

# An Invitation to Modal Logic: Lecture 3

Philosophy 150

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# Plan

- ✓ Motivating Examples
- ✓ Formalizing the muddy children puzzle, Basic Modal Logic I

11/30: More about truth of modal formulas.

12/3: Basic Modal Logic III

12/5: Dynamics in Logic I

12/7: Dynamics in Logic II

Goal for today: Understand how the basic semantics for modal logic works.

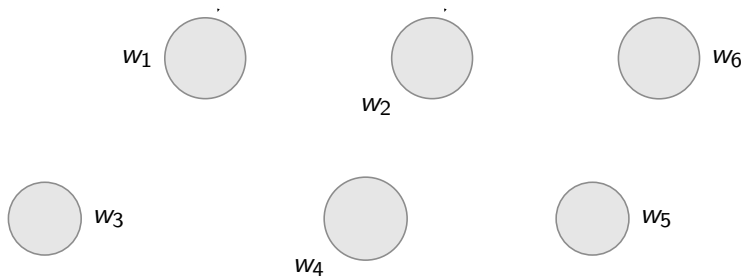
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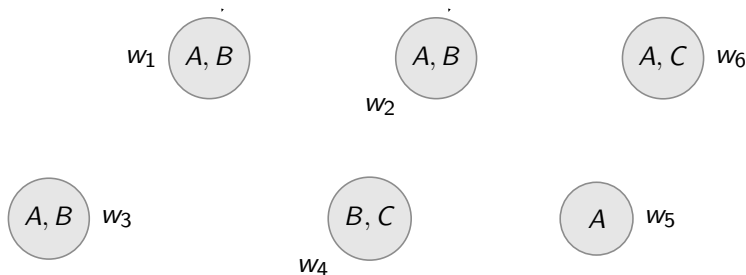
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- 2.



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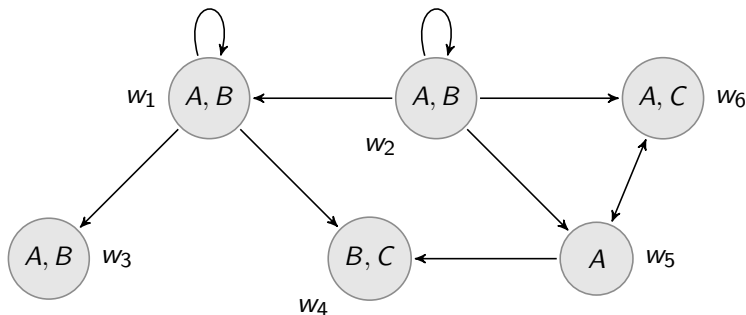
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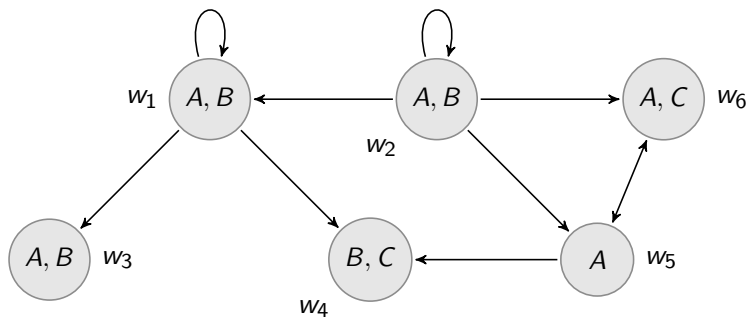


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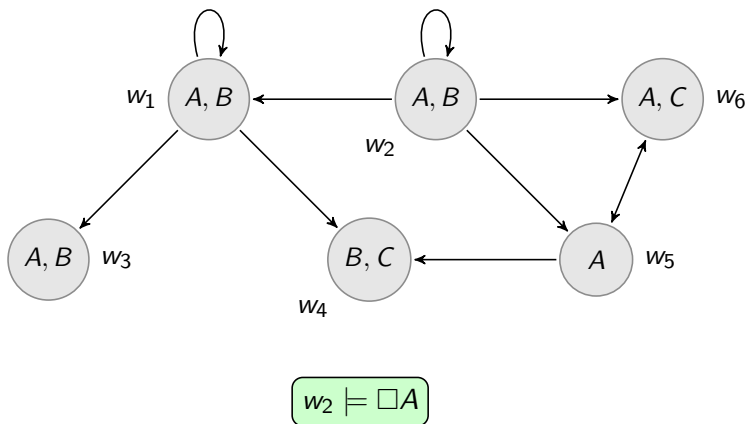
A **Kripke structure** is

1. A set of states, or worlds (each world specifies the truth value of all propositional variables)
2. A **relation** on the set of states (specifying the “relevant situations”)

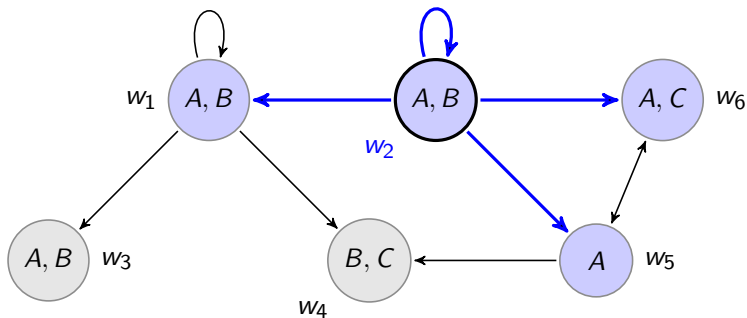






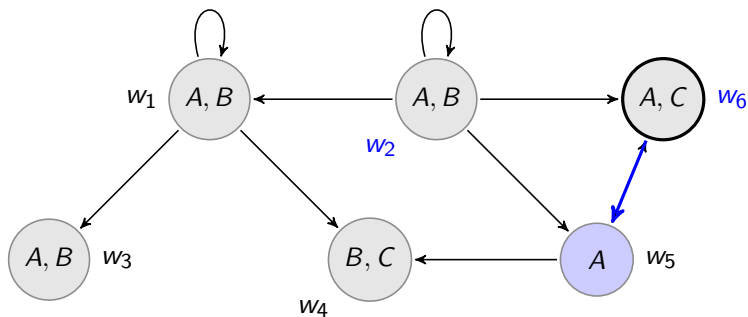


- $\Box P$  is true at state  $w$  iff  $P$  is true in **all accessible worlds**.  
 $w \models \Box P$  iff for all  $v$ , if  $wRv$  then  $v \models P$



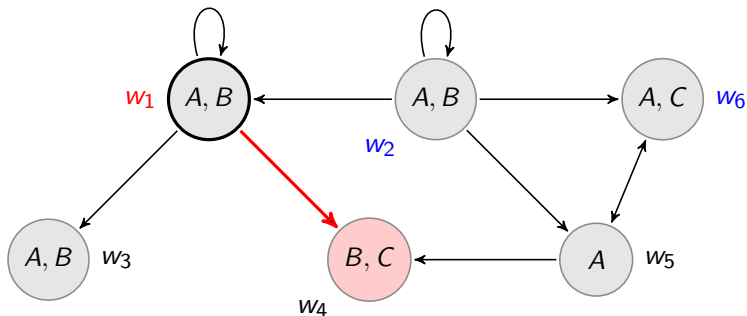
$$w_2 \models \Box A$$

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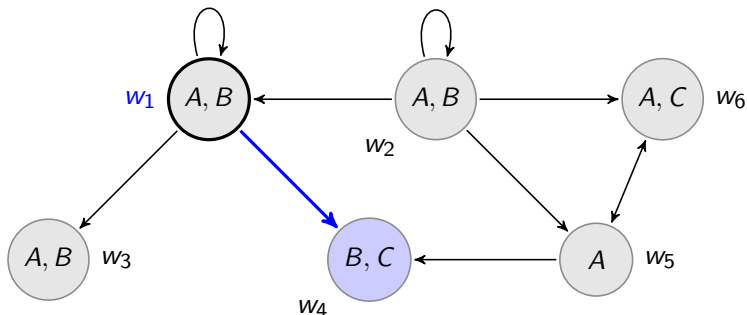
$w_2 \models \Box A$  and  $w_6 \models \Box A$

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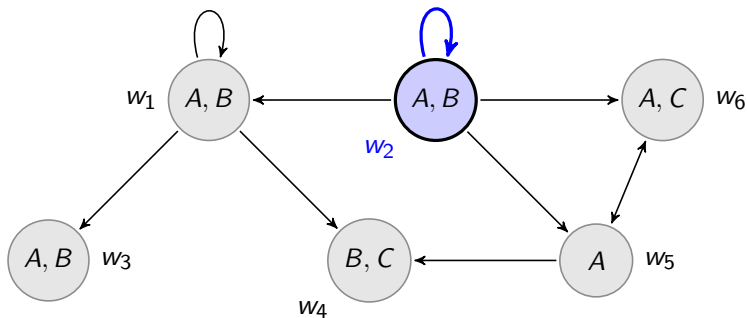
$w_2 \models \Box A$  and  $w_6 \models \Box A$  and  $w_1 \not\models \Box A$

- $\Box P$  is true at state  $w$  iff  $P$  is true in **all accessible worlds**.  
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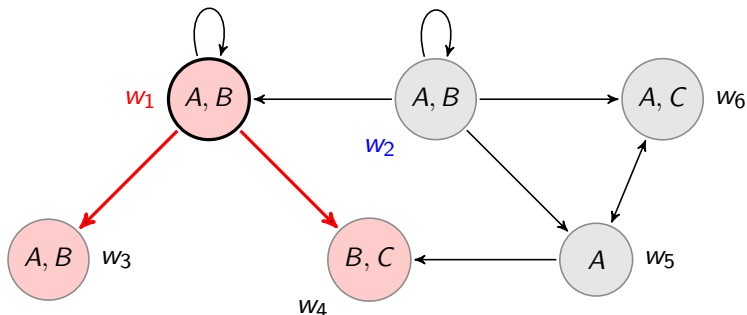
$$w_1 \models \Diamond C$$

- $\Diamond P$  is true at state  $w$  iff  $P$  is true at **some accessible world**.  
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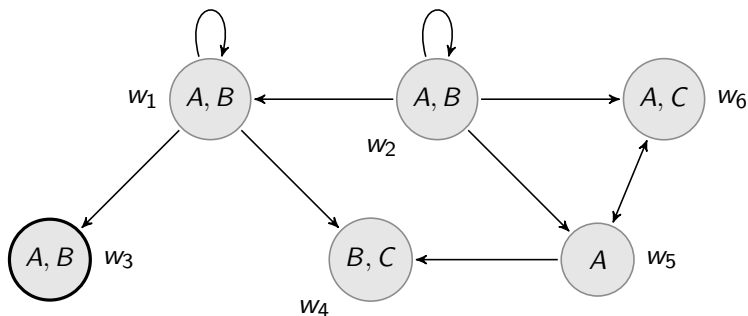
$w_1 \models \Diamond C$  and  $w_2 \models \Diamond B$

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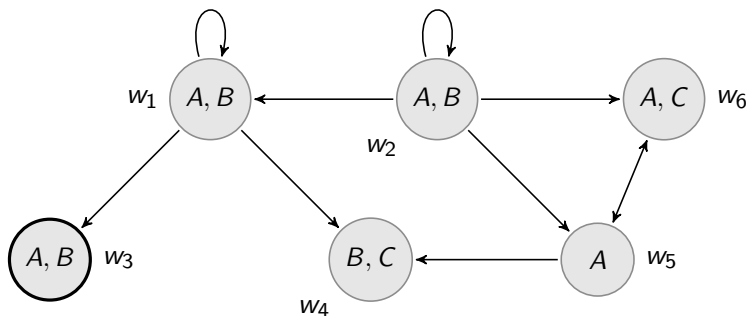
$w_1 \models \Diamond C$  and  $w_2 \models \Diamond B$  and  $w_1 \not\models \Diamond(\neg A \wedge \neg C)$

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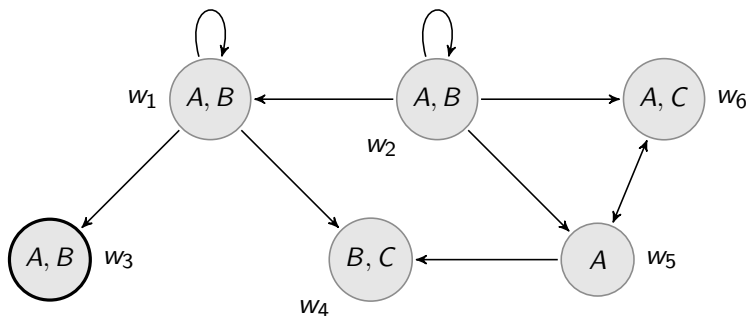
$w_3 \models A$  and  $w_3 \not\models \Diamond A$

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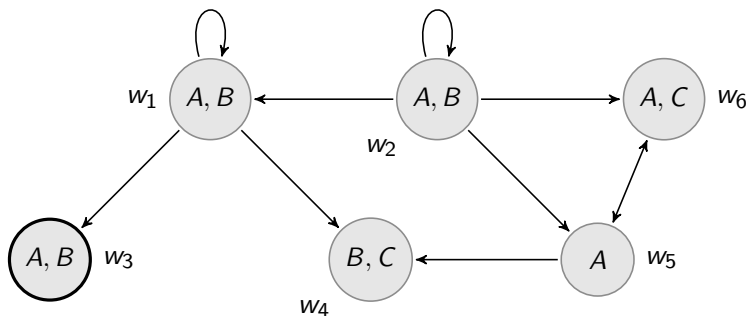
$w_3 \models A$  and  $w_3 \not\models \Diamond A$  and  $w_3 \not\models \Diamond(A \vee \neg A)$

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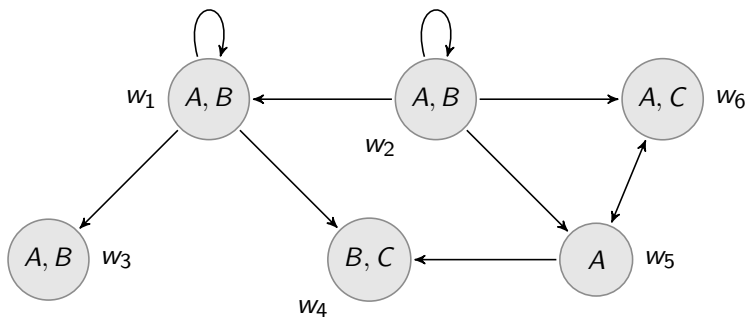
$w_3 \not\models C$  and  $w_3 \models \Box C$

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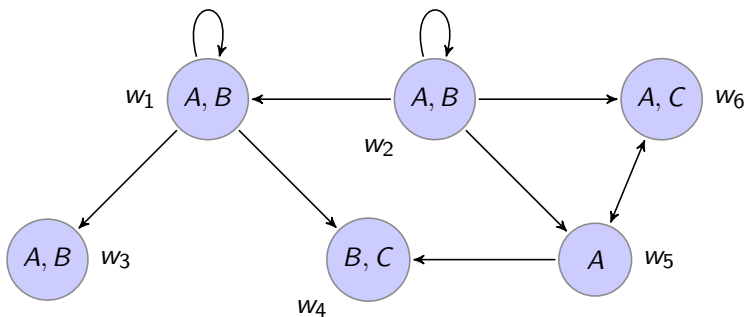
$w_3 \not\models C$  and  $w_3 \models \Box C$  and  $w_3 \models (C \wedge \neg C)$

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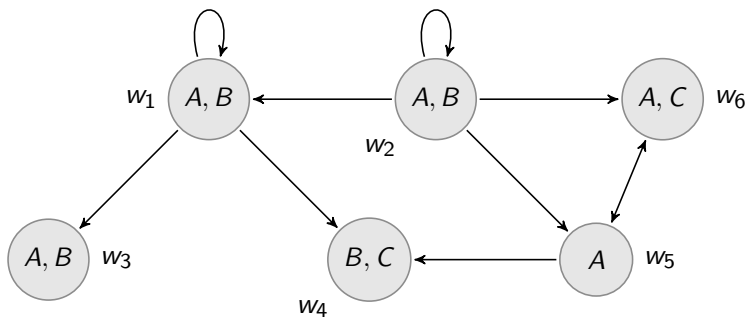
Where is  $\Box A \rightarrow A$  true?

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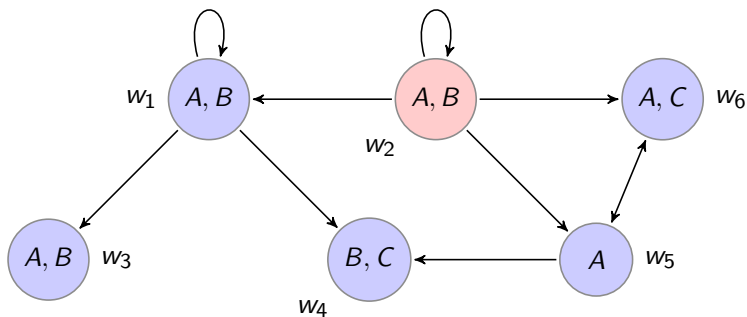
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## Some Facts

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- ▶  $\Box P \leftrightarrow \neg \Diamond \neg P$  is true at any state in any Kripke structure.

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- ▶  $\Box P \leftrightarrow \neg \Diamond \neg P$  is true at any state in any Kripke structure.
- ▶ It is not true that  $\Diamond P \rightarrow \Box P$  is true at any state in any Kripke structure.

## More Facts

Determine which of the following formulas are true at any state in any Kripke structure:

1.  $\Box P \rightarrow \Diamond P$
2.  $\Box(P \vee \neg P)$
3.  $\Box P \rightarrow P$
4.  $P \rightarrow \Box \Diamond P$
5.  $\Diamond(P \vee Q) \rightarrow \Diamond P \vee \Diamond Q$

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## Some Facts

- ▶  $\Box P \rightarrow P$  is true at any state in any Kripke structure where each state is accessible from itself.
- ▶  $\Box P \rightarrow \Diamond P$  is true at any state in any Kripke structure where each state has at least one accessible world.

Can you think of properties that force each of the following formulas to be true at any state in any appropriate Kripke structure?

1.  $\Diamond P \rightarrow \Box P$

2.  $\Box P \rightarrow \Box \Box P$

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Kripke structures, or more generally **relational structures**, are important in

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Logic is not just about formalizing arguments! It can help us study mathematical structures.

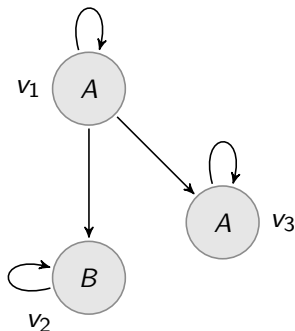
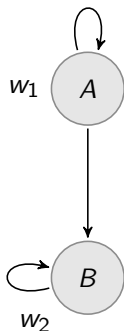
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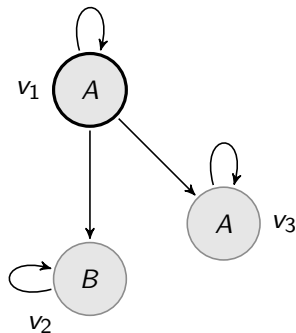
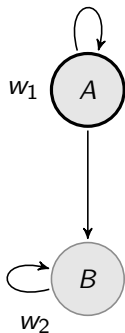
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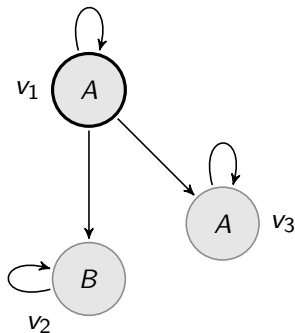
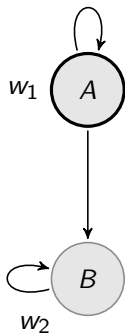
**What “good” means will be discussed in Philosophy 151.**



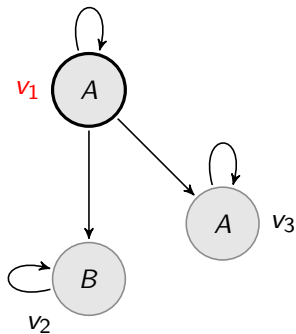
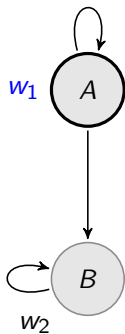
What is the difference between states  $w_1$  and  $v_1$ ?



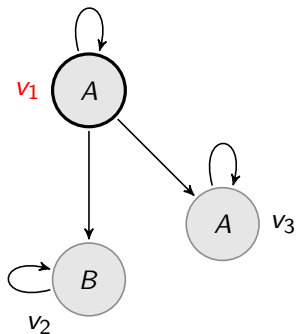
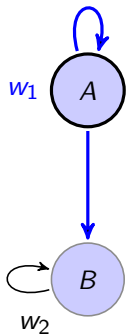
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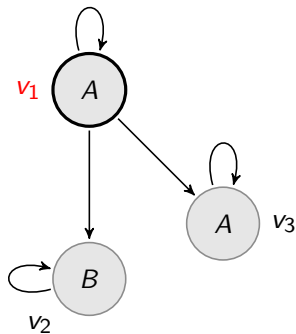
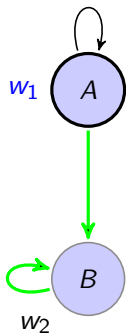
Is there a **modal formula** true at  $w_1$  but not at  $v_1$ ?



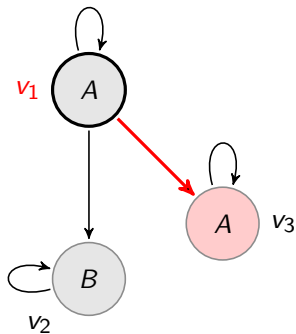
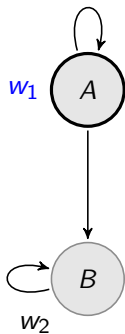
$w_1 \models \Box \Diamond \neg A$  but  $v_1 \not\models \Box \Diamond \neg A$ .



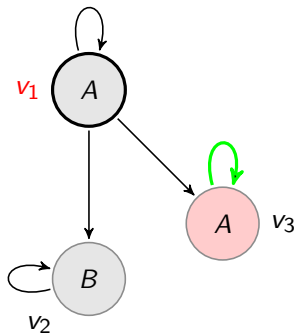
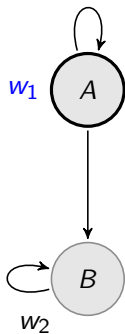
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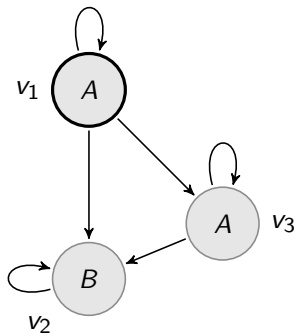
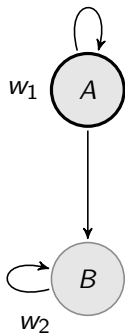
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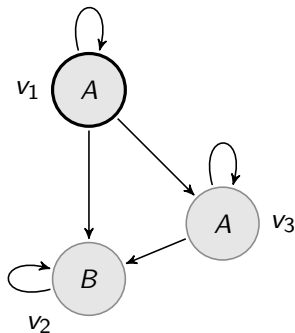
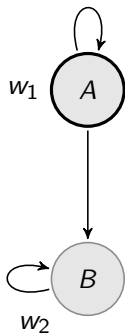
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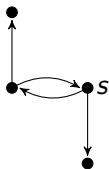
What about now? Is there a modal formula true at  $w_1$  but not  $v_1$ ?



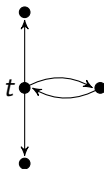
No modal formula can distinguish  $w_1$  and  $v_1$ !

## A More Complicated Example

Which pair of states cannot be distinguished by a modal formula?



$K$



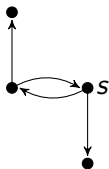
$M$



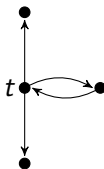
$N$

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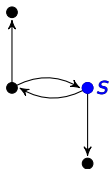


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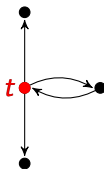
$$\Box(\Box\perp \vee \Diamond\Box\perp)$$

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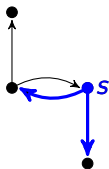


N

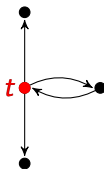
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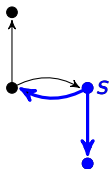


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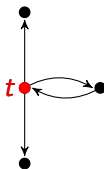
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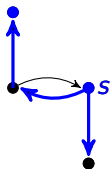


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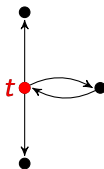
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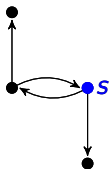


$\mathbb{N}$

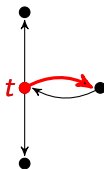
$$\Box(\Box\perp \vee \Diamond\Box\perp)$$

## A More Complicated Example

Which pair of states cannot be distinguished by a modal formula?



$\mathbb{K}$



$\mathbb{M}$

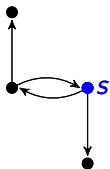


$\mathbb{N}$

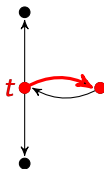
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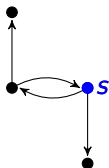


$\mathbb{N}$

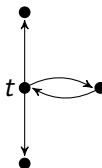
$$\Box(\Box\perp \vee \Diamond\Box\perp)$$

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$\mathbb{N}$

More about this in Philosophy 151!

**Next week:** Focus on epistemic logic.

**Homework:** available on the course website.

Questions?

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