Problem Set 6
Introduction to Modal Logic

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1. Prove any two of the following:
   - Let F be a filter over W, prove that F is an ultrafilter if and only if it is proper and maximal, that is, it has no proper extension
   - Prove that a filter F (over W) is an ultrafilter if and only if it is proper and for each pair of subsets X, Y of W we have that $X \cup Y \in F$ iff $X \in F$ or $Y \in F$.
   - Prove that an ultarfilter is non-principal if and only if it contains only infinite sets if and only if it contains all co-finite sets.
2. (**Exercise 2.5.5, pg. 99**) Given a model $\mathfrak{M} = \langle W, R, V \rangle$ and two ultrafilters u and v over W, show that $uR^{ue}v$ if and only if $\{Y \mid l_R(Y) \in u\} \subseteq v$.
3. We first need some definitions. Let $\mathcal{F} = \langle W, R \rangle$ be any frame.

   $\mathcal{F}$ is **weakly dense** if $\mathcal{F}$ satisfies $\forall x \forall y (xRy \rightarrow \exists z (xRz \wedge zRy))$.

   $\mathcal{F}$ is **weakly connected** if $\mathcal{F}$ satisfies $\forall x \forall y \forall z (xRy \wedge xRz \rightarrow zRy \vee y = z \vee yRz$.

   $\mathcal{F}$ is **weakly directed** if $\mathcal{F}$ satisfies $\forall x \forall y \forall z (xRy \wedge xRz \exists u (yRu \wedge zRu))$.

   Prove any two of the following:
   - A frame $\mathcal{F}$ is weakly dense if and only if $\Box\Box\phi \rightarrow \Box\phi$ is valid on $\mathcal{F}$.
   - A frame $\mathcal{F}$ is weakly connected if and only if $\Box(\phi \wedge \Box\phi \rightarrow \psi) \vee \Box(\psi \wedge \Box\psi \rightarrow \phi)$ is valid on $\mathcal{F}$.
   - A frame $\mathcal{F}$ is weakly directed if and only if $\Diamond\Box\phi \rightarrow \Box\Diamond\phi$ is valid on $\mathcal{F}$.