An Invitation to Modal Logic: Lecture 5

Philosophy 150

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Plan

✓ Motivating Examples

✓ Formalizing the muddy children puzzle, Introduction to Modal Logic

✓ More about truth of modal formulas

✓ Summary so far.
Digression: A small experiment.

12/5: Focus on Epistemic Logic, Dynamics in Logic

12/7: Dynamics in Logic II

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LP: "P is an epistemic possibility"

KLP: "Ann knows that she thinks P is possible"

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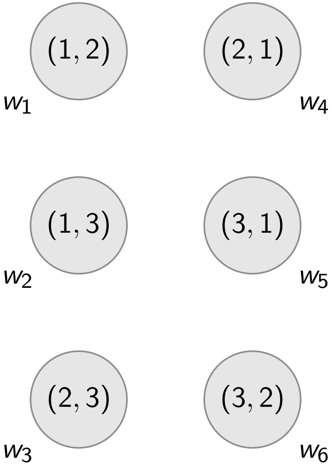

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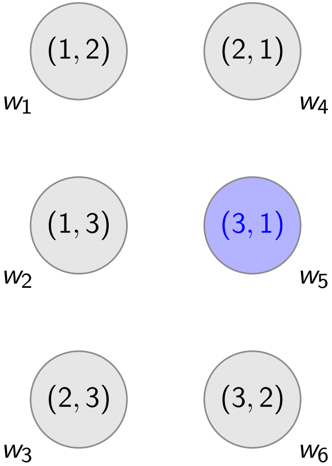

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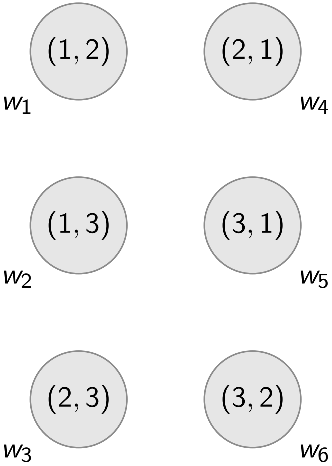

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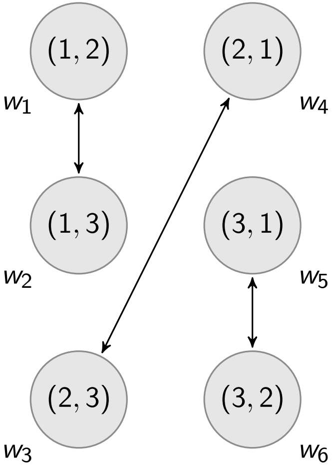

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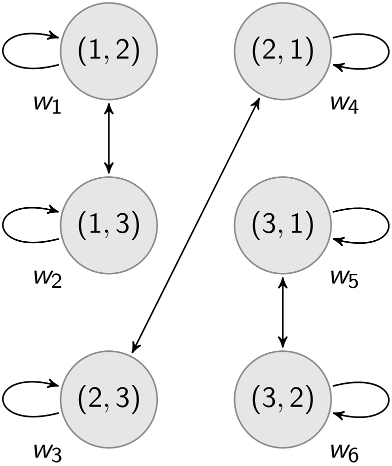

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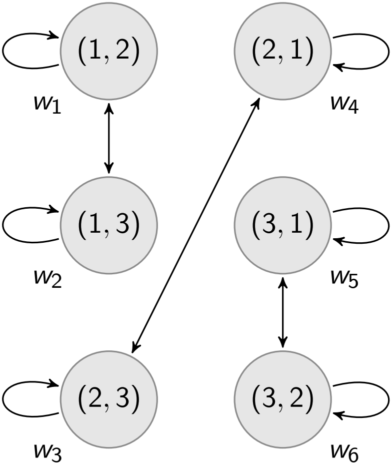

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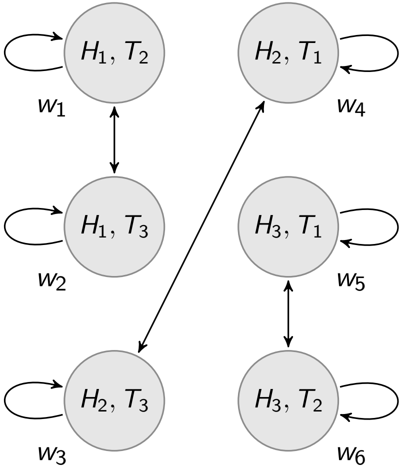

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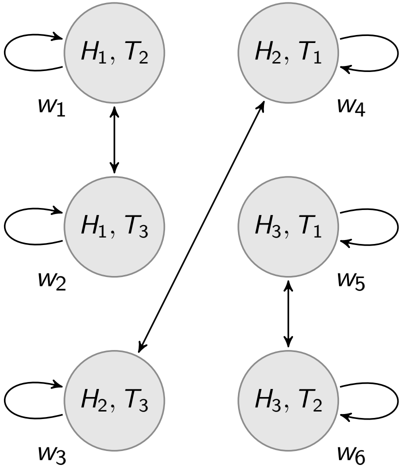

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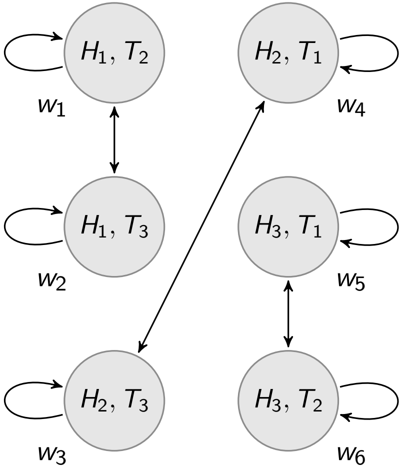

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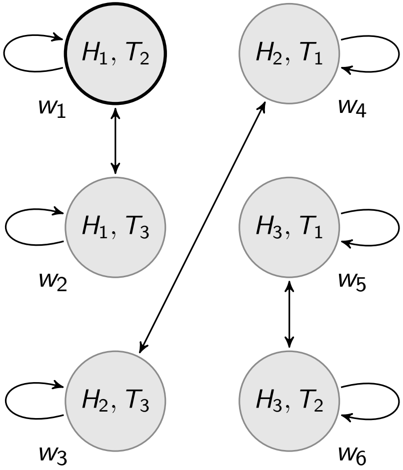

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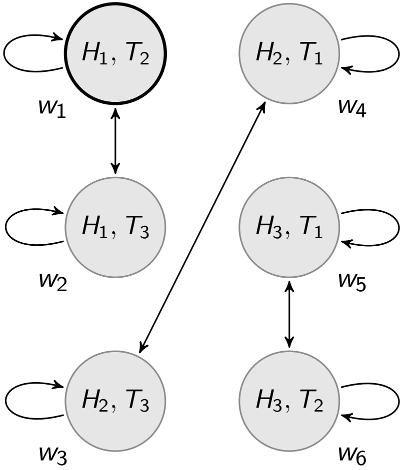

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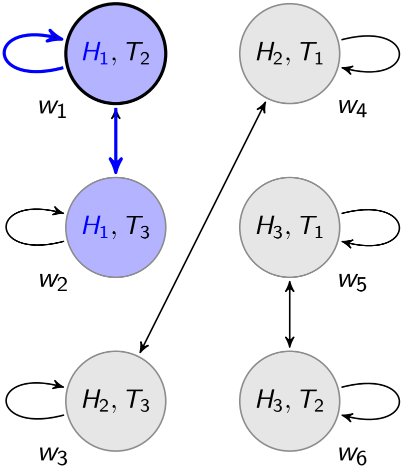

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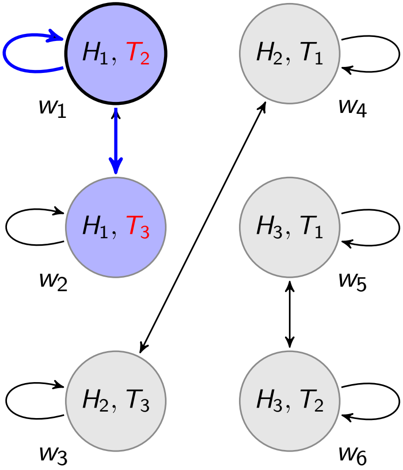

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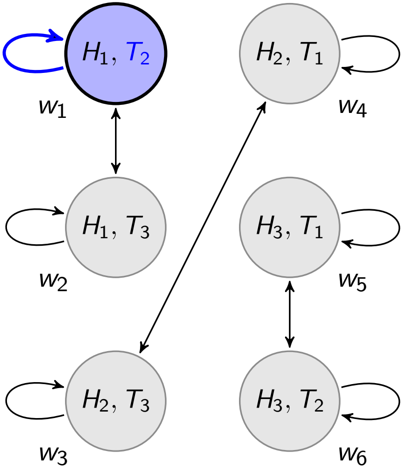

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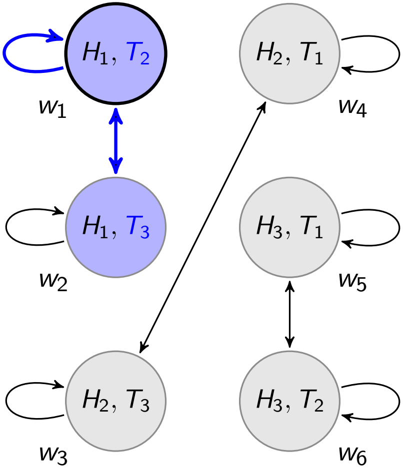

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What idealizations have we made?

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$\neg K\bot$	Serial	Consistency

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- $K_A K_B P$: "Ann knows that Bob knows P"
- $K_A(K_B P \vee K_B \neg P)$: "Ann knows that Bob knows whether P
- $\neg K_B K_A K_B(P)$: "Bob does not know that Ann knows that Bob knows that P"

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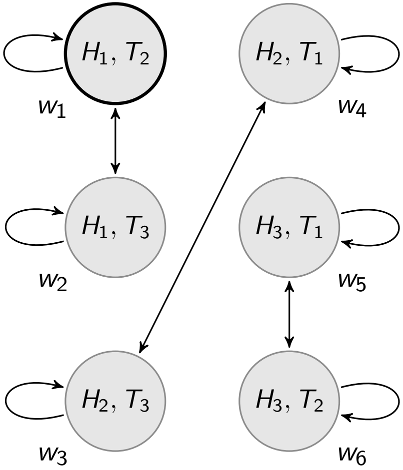

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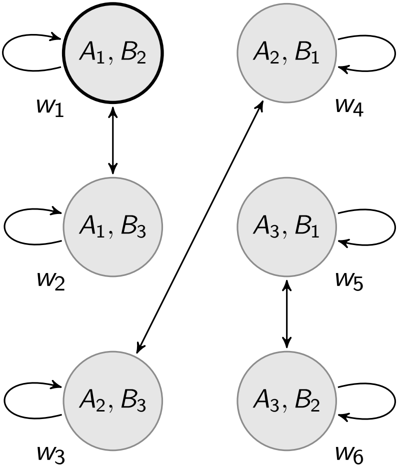

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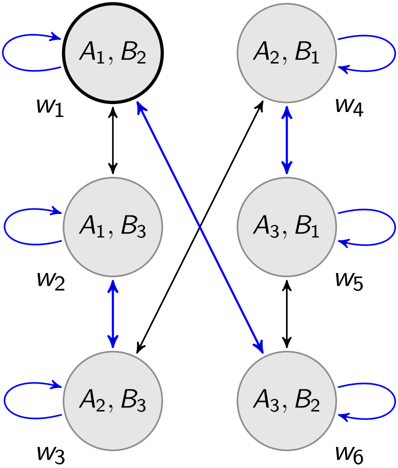

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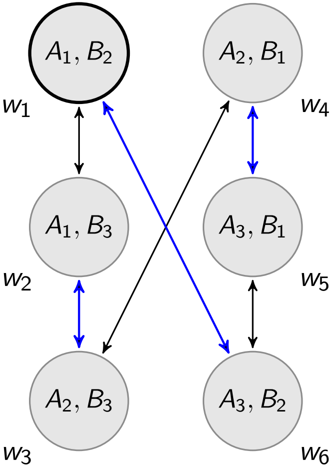

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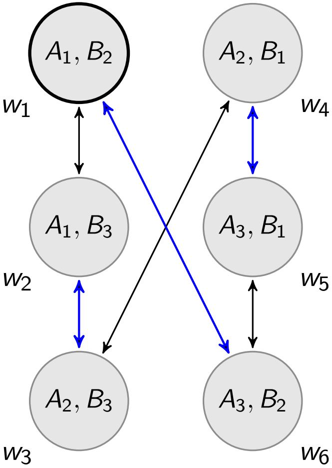

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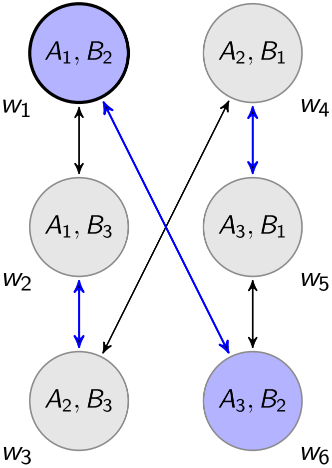

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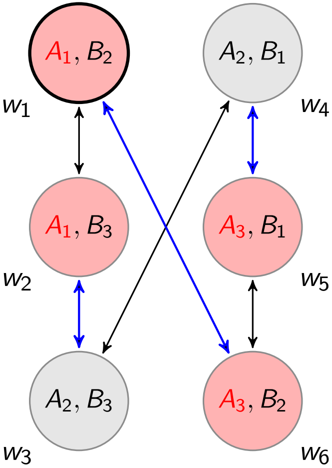

Two issues

Suppose we want to be completely formal in our solution to the muddy children puzzle, what is Epistemic Logic missing?

1. Group knowledge (all the children know there is at least one muddy child, they all know this fact, they all know that they know this fact, etc.).
2. Public announcements (various statements are publicly announced in the course of the puzzle).

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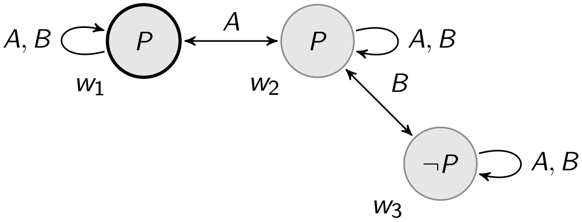

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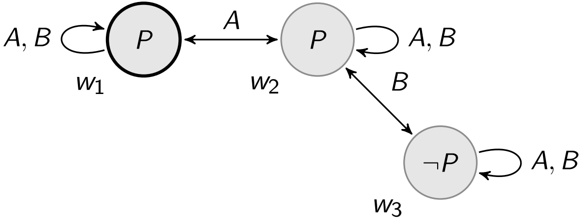

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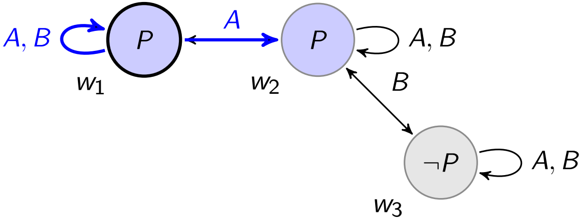

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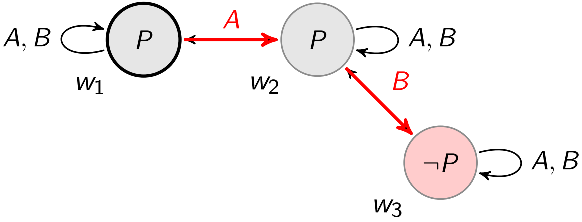

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- So G_1 sends a message m'', ...

Spreading Gossip

Suppose that there are three friends, Ann, Bob and Charles, and Ann learns a interesting piece of news (P). If each of the friends are at home, how many calls are needed to create common knowledge that P?

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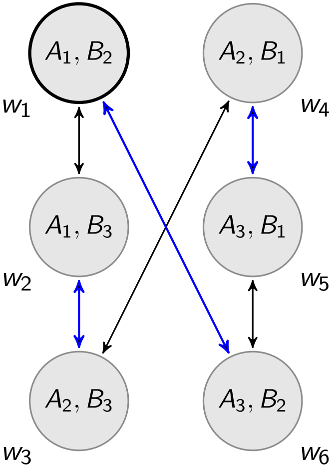
$\langle !(\neg(K_A \vee K_A \neg P)) \rangle K_B P$: "After it is announced that A does not know whether P, then B knows P."

Example

Suppose there are three cards: 1, 2 and 3.

Ann is dealt one of the cards, Bob is given one of the cards and the third card is put back in the deck.

Suppose that Ann receives card 1 and Bob receives card 2.



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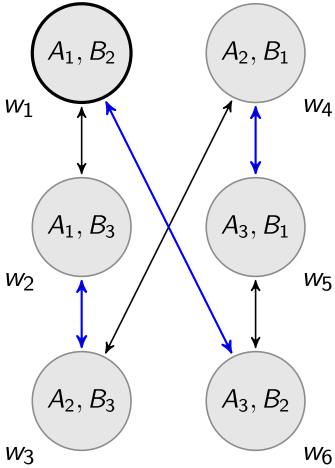

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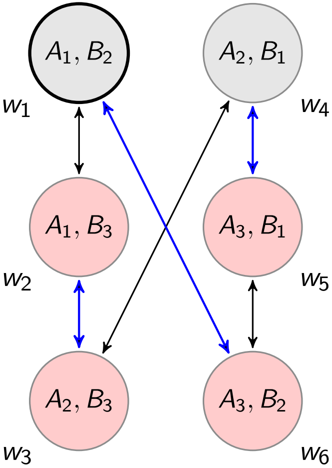

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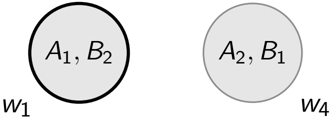

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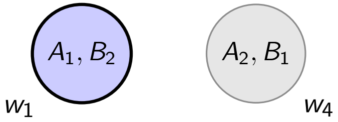

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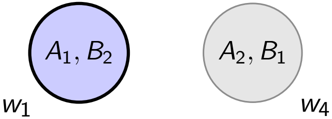

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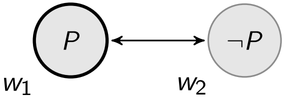

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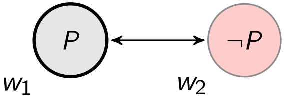

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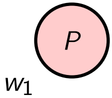

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Something to Think About

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2. **Russian Cards Problem**: From a deck of seven cards Ann and Bob each receive three cards and Charles the remaining card. How can Ann and Bob openly inform each other about their cards, without informing Charles who holds which card?

Next lecture: Dynamics in logic.

Questions?

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