Lecture 11: Topics in Formal Epistemology

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Caput Logic, Language and Information: Social Software

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Introduction

- What are we trying to model?

- Formal Models
  - Epistemic Logic; Aumann Structures
  - (Type Spaces, Knowledge Structures, Bayesian Structures)

- Common Knowledge
  - Levels of Knowledge

- A Model of Knowledge for Social Software

- Communication Graphs

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Gettier Cases

Suppose that Smith and Jones have applied for a certain job.

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(a) Jones is the man who will get the job, and Jones has ten coins in his pocket.

Hence, Smith is justified in believing

(b) The man who will get the job has ten coins in his pocket

Imagine, further, that **unknown** *to Smith, he himself, not Jones will get the job. And, also,* **unknown** *to Smith, he himself has ten coins in his pocket.*

Gettier Cases

Let $P :=$ *The man who will get the job has ten coins in his pocket*

Unknown to Smith, he himself, not Jones will get the job. And, also, **unknown** to Smith, he himself has ten coins in his pocket.

- It is true that P
- Smith believes that P
- Smith is justified in believing that P
- But we do not say that Smith **knows** that P

E. Gettier. *Is Justified True Belief Knowledge?*. 1963.

Digression: What does Amazon.com know?

Does *Amazon.com* **know** my address?

X knows p iff

1. X has *ready* access to p
2. p is true
3. X can make use of the informational content of p (i.e., X can exercise certain capacities dependent on its knowing p).

S. Chopra and L. White. *Attribution of Knowledge to Artificial Agents and their Principals*. 2005.

Formal Models of Knowledge

There is a huge literature!

FHMV. *Reasoning about Knowledge.* 1995.

- Epistemic Logic
- Aumann Structures
- History Based Structures
- (Knowledge Structures)
- (Type Spaces)
- (Probability Spaces)

Possible Worlds

A state of the worlds is a **complete** description of the relevant information — including all ground facts, what each agent knows, what every agent knows the other agent knows, and so on....

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Is there a *vicious circle?*

J. Barwise and L. Moss. *Vicious Circles.* 1996.

Epistemic Logic

Let $\mathcal{A}$ be a set of agents and At a set of atomic propositions.

Modal Language the smallest set generated by ($p \in \mathsf{At}$)

$$\phi := p \mid \neg\phi \mid \phi \wedge \psi \mid \Box_i \phi$$

Kripke Frame $\langle W, \{R_i\}_{i \in \mathcal{A}} \rangle$ where $R_i \subseteq W \times W$

Kripke Model $\langle W, \{R_i\}_{i \in \mathcal{A}}, V \rangle$ where $V : \mathsf{At} \rightarrow 2^W$

Truth in a Model

1. $\mathcal{M}, s \models p$ if $p \in V(s)$
2. $\mathcal{M}, s \models \phi \wedge \psi$ if $\mathcal{M}, s \models \phi$ and $\mathcal{M}, s \models \psi$
3. $\mathcal{M}, s \models \neg\phi$ if $\mathcal{M}, s \not\models \phi$
4. $\mathcal{M}, s \models \Box_i \phi$ if for each $v \in W$, if wRv, then $\mathcal{M}, v \models \phi$

Some Epistemic Axioms

Axiom	Property	Frame Property
$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$		(valid in all frames)
$\Box\phi \to \phi$	Truth Axiom	Reflexivity
$\Box\phi \to \Box\Box\phi$	Positive Introspection	Transitivity
$\neg\Box\phi \to \Box\neg\Box\phi$	Negative Introspection	Euclidean
$\neg\Box\bot$	Consistency	Serial

FHMV. *Reasoning about Knowledge*. 1995.

Aumann Structures

R. Aumann. *Interactive Epistemology I: Knowledge*. 1999.

Knowledge function: κ with domain W and range an abstract set. for $w \in W$, $\kappa(w)$ is the signal the agent receives from the outside world when the true state of the world is w.

Information function: $\mathcal{I}(w) := \{w' \in W \mid \kappa(w) = \kappa(w')\}$

- $w \in \mathcal{I}(w)$
- $\mathcal{I}(w)$ and $\mathcal{I}(w')$ are either disjoint or identical

So $\mathcal{I}$ forms a partition of W.

Aumann Structures

Let Σ be the family of all events $(= 2^W)$. Define $K : \Sigma \to \Sigma$ as follows

- $w \in KE$ iff $\mathcal{I}(w) \subseteq E$

K has the following properties:

1. $KE \subset E$
2. $E \subset F$ implies $KE \subset KF$
3. $\overline{KE} \subset K\overline{KE}$

We can take any of the $\kappa, \mathcal{I}, K$ as primitive. Eg. Take $K : \Sigma \to \Sigma$ as primitive.

$$\mathcal{I}(w) := \overline{K\overline{\{w\}}}$$

Epistemic Logic with Different Terminology

Given an arbitrary set operator $\mathsf{P} : 2^W \to 2^W$, the following properties can be assumed of P. Let E, F be any two subsets W

P1 $\mathsf{P}(E) \cap \mathsf{P}(F) = \mathsf{P}(E \cap F)$

P2 $\cap_{j \in J} \mathsf{P}(E_j) = \mathsf{P}(\cap_{j \in J} E_j)$, for any index set J[a]

P3 $\mathsf{P}(E) \subseteq E$

P4 $\mathsf{P}(E) \subseteq \mathsf{P}(\mathsf{P}(E))$

P5 $\overline{\mathsf{P}(E)} \subseteq \mathsf{P}(\overline{\mathsf{P}(E)})$

P6 $\mathsf{P}(E) \subseteq \overline{\mathsf{P}(\overline{E})}$

---

[a]When $J = \varnothing$, we get $\mathsf{K}(\Omega) = \Omega$

Let $P : W \to 2^W$ be any function. We define the following properties of P:

Reflexive $\forall w \in W,\ w \in P(w)$

Transitive $\forall w, v \in W,\ v \in P(w) \Rightarrow P(v) \subseteq P(w)$

Euclidean $\forall w, v \in W,\ v \in P(w) \Rightarrow P(w) \subseteq P(v)$

Serial $\forall w \in W,\ P(w) \neq \varnothing$

J. Halpern. *Set Theoretic Completeness of Epistemic and Conditional Logic.* 2001.

Common Knowledge

Given two agents i and j, how do we formalize the statement

$$\phi \text{ is common knowledge between } i \text{ and } j$$

in terms of i, j, ϕ and (private knowledge)?

Let γ be the statement *it is common knowledge between* i *and* j *that* ϕ.

For a good exposition see,

M. Chwe. *Rational Ritual*. 2004.

Three Views of Common Knowledge

1. $\gamma :=$ i knows that ϕ, j knows that ϕ, i knows that j knows that ϕ, j knows that i knows that ϕ, i knows that j knows that i knows that ϕ, ...

   D. Lewis. *Convention, A Philosophical Study.* 1969.

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3. There is a *shared situation* s such that

   - s entails ϕ
   - s entails i knows ϕ
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Common Knowledge: Iterated View

Define $K^m : 2^W \to 2^W$ for $m \geq 1$ by

$$K^1 E := \cap_{i \in \mathcal{A}} K_i E$$

$$K^{m+1} E := K^1(K^m(E))$$

$K^1 E$ means *everyone knows* E

$K^2 E$ means *everyone knows that everyone knows* E

Define $K^\infty : 2^W \to 2^W$

$$K^\infty E := K^1 E \cap K^2 E \cap \cdots \cap K^m E \cap \cdots$$

It is easy to verify that $K_i K^\infty E = K^\infty E$ for all i

Three Views of Common Knowledge

In Relational Models (including Aumann Structures), the first two views are simply equivalent and it is not clear how to represent the third view.

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$C^0_{1,2}E := E$

$C^{\kappa+1}_{1,2}E := E \cap K_1(C^{\kappa}_{1,2}E) \cap K_2(C^{\kappa}_{1,2}E)$

$C^{\lambda}_{1,2}E := \cap_{\kappa<\lambda} C^{\lambda}_{1,2}$

Let $C_{1,2}E = C^{\kappa}_{1,2}E$ where κ is the least ordinal where $C^{\kappa}_{1,2}E = C^{\kappa+1}E$

Fact In every Relational Model, the above procedure stabilizing at $\kappa \leq \omega$.

Three Views of Common Knowledge

In topological models the first two views can be distinguished.

J. van Benthem and D. Sarenac. *The Geometry of Knowledge.* 2005.

All three can be separated in the *situation calculus.*

J. Barwise. *Three Views of Common Knowledge.* TARK, 1987.

Models of Knowledge for Social Software

Let P be a *plan* the agent is carrying out.

$\pi(P) = \{w \mid w \in C \text{ and } w \text{ enables } P\}$

where C is the *context of the plan* P (decided by an observer).

An agent believes ϕ, provided $\pi(P) \subseteq (\phi)^{\mathcal{M}}$.

An agent knows ϕ provide the agent believes ϕ and ϕ is true.

Models of Knowledge for Social Software

If an agent comes to a point in her plan where her appropriate action is

$$\text{If } \phi \text{ do } \alpha \text{ else do } \beta$$

and she does α, then we will say that she i-believes ϕ.

R. Parikh. *WHAT do we know and what do WE know.* TARK, 2005.

Recall the following example

Ann would like Bob to attend her talk; however, she only wants Bob to attend if he is interested in the subject of her talk, not because he is just being polite.

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Taking a cue from computer science, we can ask is this procedure correct?

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Yes, if

1. Ann knows about the talk.

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Yes, if

1. Ann knows about the talk.

2. Bob knows about the talk.

Recall the following example

Yes, if

1. Ann knows about the talk.

2. Bob knows about the talk.

3. Ann knows that Bob knows about the talk.

Recall the following example

Yes, if

1. Ann knows about the talk.
2. Bob knows about the talk.
3. Ann knows that Bob knows about the talk.
4. Bob *does not* know that Ann knows that he knows about the talk.

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1. Ann knows about the talk.
2. Bob knows about the talk.
3. Ann knows that Bob knows about the talk.
4. Bob *does not* know that Ann knows that he knows about the talk.
5. *And nothing else.*

Basic Definitions

R. Parikh and R. Ramanujam. *A History Based Semantics of Messages.* 2003.

- Let E be a fixed set of events.
- Elements of $E^* \cup E^\omega$ will be called histories. Three types: 1. local histories, 2. finite global histories and 3. infinite global histories.
- Given two histories H' and H, write $H \preceq H'$ to mean H is a finite prefix of H'.
- Define $\mathsf{FinPre}(\mathcal{H}) = \{H \mid H \in E^*, H \preceq H', \ \& \ H' \in \mathcal{H}\}$.
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Protocols

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Note that any (social) procedure will generate a protocol, but not all protocols are generated by some procedure.

History Based Structures

A history based structure based on a set of events E is a tuple $\langle \mathcal{H}, E_1, \ldots, E_n \rangle$ where $\mathcal{H}$ is a protocol and $E_i \subseteq E$ for each $i \in \mathcal{A}$.

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History based structures are "low-level descriptions" of a social situation.

History Based Frames

For each $i \in \mathcal{A}$ define $\lambda_i : \mathsf{FinPre}(\mathcal{H}) \longrightarrow E_i^*$ to be the local view function of agent i.

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A history based frame is a tuple $\langle \mathcal{H}, E_1, \ldots, E_n, \lambda_1, \ldots, \lambda_n \rangle$.

History Based Frames: Local View Functions

Note that for each $i \in \mathcal{A}$, λ_i is *any* function on $\mathsf{FinPre}\mathcal{H}$. Possible

Assumptions:

- The agents' local clock is *consistent* with the global clock:

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  $\lambda_i(H)$ is *embeddable* in H.

- Assumptions about the agents reasoning capabilities, i.e., *perfect recall* :
  $\lambda_i(H)$ is obtained by mapping each event "seen" by i into itself and each event not seen by i to the empty string.

Temporal Epistemic Logic

- A formula $\phi \in {}_{n}^{KT}$ can have the following form

$$\phi := p \mid \neg\phi \mid \phi \wedge \psi \mid K_i\phi \mid \bigcirc \phi \mid \phi \, U \psi$$

Temporal Epistemic Logic

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$$\phi := p \mid \neg\phi \mid \phi \wedge \psi \mid K_i\phi \mid \bigcirc \phi \mid \phi\, U \psi$$

  - $K_i\phi$: "agent i knows ϕ"
  - $\bigcirc\phi$: "ϕ is true at the next moment"
  - $\phi U \psi$: "ϕ is true until ψ becomes true"
  - $F\phi$: $\top U \phi$: "ϕ is true sometime in the future"
  - $G\phi$: $\neg F \neg \phi$: "ϕ is always true"

Temporal Epistemic Logic: Truth

If $H \in \mathcal{H}$ is a **infinite** global history, then $H, t \models \phi$ is intended to mean ϕ *is true at time* t *in history* H:

1. $H, t \models \phi \, U \psi$ iff there exists $m \geq t$ such that $H, m \models \psi$ and for all l such that $t \leq l < m$, $H, l \models \phi$

2. $H, t \models K_i \phi$ iff for all $H' \in \mathcal{H}$ such that $H_t \sim_i H'_t$, $H', t \models \phi$

2′. $H, t \models K_i \phi$ iff for all $m \geq 0$, for all $H' \in \mathcal{H}$ such that $H_t \sim_i H'_m$, $H', m \models \phi$

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Returning to the Example

Let m_{AC} be the event that represents the message from Ann to Carol

Let m_{CB} be the event that represents the message from Carol to Bob

Let P be the proposition that means “Ann’s talk is at 2 PM".

$$H m_{AC} H' m_{CB} \models \neg K_B K_A K_B P$$

Temporal Epistemic Logic: Axiomatizations

Sound and complete axiomatizations (using a different semantics)

Temporal Epistemic Logic: Axiomatizations

Sound and complete axiomatizations (using a different semantics)

1. Linear time: [HvdMV, 2003]

2. Branching time: [vdMW, 2004]

3. Past time operators: [FvdMR, 2004]

4. Group strategies (ATL): [GvD, 2002]

Histories or Runs?

See [FHMV] and [HvdMV] for more information.

- Let L be a set of local states.
- A run $r \in \mathcal{R}$ is a function $r : \mathbb{N} \rightarrow L^{n+1}$
- A system for n agents is a set $\mathcal{R}$.
- $r(t)$ has the form $\langle l_e, l_1, \ldots, l_n \rangle$, where l_e is the state of the environment, l_i for $i = 1, \ldots, n$ is the local state of each agent.
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- An interpreted system $\mathcal{I} = (\mathcal{R}, \pi)$, where $\mathcal{R}$ is a system and $\pi : (\mathcal{R} \times \mathbb{N}) \times \mathsf{At} \rightarrow \{\mathbf{true}, \mathbf{false}\}$
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(Some!) Relevant Papers

- Histories or Runs?

  EP. *Comments on History Based Structures.* JAL, 2006.

- Bringing in common knowledge creates *a lot* of complexity.

  Halpern, van der Meyden and Vardi. *Reasoning about Knowledge and Common Knowledge in a Distributed Environment.* 1989.

  J. van Benthem and EP. *The Tree of Knowledge: Towards a Common Perspective.* 2006.

- Extending the language — stit operators

  J. Horty. *Agency and Deontic Logic.* 2001.

  EP, R. Parikh and E. Cogan. *The Logic of Knowledge Based Obligation.* KRA 2006.

Knowledge, Time and Communication

The occurrence of an event, such as communication, changes the agents' states of knowledge. How should we model this *change* of information?

There have been a number of recent papers concerned with adding a notion of update to otherwise static Kripke structures (Dynamic Epistemic Semantics).

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Assume:

- Agents are assumed to communicate (only) according to a *communication graph*
- The starting point is not Kripke structures, but rather subset models (Topologic models).

EP and R. Parikh. *The Logic of Communication Graphs.* 2005.

Background: Topologic

Topologic is a bimodal logic with a knowledge modality (K) and an effort modality ($\Diamond$) first studied by [MP]

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Subset Frame: $\langle W, \rangle$

- W is a set of states
- $\subseteq 2^W$ is a set of subsets of W, i.e., a set of *observations*

Neighborhood Situation: Given a subset frame $\langle W, \rangle$, (w, U) is called a neighborhood situation, where $w \in U$ and $U \in$.

Model: $\langle W, , V \rangle$, where $V : \Phi_0 \to 2^W$ is a valuation function.

Topologic: Truth

Truth: Formulas are interpreted at neighborhood situations.

- $w, U \models p$ iff $w \in V(p)$
- $w, U \models \neg\phi$ iff $w, U \not\models \phi$
- $w, U \models \phi \wedge \psi$ iff $w, U \models \phi$ and $w, U \models \psi$
- $w, U \models K\phi$ iff for all $v \in U$, $v, U \models \phi$
- $w, U \models \Diamond\phi$ iff there is a $V \in$ such that $w \in V \subseteq U$ and $w, V \models \phi$

From Topologic to Communication Graphs

- *Topologic*: a bimodal logic with a knowledge modality (K) and an effort modality ($\Diamond$).

$$\phi \rightarrow \Diamond K \phi$$

"if ϕ is true then after some 'effort' ϕ is known"

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$$K_i \phi \rightarrow \Diamond K_j \phi$$

"if i knows ϕ then after some communication j knows ϕ"

An Example

If Bush wants some information from a particular CIA operative, say Bob, he must get this information through Goss. Suppose that ϕ is the exact whereabouts of Bin Laden and

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So, there is a connection between the fact that the CIA restricts the flow of information and the truth of certain formulas in our language, such as

$$\bigwedge_{i \in \mathsf{RoomD.028}} \neg\Diamond K_i \phi \qquad \text{and} \qquad \neg K_{\text{Bush}}\phi \wedge \Box(K_{\text{Bush}}\phi \rightarrow K_{\text{Goss}}\phi)$$

Communication Graphs

Let $\mathcal{A}$ be a set of agents. A **communication graph** is a directed graph $G_{\mathcal{A}} = (\mathcal{A}, E)$ such that for all $i \in \mathcal{A}$, $(i, i) \notin E$

$(i, j) \in E$ means that i can directly receive information from agent j, *without* j knowing this fact.

Assume that:

1. all the agents share a common language
2. the agents make available all possible pieces of information

The Logic of Communication Graphs: Introduction

Suppose that $\mathcal{G} = (\mathcal{A}, E_{\mathcal{G}})$ is a fixed communication graph.

Assume that agents are initially given some private information and communicate according to the communication graph $\mathcal{G}$.

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Assume that agents are initially given some private information and communicate according to the communication graph $\mathcal{G}$.

More formally, initially, each agent i knows or is informed (say by nature) of the truth values of a certain subset At_i of propositional variables, and the At_i *as well as this fact are common knowledge*.

Thus the other agents know that i knows the truth values of elements of At_i, but, typically, not what these values actually are.

The Logic of Communication Graphs: Introduction

Let W be the set of boolean valuations on At. An element $v \in W$ is called a state.

Call any vector of partial boolean valuations $\vec{v} = (v_1, \ldots, v_n)$ consistent if for each $p \in dom(v_i) \cap dom(v_j)$, $v_i(p) = v_j(p)$ for all $i, j = 1, \ldots, n$.

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All this information (including the structure of the communication graph) is common knowledge and only the precise values of the v_i are private.

A Logic for Communication Graphs: Semantics

Fix a communication graph $\mathcal{G} = (\mathcal{A}, E)$ and a set of states W.

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A communication event is a tuple (i, j, ϕ) intended to mean that i learns information ϕ from j, where

- ϕ is a ground (propositional) formula
- there is an edge between i and j in $\mathcal{G}$

Let $\Sigma_{\mathcal{G}}$ be the set of all such events (based on communication graph $\mathcal{G}$).

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A history is a finite sequence of events, $\lambda_i(H)$ be i's local history.

Assume that λ_i satisfies the perfect recall assumption.

A Logic for Communication Graphs: Semantics

A communication graph frame is a pair $\langle \mathcal{G}, \vec{\mathsf{At}} \rangle$ where $\mathcal{G}$ is a communication graph, and $\vec{\mathsf{At}} = (\mathsf{At}_1, ..., \mathsf{At}_n)$ is an assignment of sub-languages to the agents.

A communication graph model based on a frame $\langle \mathcal{G}, \vec{\mathsf{At}} \rangle$ is a triple $\langle \mathcal{G}, \vec{\mathsf{At}}, \vec{v} \rangle$, where $\vec{v}$ is a consistent vector of partial boolean valuations for $\vec{\mathsf{At}}$.

Uncertainty of the Agents

1. Agents are uncertain about the actual state of the world.
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Let w be a state and H a finite history. Define the relation $\sim_i$ as follows: $(w, H) \sim_i (v, H')$ iff $w_{|\mathsf{At}_i} = v_{|\mathsf{At}_i}$ and $\lambda_i(H) = \lambda_i(H')$.

Definition of Truth I

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   Suppose that i learns $p \vee q$ from j, but j is not connected, directly or indirectly, to anyone who might know the initial truth value of q. In this case i has learned *more* than $p \vee q$, i has learned p as well.

Definition of Truth II

Introduce a propositional symbol L which is satisfied only by legal pairs (w, H), denoted $L(w, H)$.

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- $w, \epsilon \models_{\mathcal{M}} L$
- $w, H; (i, j, \phi) \models_{\mathcal{M}} L$ iff $w, H \models_{\mathcal{M}} L$ and $w, H \models_{\mathcal{M}} K_j\phi$
- $w, H \models_{\mathcal{M}} p$ iff $w(p) = 1$, where $p \in \mathsf{At}$
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- $w, H \models_{\mathcal{M}} \Diamond\phi$ iff $\exists H', \ H \preceq H', L(w, H')$, and $w, H' \models_{\mathcal{M}} \phi$
- $w, H \models_{\mathcal{M}} K_i\phi$ iff $\forall (v, H')$ if $(w, H) \sim_i (v, H')$, and $L(v, H')$, then $v, H' \models_{\mathcal{M}} \phi$

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The agents learn at least as much as what is implied by the sequence of communications.

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2. $X_i(w, H; (i, j, \phi)) = X_i(w, H) \cap \hat{\phi}$
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Theorem Let $\mathcal{M} = \langle \mathcal{G}, \vec{\mathsf{At}}, \vec{v} \rangle$ be any communication graph model and ϕ a ground formula. If $X_i(w, H) \subseteq \widehat{\phi}$, then $(w, H) \models_{\mathcal{M}} K_i(\phi)$.

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The converse is not true

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The structure of the communication graph is reflected in the set of valid formulas

Theorem Let $\mathcal{G} = (\mathcal{A}, E)$ be a communication graph. Then $(i, j) \in E$ if and only if, for all $l \in \mathcal{A}$ such that $l \neq i$ and $l \neq j$ and all ground formulas ϕ, the scheme

$$K_j\phi \wedge \neg K_l\phi \rightarrow \Diamond(K_i\phi \wedge \neg K_l\phi)$$

is valid in all communication graph models based on $\mathcal{G}$.

Next Week: (**Last Class**) Topics in Voting Theory