (eg., knowledge, belief, certainty)
✓ time, actions and ability
✓ motivational attitudes (eg., preferences)
✓ group notions (eg., common knowledge and coalitional ability)
✓ normative attitudes (eg., obligations)
End of lecture 1.