Practice Problems 1
Recursion Theory

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[CT] is *Computability Theory* by Barry Cooper.

1. Prove that each of the following functions is an element Prim.

   (a) For each $k \in \mathbb{N}$, the constant function: $\mathbf{k}(n) = k$ (Example 2.1.2)

   (b) Addition: $p(m, n) = m + n$ (Example 2.1.3)

   (c) Multiplication: $t(m, n) = m \times n$ (Example 2.1.4)

   (d) Exponentiation: $e(m, n) = m^n$ (Exercise 2.1.5)

   (e) Predecessor function (Example 2.1.6):

   $$\delta(m) = \begin{cases} m - 1 & \text{if } m > 0 \\ 0 & \text{if } m = 0 \end{cases}$$

   (f) $sg(n)$ and $\overline{sg}(n)$ defined as follows:

   $$sg(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

   $$\overline{sg}(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n \neq 0 \end{cases}$$

   (g) Divisibility relation (Exercise 2.1.11):

   $$\begin{cases} 1 & \text{if } m \text{ divides } n \\ 0 & \text{otherwise} \end{cases}$$

   (h) Number of divisors: (Exercies 2.1.14) $D(m) = |\{n \mid n \text{ divides } m\}|$

   (i) $f(n_1, n_2, \ldots, n_k) = \max\{n_1, \ldots, n_k\}$ (the largest of the number $n_1, \ldots, n_k$)

2. Suppose that $h(x)$ is primitive recursive. Show $f(x) = h^n(x)$ is primitive recursive.

3. Suppose that $f(x, y)$ is primitive recursive. Show that $g(x, y) = f(y, x)$ is primitive recursive.

4. Suppose that g and h are primitive recursive. Show that the function defined by

$$\begin{aligned} f(0,m) &= g(m) \\ f(n+1,m) &= h(m,n,f(n,m)) \end{aligned}$$

is also primitive recursive.

5. Let A be the Ackermann function defined by (see pg. 17 of [CT])

$$\begin{aligned} A(m,0) &= m+1 \\ A(0,n+1) &= A(1,n) \\ A(m+1,n+1) &= A(A(m,n+1),n) \end{aligned}$$

Prove

(a) $A(m,0) = m+1$

(b) $A(m,1) = m+2$

(c) $A(m,2) = 2m+3$

A formal proof that $A \notin \textsc{Prim}$ can be found in Mendelson *Introduction to Logic*, pg. 247 Exercises 1 - 11. A sketch of the argument is

(a) Prove that for each $n \in \mathbb{N}$, $A(n,m) > n$.

(b) Prove A is monotonic in each variable, eg. if $x < z$, then $A(x,y) < A(z,y)$

(c) Prove $A(n,m+1) \geq A(n+1,m)$

(d) Prove that for each primitive recursive function f there exists m such that $f(x_1,\ldots,x_n) < A(\max\{x_1,\ldots,x_n\},m)$ for each $x_1,\ldots,x_n$

(e) Prove $A(x,x)+1$ is not primitive recursive.