Clarification for Question #12 on the Final

Let K be a relational structure. That is, K is a set of states W and a relation R on W. A **valuation** is a way of assigning propositional variables to states (formally, V is a function from the set of atomic propositions to the powerset of W). Let V be a valuation. We write

1. $K, V, w \models P$ if P is true at w under valuation V
2. $K, V \models P$ if P is true at all states w under valuation V (i.e., for all w, $K, V, w \models P$)
3. $K, w \models P$ if P is true at w under all valuations V (i.e., for all V, $K, V, w \models P$)
4. $K \models P$ if P is true at all states under all valuations (i.e., for all V, $K, V \models P$).

We say that a relational structure K is **reflexive** if for all w, wRw.

Claim. K is reflexive iff $K \models \Box P \rightarrow P$.

Informal Proof. Suppose that K is reflexive. Let V be any valuation and w any state. Suppose that $K, V, w \models \Box P$. Then P is true in all worlds accessible from w. Since K is reflexive, w is accessible from itself. So, P must be true at w. We have shown that if $K, V, w \models \Box P$ then $K, V, w \models P$, so $K, V, w \models \Box P \rightarrow P$.

Suppose that K is not reflexive. Then there is some world w such that it is not the case that wRw. Now we want to show that $K \not\models \Box P \rightarrow P$. To do this, we must find a valuation V and state w such that $K, V, w \models \Box P \wedge \neg P$. Let V be the valuation that makes P false at w and true everywhere else. Then $K, V, w \models \Box P$ since the only place where P is false is w and w is not related to itself. Furthermore, we defined V so that $K, V, w \models \neg P$. Q.E.D.

A Kripke structure is **transitive** iff for all w, v, u (if wRv and vRu then wRu).

Claim. K is transitive iff $K \models \Box P \rightarrow \Box\Box P$.

Informal Proof. Suppose that K is transitive. Let V be any valuation and w any state. Suppose that $K, V, w \models \Box P$. Then P is true in all worlds accessible from w. We want to show that $K, V, w \models \Box\Box P$. In other words, we must show that $\Box P$ is true at all worlds accessible from w. Consider any world accessible from w and call it v (so wRv). To see that $K, V, v \models \Box P$ we must show that any world accessible from v makes P true. Take any such world and call it u (so vRu). Since K is transitive and (wRv and vRu), it must be the case that wRu. Therefore, since P is true in all worlds accessible from w and u is such a world, u must satisfy P. Therefore, $K, V, w \models \Box\Box P$. We have shown that if $K, V, w \models \Box P$ then $K, V, w \models \Box\Box P$, so $K, V, w \models \Box P \to \Box\Box P$.

Suppose that K is not transitive. Then there are three worlds w, v and u such that wRv and vRu but it is not the case that wRu. Now we we want to show that $K \not\models \Box P \to \Box\Box P$. To do this, we must find a valuation V and state w such that $K, V, w \models \Box P \wedge \neg\Box\Box P$. Let V be the valuation that makes P false at u and true everywhere else. Then $K, V, w \models \Box P$ since the only place where P is false is u and u is not accessible from w. Furthermore, since vRu and P is false at u, we have $\Box P$ is false at v and so $\Box\Box P$ is false at w. Q.E.D.