Game Equivalence: *Handout*

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1 Background on Games

Definition 1.1 (Strategic Games) A *strategic game* is a tuple $(N, \{A_i\}_{i \in N}, \{\preceq_i\}_{i \in N})$ where:

- N is a set of players
- A_i is the set of actions available to player i
- $\preceq_i$ is an ordering on $A = \times_{j \in N} A_j$ representing player i's preference over possible outcomes $\lhd$

Intuitively, a strategic game identifies moves ("strategies") with the set of outcomes an agent can force by moving. The outcome of the game is uniquely determined by the intersection of all strategies. A solution concept (such as the *Nash Equilibrium*) characterizes the strategy a *rational* agent should follow. Here are two familiar examples:

	dove	hawk
dove	3, 3	1, 4
hawk	4, 1	0, 0

Figure 1: "Hawk - Dove"

	heads	tails
heads	1, -1	-1, 1
tails	-1, 1	1, -1

Figure 2: "Matching Pennies"

Definition 1.2 (Extensive Games) An *extensive game* is a tuple $\langle N, H, P, \{\mathcal{I}_i\}_{i \in N}, \{\preceq_i\}_{i \in N}\rangle$ where:

- N is a set of players
- H is a set of sequences (histories) which form a tree, this means
  - $\emptyset \in H$
  - if $(a^k)_{k=1,\ldots,n} \in H$ and $m < n$, then $(a^k)_{k=1,\ldots,m} \in H$

The a history $(a^k)_{k=1,\ldots,n} \in H$ is *terminal* if there is no a^{n+1} such that $(a^k)_{k=1,\ldots,n+1} \in H$. Call the set of all terminal histories Z.

- $P : H \setminus Z \longrightarrow N$ assigns a player to each non-terminal node of the tree; intuitively, the player whose turn it is to move.
- $\mathcal{I}_i$ is a partition on $\{h \in H | P(h) = i\}$ which satisfies the constraint that the same actions are available to i from any node in a member of $\mathcal{I}_i$; intuitively, these are player i's information sets.
- $\preceq_i$ is an ordering on Z representing player i's preference over possible outcomes. $\triangleleft$

Essentially, an extensive game is just a tree decorated with equivalence relations (*not unlike a model in epistemic temporal logic!*). These equivalence relations capture each player's state of uncertainty about the moves that have been made so far in the game. We can also use them to define scenarios such as *imperfect recall*, where an agent forgets how she moved earlier in the game, and *absentmindedness*, where an agent forgets whether or not she has made an earlier move.

2 Background on Game Equivalence

F. B. Thompson (1952) "Equivalence of Games in Extensive Form" considers the idea that two extensive form games are equivalent if they share the same reduced strategic form. (Reduced strategic form is just a strategic game with all redundant moves removed.)

Observation 2.1 *We can simply read the corresponding strategic game off of an extensive game by suppressing temporal information and focussing only on the set of outcomes each player can force to obtain.*

Thompson defines four transformations of extensive form games, each of which preserves strategic form (see attached for a visual representation of these transformations). He then proves this theorem:

Theorem 2.2 *Any two extensive games* Γ *and* Γ' *share the same strategic form* if and only if Γ' *can be obtained from* Γ *by the stepwise application of some sequence of the defined transformations.*

G. Bonanno (*speaking at CSLI, May 31!*) (1992) "Set-Theoretic Equivalence of Extensive-Form Games" offers a refinement of Thompson's result. He argues that extensive games contain *too much* structure because they treat as ordered sequences of moves which, conceptually, should be treated as simultaneous. Strategic games, however, contain *too little* structure as they suppress all temporal information. In response to this difficulty, Bonanno defines a new game form, *set-theoretic games*.

Definition 2.3 (Set-Theoretic Games) A *set-theoretic game* is a tuple

$$\left\langle N, Z, P, \{\pi_i\}_{i \in N}, \{\mathcal{A}_i\}_{i \in N}, \{\Sigma_i\}_{i \in N} \right\rangle$$

where:

- N is a set of players
- Z is a set of outcomes
- $\pi_i : Z \to \mathbf{Re}$ is the payoff function for each player $i \in N$
- $\mathcal{A}_i$ is a collection of non-empty subsets of Z; intuitively, each $A \in \mathcal{I}_i$ is an action available to player i. At each stage in the game where player i can act, she is presented with a situation, or set of available actions. By picking one, she forces the outcome of the game to fall within that set.
- $\preceq_i$ is an ordering on Z representing player i's preference over possible outcomes $\lhd$

Intuitively, set-theoretic games are something like a sequence of alternating or simultaneous strategic games which successively narrow the space of possible outcomes. Having defined this new game form, Bonanno then proves this theorem.

Theorem 2.4 *Any extensive game* Γ *determines a* unique *set-theoretic game* $\mathcal{S}$.

This partitions the space of extensive games just as Thompson had done, now with respect to their corresponding set-theoretic forms. Bonanno then generalizes one of Thompson's transformations, the *interchange of contiguous moves*. This generalization allows him to prove the following theorem, corresponding to Thompson's theorem 2.2:

Theorem 2.5 *Any two extensive games* Γ *and* Γ' *share the same set-theoretic form* if and only if Γ' *can be obtained from* Γ *by a sequence of applications of the interchange of contiguous moves transformation.*

3 Extending Game Equivalence

The aim of the present project is to unify and extend the results of Thompson and Bonanno. First, note that there are two faces to the notion of equivalence:

SEMANTIC	SYNTACTIC
invariance under isomorphism, bisimulation, or other transformations	*logic* - satisfaction of axioms; *game theory* - application of solution concepts

Ideally, a unified approach to the equivalence of games would provide both sides to this story. The first step, however, is to define the space of games over which the transformations will apply. In order to subsume the results of Thompson and Bonanno, we need to consider games at least as expressive as extensive games. However, three considerations indicate that an even stronger space might be required. First, we also wish to include cases of absentmindedness, which require a game form stronger than extensive form. Second, Bonanno's argument that certain instances of ordering do not matter is interesting, but what it really indicates is that a maximally expressive game form should be able to model both ordered and simultaneous moves, even when players are ignorant of each other's moves. Third, Bonanno's treatment and Thompson's are not strictly speaking compatible as Bonanno's formalism does not allow for identity between outcomes resulting from distinct histories, while Thompson's does; in order to fix this problem, we need to include an apparatus for defining equivalence of outcomes.

Definition 3.1 (Causal+Epistemic Games) A *causal+epistemic game* is a tuple

$$\Gamma = \left\langle N, H, \{\pi_i\}_{i \in N}, \equiv, \{A_i\}_{i \in N}, \{E_i\}_{i \in N} \right\rangle$$

where

- N is the set of players
- H is a set of sequences (histories) which form a tree, this means
  - $\emptyset \in H$
  - if $(a^k)_{k=1,\dots,n} \in H$ and $m < n$, then $(a^k)_{k=1,\dots,m} \in H$

  The a history $(a^k)_{k=1,\dots,n} \in H$ is *terminal* if there is no a^{n+1} such that $(a^k)_{k=1,\dots,n+1} \in H$. Call the set of all terminal histories Z and the set of all non-terminal histories $H \setminus Z$.
- $\pi_i : Z \to \mathbf{Re}$ is the payoff function for each player $i \in N$
- $\equiv$ is an equivalence relation on Z such that for each $x, y \in Z$, $x \equiv y$ *iff* for all $j \in N$, $\pi_j(x) = \pi_j(y)$. This gives us a notion of identity between outcomes with the same payoffs for all agents. Thompson's transformations assume agents are interested only in the quotient set $Z/\equiv$, this is why they preserve *reduced* strategic form.
- A_i is a family of partitioned subsets of Z. Each subset represents the set of outcomes player i considers still possible at some step in the game, and the partition represents the actions which player i perceives as available to her to reduce the remaining possibilia.
- $E_i : j \times H \setminus Z \to A_j$ for all $i, j \in N$. E_i captures player i's beliefs about the perceptions of each player at each stage in the causal structure of the game. ◁