An Invitation to Modal Logic: Lecture 3
Philosophy 150

Eric Pacuit
Stanford University
ai.stanford.edu/~epacuit

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Plan

✓ Motivating Examples

✓ Formalizing the muddy children puzzle, Basic Modal Logic I

11/30: More about truth of modal formulas.

12/3: Basic Modal Logic III

12/5: Dynamics in Logic I

12/7: Dynamics in Logic II
Goal for today: Understand how the basic semantics for modal logic works.
Kripke Structures

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1. A set of states, or worlds

2.
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1. A set of states, or worlds (each world specifies the truth value of all propositional variables)
2. A relation on the set of states (specifying the "relevant situations")

\[
\begin{align*}
A, B &\quad w_1 \\
A, B &\quad w_2 \\
A, B &\quad w_3 \\
B, C &\quad w_4 \\
A, C &\quad w_5 \\
A, C &\quad w_6
\end{align*}
\]
Kripke Structures

A Kripke structure is

1. A set of states, or worlds (each world specifies the truth value of all propositional variables)
2. A relation on the set of states (specifying the “relevant situations”)

\[ A, B \overset{w_1}{\rightarrow} A, B \]
\[ A, B \overset{w_2}{\rightarrow} B, C \]
\[ A, B \overset{w_3}{\rightarrow} A, B \]
\[ A \overset{w_4}{\rightarrow} B, C \]
\[ B, C \overset{w_5}{\rightarrow} A \]
\[ A, C \overset{w_6}{\rightarrow} A, B \]
\( P \) is true at state \( w \) iff \( P \) is true in all accessible worlds.

\( \square P \) iff for all \( v \), if \( wRv \) then \( v \) is accessible from \( w \).
\[ \Box P \text{ is true at state } w \text{ iff } P \text{ is true in all accessible worlds.} \]

\[ w \models \Box P \text{ iff for all } v, \text{ if } wRv \text{ then } v \models P \]
\( \Box P \) is true at state \( w \) iff \( P \) is true in all accessible worlds.

\( w \models \Box P \) iff for all \( v \), if \( wRv \) then \( v \models P \)
\( \Box P \) is true at state \( w \) iff \( P \) is true in all accessible worlds.

\[ \mathcal{M} \models \Box P \text{ iff } \forall v, \text{ if } wRv \text{ then } v \models P \]
\( w_2 \models \Box A \) and \( w_6 \models \Box A \)

- \( \Box P \) is true at state \( w \) iff \( P \) is true in all accessible worlds.
- \( w \models \Box P \) iff for all \( v \), if \( wRv \) then \( v \models P \)
\( P \) is true at state \( w \) iff \( P \) is true in all accessible worlds.

\( w \models \Box P \) iff for all \( v \), if \( w \mathrel{R} v \) then \( v \models P \)
\[ w_1 \models \lozenge C \]

\[ \lozenge P \text{ is true at state } w \text{ iff } P \text{ is true at some accessible world.} \]

\[ w \models \lozenge P \text{ iff there exists } v \text{ such that } wRv \text{ and } v \models P. \]
$w_1 \models \lozenge C$ and $w_2 \models \lozenge B$.

\[\lozenge P \text{ is true at state } w \text{ iff } P \text{ is true at some accessible world.}\]

\[w \models \lozenge P \text{ iff there exists } v \text{ such that } wRv \text{ and } v \models P.\]
$w_1 \models \Box C$ and $w_2 \models \Box B$ and $w_1 \not\models \Box (\neg A \land \neg C)$

$\Diamond P$ is true at state $w$ iff $P$ is true at some accessible world.

$w \models \Diamond P$ iff there exists $v$ such that $wRv$ and $v \models P$. 
$w_3 \models A$ and $w_3 \not\models \Diamond A$

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  - $w \models \Diamond P$ iff there exists $v$ such that $wRv$ and $v \models P$. 

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\[ w_3 \models A \text{ and } w_3 \not\models \Diamond A \text{ and } w_3 \not\models \Diamond (A \lor \neg A) \]

- \( \Diamond P \) is true at state \( w \) iff \( P \) is true at some accessible world.
- \( w \models \Diamond P \) iff there exists \( v \) such that \( wRv \) and \( v \models P \).
$w_3 \not|= C$ and $w_3 |= \Box C$

- $\Box P$ is true at state $w$ iff $P$ is true in all accessible worlds.
  
  $w |= \Box P$ iff for all $v$, if $wRv$ then $v |= P$
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\[ w_3 \not\models C \text{ and } w_3 \models \Box C \text{ and } w_3 \models (C \land \neg C) \]

- \( \Box P \) is true at state \( w \) iff \( P \) is true in all accessible worlds.

\[ w \models \Box P \text{ iff for all } v, \text{ if } wRv \text{ then } v \models P \]
Where is □A → A true?

- □P is true at state w iff P is true in all accessible worlds.
  
  \[ w \models □P \iff \text{for all } v, \text{ if } wRv \text{ then } v \models P \]
Where is □A → A true?

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  \[ w \models □P \text{ iff for all } v, \text{ if } wRv \text{ then } v \models P \]
Where is $\Box A \rightarrow \Box \Box A$ true?

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$w \models \Box P$ iff for all $v$, if $wRv$ then $v \models P$
Where is $\square A \rightarrow \square \square A$ true?

- $\square P$ is true at state $w$ iff $P$ is true in all accessible worlds.
  - $w \models \square P$ iff for all $v$, if $wRv$ then $v \models P$
Some Facts

- □P ∨ ¬□P is always true (i.e., true at any state in any Kripke structure), but what about □P ∨ □¬P?
Some Facts

- $\Box P \lor \neg \Box P$ is always true (i.e., true at any state in any Kripke structure), but what about $\Box P \lor \Box \neg P$?

- $\Box P \land \Box Q \rightarrow \Box (P \land Q)$ is true at any state in any Kripke structure.
Some Facts

- □P ∨ ¬□P is always true (i.e., true at any state in any Kripke structure), but what about □P ∨ □¬P?

- □P ∧ □Q → □(P ∧ Q) is true at any state in any Kripke structure. What about □(P ∨ Q) → (□P ∨ □Q)?
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- □P ↔ ¬◊¬P is true at any state in any Kripke structure.
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- □P ↔ ¬◊¬P is true at any state in any Kripke structure.

- It is not true that ◊P → □P is true at any state in any Kripke structure.
More Facts

Determine which of the following formulas are true at any state in any Kripke structure:

1. $\Box P \to \Diamond P$
2. $\Box (P \lor \neg P)$
3. $\Box P \to P$
4. $P \to \Box \Diamond P$
5. $\Diamond (P \lor Q) \to \Diamond P \lor \Diamond Q$
But, we are not always interested in all Kripke structures.
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For example, consider the epistemic interpretation: A state $\nu$ is accessible from $w$ ($wR\nu$) provided “given the agents information, $w$ and $\nu$ are indistinguishable”.

Some Facts

$\Box P \rightarrow P$ is true at any state in any Kripke structure where each state is accessible from itself.

$\Box P \rightarrow \Diamond P$ is true at any state in any Kripke structure where each state has at least one accessible world.
But, we are not always interested in all Kripke structures.

For example, consider the epistemic interpretation: A state $v$ is accessible from $w$ ($wRv$) provided “given the agents information, $w$ and $v$ are indistinguishable”. What are natural properties?
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For example, consider the epistemic interpretation: A state \( v \) is accessible from \( w \) (\( wRv \)) provided “given the agents information, \( w \) and \( v \) are indistinguishable”. **What are natural properties?**

Eg., for each state \( w \), \( w \) is accessible from itself (\( R \) is a reflexive relation).
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For example, consider the epistemic interpretation: A state $v$ is accessible from $w$ ($wRv$) provided “given the agents information, $w$ and $v$ are indistinguishable”. What are natural properties?

Eg., for each state $w$, $w$ is accessible from itself ($R$ is a reflexive relation).

**Some Facts**

- $\Box P \rightarrow P$ is true at any state in any Kripke structure where each state is accessible from itself.
- $\Box P \rightarrow \Diamond P$ is true at any state in any Kripke structure where each state has at least one accessible world.
Can you think of properties that force each of the following formulas to be true at any state in any appropriate Kripke structure?

1. $\diamond P \rightarrow \Box P$
2. $\Box P \rightarrow \Box \Box P$
Modal logic is a good formal language for “talking about” Kripke structures!
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Kripke structures, or more generally relational structures, are important in

- Philosophical logic
- Linguistics
- Theoretical Computer Science
- Game Theory
- ...
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Logic is not just about formalizing arguments! It can help us study mathematical structures.
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**What “good” means will be discussed in Philosophy 151.**
What is the difference between states $w_1$ and $v_1$?
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Is there a modal formula true at $w_1$ but not at $v_1$?
$w_1 \models \lozenge \lozenge \neg A$ but $v_1 \not\models \lozenge \lozenge \neg A$. 
$w_1 \models \Box \Diamond \neg A$ but $v_1 \not\models \Box \Diamond \neg A$. 
\[ w_1 \models \Box \diamond \neg A \text{ but } v_1 \not\models \Box \diamond \neg A. \]
$w_1 \models \Box \Diamond \neg A$ but $v_1 \not\models \Box \Diamond \neg A$. 
\(w_1 \models \lozenge \Diamond \neg A\) but \(v_1 \not\models \lozenge \Diamond \neg A\).
What about now? Is there a modal formula true at $w_1$ but not $v_1$?
No modal formula can distinguish $w_1$ and $v_1$!
A More Complicated Example

Which pair of states cannot be distinguished by a modal formula?
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Which pair of states cannot be distinguished by a modal formula?

Which pair of states cannot be distinguished by a modal formula?
More about this in Philosophy 151!
Next week: Focus on epistemic logic.

Homework: available on the course website.

Questions?
Email: epacuit@stanford.edu
Website: ai.stanford.edu/~epacuit
Office: Gates 258