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Merging Frameworks for Interaction

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Abstract A variety of logical frameworks have been developed to study rational agents interacting over time. This paper takes a closer look at one particular interface, between two systems that both address the dynamics of knowledge and information flow. The first is Epistemic Temporal Logic (ETL) which uses linear or branching time models with added epistemic structure induced by agents' different capabilities for observing events. The second framework is Dynamic Epistemic Logic (DEL) that describes interactive processes in terms of epistemic event models which may occur inside modalities of the language. This paper systematically and rigorously relates the DEL framework with the ETL framework. The precise relationship between DEL and ETL is explored via a new *representation theorem* characterizing the largest class of

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ETL models corresponding to DEL protocols in terms of notions of *Perfect Recall*, *No Miracles*, and *Bisimulation Invariance*. We then focus on new issues of *completeness*. One contribution is an axiomatization for the dynamic logic of public announcements constrained by protocols, which has been an open problem for some years, as it does not fit the usual 'reduction axiom' format of DEL. Finally, we provide a number of examples that show how DEL suggests an interesting fine-structure inside ETL.

Keywords Dynamic epistemic logic · Epistemic temporal logic · Epistemic logic

1 Introduction

Many logical systems today describe intelligent interacting agents over time. Frameworks include Interpreted Systems (IS, Fagin et al. [22]), Epistemic Temporal Logic (ETL, Parikh & Ramanujam [37]), Logics of Agency (STIT, Belnap et al. [6]), Process Algebra and Game Semantics (Abramsky [1]). This variety is an asset, as different modeling tools can be fine-tuned to specific applications. But it may also be an obstacle, when barriers between paradigms and schools go up.

This paper takes a closer look at one particular interface, between two systems that both address the dynamics of knowledge and information flow in multi-agent systems. One is IS/ETL (IS and ETL are, from a technical point of view, the same up to model transformations, cf. [35]), which uses linear or branching time models with added epistemic structure induced by agents' different capabilities for observing events. These models provide a Grand Stage where histories of some process unfold constrained by a protocol, and a matching epistemic temporal language describes what happens. The other framework is Dynamic Epistemic Logic (DEL, [5, 20, 25]) that describes interactive processes in terms of epistemic event models which may occur inside modalities of the language. Temporal evolution is then computed from some initial epistemic model through a process of successive 'product updates'. It has long been unclear how to best compare IS/ETL and DEL. Various aspects have been investigated in [15, 17, 23], but in this paper, we study the interface in a more systematic way.

Often, DEL and ETL are presented as *alternative* ways of adding dynamics to multi-agent epistemic models. In this paper, we rather focus on how *merging* the two different modeling choices leads to interesting new questions. Our leading interest here will be a view of informational processes as evolving over time.

To see what we mean, consider the simplest version of DEL, viz. the logic of *public announcements* PAL [38] which adds a very specific type of communicative action to epistemic models: a public announcement. Formulas of the form $\langle P \rangle \varphi$ are intended to mean "after a public announcement of P, φ is true". The '$\langle P \rangle$' is interpreted as a *restriction* of the current model to the states satisfying P. Now, in many real interactions between agents, protocol or social convention dictates that some announcements that *can* happen may not be allowed. For example, in a conversation, it is typically not polite to "blurt everything out at the beginning", as we must speak in small chunks. Other natural protocol rules include "do not repeat yourself", "let others speak in turn", "be honest", and so on. Imposing these rules *restricts* the legitimate sequences of possible announcements, and this immediately affects the standard validities of PAL. For instance, consider the PAL-validity stating that the effect of two consecutive announcements, expressed in $\langle P\rangle\langle Q\rangle\varphi$, is the same as the effect of one single 'two-in-one' announcement: $\langle (P \wedge \langle P \rangle Q)\rangle\varphi$. This equivalence will no longer hold in general protocol-based models, as will be discussed in more detail in Section 4. Other examples of protocols occur in puzzles (the ever-present muddy children are only allowed to make epistemic assertions), while interaction with a database, or some physical measuring device, involves only *factual* assertions.

In a sense then, this paper is about 'logics of conversation' as governed by protocols. But our results apply much more generally to any sort of informational process, whether linguistically encoded or not. In particular, we want to emphasize another, equally valid interpretation at the outset, which applies to all our notions and results. PAL may also be viewed as a logic of general *observation* [9, 16], without any linguistic communication at all. And then, the protocol setting describes knowledge growth in various *learning scenarios*, moving closer to formalizing part of the temporal logic of formal learning theory (cf. [31]).

We have two main objectives in this paper. The first is to systematically relate the DEL framework with the ETL framework. The key idea is that repeatedly applying product update with sequences of event models creates an ETL model (details are given in the next section). In other words, given an initial epistemic model and sequences of DEL event models we can *generate* an ETL model, and thus transform the DEL dynamic modalities into ETL (labeled) temporal modalities. This provides a concrete way of relating DEL and ETL, but it is not the whole story. The precise relationship between DEL and ETL is explored further in Section 3. We prove a new *representation theorem* characterizing the largest class of ETL models corresponding to DEL protocols in terms of notions of *Perfect Recall*, *No Miracles*, and *Bisimulation Invariance*. These describe the sort of idealized agent presupposed in standard DEL.

Our second objective is to show how ETL and DEL lead to interesting new issues when merged as accounts of intelligent interacting agents. In particular, we focus on new issues of *completeness*. One contribution is an axiomatization for the dynamic logic of public announcements constrained by protocols, which has been an open problem for some years, as it does not fit the usual 'reduction axiom' format of DEL (cf. Section 4.1). More generally, Section 6 provides a number of examples that show how DEL suggests an interesting fine-structure inside ETL.

2 Preliminaries

This Section provides the formal details of the two frameworks (ETL and DEL) that we investigate in this paper. We only give the bare necessities needed for our results and the reader is referred to the relevant references listed below for more information.

2.1 Epistemic Temporal Logic

We start by fixing a finite set of agents $\mathscr{A}$ and a (possibly infinite) set of events Σ. A **history** is a finite sequence of events from Σ. We write Σ^* for the set of histories built from elements of Σ. For a history h, we write he for the history h followed by the event e. Given $h, h' \in \Sigma^*$, we write $h \preceq h'$ if h is a prefix of h', and $h \prec_e h'$ if $h' = he$ for some event e.

Definition 1 (ETL Frames) Let Σ be a set of events. A **protocol** is a set $\mathsf{H} \subseteq \Sigma^*$ closed under non-empty prefixes. An **ETL frame** is a tuple $\langle \Sigma, \mathsf{H}, \{\sim_i\}_{i\in\mathscr{A}} \rangle$ with H a protocol, and for each $i \in \mathscr{A}$, a binary relation $\sim_i$ on[1] H.

An ETL frame describes how knowledge evolves over time in some informational process. The protocol captures the temporal structure, with h' such that $h \prec_e h'$ representing the point in time after e has happened in h. The relations $\sim_i$ represent the uncertainty of the agents about how the current history has evolved. Thus, $h \sim_i h'$ means that from agent i's point of view, the history h' looks the same as the history h.

Different modal languages describe these structures (see, for example, [22, 28]), with 'branching' or 'linear' variants. Here we give the bare necessities to facilitate comparisons with DEL (further language extensions are explored in Section 6.1). Let At be a countable set of atomic propositions. The language $\mathscr{L}_{\mathrm{ETL}}$ is generated by the following grammar:

$$P \mid \neg\varphi \mid \varphi \wedge \psi \mid [\,i\,]\varphi \mid \langle e\rangle\varphi$$

where $i \in \mathscr{A}$, $e \in \Sigma$ and $P \in \mathsf{At}$. Boolean connectives ($\vee, \rightarrow, \leftrightarrow$) and dual modal operators ($\langle i\rangle$, $[e]$) are defined as usual. The pure epistemic language,

[1] Although we will not do so here, typically it is assumed that $\sim_i$ is an equivalence relation.

denoted $\mathcal{L}_{\text{EL}}$, is the fragment of $\mathcal{L}_{\text{ETL}}$ with only epistemic modalities. Formulas are interpreted at states in an **ETL model**:

Definition 2 (ETL Model) An **ETL model** is a tuple $\langle \Sigma, \mathsf{H}, \{\sim_i\}_{i\in\mathcal{A}}, V\rangle$ with $\langle \Sigma, \mathsf{H}, \{\sim_i\}_{i\in\mathcal{A}}\rangle$ an ETL frame and V a valuation function ($V : \mathsf{At} \to 2^{\mathsf{H}}$).

Definition 3 (Truth of $\mathcal{L}_{\text{ETL}}$ Formulas) Let $\mathcal{H} = \langle \Sigma, \mathsf{H}, \{\sim_i\}_{i\in\mathcal{A}}, V\rangle$ be an ETL model. The truth of a formula φ at a history $h \in \mathsf{H}$, denoted $\mathcal{H}, h \models \varphi$, is defined as follows:

1. $\mathcal{H}, h \models P$ iff $h \in V(P)$
2. $\mathcal{H}, h \models \neg\varphi$ iff $\mathcal{H}, h \not\models \varphi$
3. $\mathcal{H}, h \models \varphi \wedge \psi$ iff $\mathcal{H}, h \models \varphi$ and $\mathcal{H}, h \models \psi$
4. $\mathcal{H}, h \models [i]\varphi$ iff for each $h' \in \mathsf{H}$, if $h \sim_i h'$ then $\mathcal{H}, h' \models \varphi$
5. $\mathcal{H}, h \models \langle e\rangle\varphi$ iff there exists $h' \in \mathsf{H}$ such that $h \prec_e h'$ and $\mathcal{H}, h' \models \varphi$

It is often natural to extend the language $\mathcal{L}_{\text{ETL}}$ with group knowledge operators (e.g., common or distributed knowledge) and more expressive temporal operators (e.g., arbitrary future or past modalities). This may lead to high complexity of the validity problem (cf. [17, 27] and Section 6.1).

2.2 Dynamic Epistemic Logic

An alternative account of interactive dynamics was elaborated by [5, 9, 12, 23] and others. From an initial epistemic model, temporal structure is generated by explicitly triggered by informative events.

Definition 4 (Epistemic Model) Let $\mathcal{A}$ be a finite set of agents and At a set of atomic propositions. An **epistemic model** is a tuple $\langle W, \{R_i\}_{i\in\mathcal{A}}, V\rangle$ where W is a non-empty set, for each $i \in \mathcal{A}$, R_i is a relation[2] on W ($R_i \subseteq W \times W$) and V a valuation function ($V : \mathsf{At} \to 2^W$). We call the set W the domain of $\mathcal{M}$, denoted by $D(\mathcal{M})$. A pair $\mathcal{M}, w$ where $\mathcal{M}$ is an epistemic model and $w \in D(\mathcal{M})$ is called a **pointed epistemic model**.

The epistemic language, $\mathcal{L}_{\text{EL}}$, defined above is interpreted at states in an epistemic model as usual: see [18] for details. We only recall the definition of the knowledge operators:

$$\mathcal{M}, w \models [i]\varphi \;\text{ iff for each }\; w' \in W, \text{if }\; wR_iw' \;\text{ then } \mathcal{M}, w' \models \varphi$$

Whereas an ETL frame describes the agents' information at all moments, **event models** are used to build new epistemic models as needed.

[2] Again, the R_i are often taken to be equivalence relations on W - but we do not commit.

Definition 5 (Event Model, Product Update) An **event model** $\mathscr{E}$ is a tuple $\langle S, \{\longrightarrow_i\}_{i\in\mathscr{A}}, \mathsf{pre}\rangle$, where S is a nonempty set, for each $i \in \mathscr{A}$, $\longrightarrow_i \subseteq S \times S$ and $\mathsf{pre} : S \to \mathscr{L}_{\mathrm{EL}}$ is the **pre-condition function**. The set S is called the domain of $\mathscr{E}$, denoted $D(\mathscr{E})$.

The **product update** $\mathscr{M} \otimes \mathscr{E}$ of an epistemic model $\mathscr{M} = \langle W, \{R_i\}_{i\in\mathscr{A}}, V\rangle$ and event model $\mathscr{E} = \langle S, \{\longrightarrow_i\}_{i\in\mathscr{A}}, \mathsf{pre}\rangle$ is the epistemic model $\langle W', R_i', V'\rangle$ with

1. $W' = \{(w, e) \mid w \in W, e \in S \text{ and } \mathscr{M}, w \models \mathsf{pre}(e)\}$,
2. $(w, e) R_i' (w', e')$ iff $w R_i w'$ in $\mathscr{M}$ and $e \longrightarrow_i e'$ in $\mathscr{E}$, and
3. For all $P \in \mathsf{At}$, $(s, e) \in V'(P)$ iff $s \in V(P)$

The language $\mathscr{L}_{\mathrm{DEL}}$ extends $\mathscr{L}_{\mathrm{EL}}$ with operators $\langle \mathscr{E}, e\rangle$ for each pair of event models $\mathscr{E}$ and event e in the domain of $\mathscr{E}$. Truth for $\mathscr{L}_{\mathrm{DEL}}$ is defined as usual. We only define the typical DEL modalities: $\mathscr{M}, w \models \langle \mathscr{E}, e\rangle\varphi$ iff $\mathscr{M}, w \models \mathsf{pre}(e)$ and $\mathscr{M} \otimes \mathscr{E}, (w, e) \models \varphi$ (see [4] for more details, and [12] for extended versions of product update allowing factual change).

Remark 1 (Size of the Event Models) Although Definition 5 does not assume that event models are finite, it is often convenient to make such an assumption. The main reason is that the usual *reduction axiom* for the DEL modality $[\mathscr{E}, e]$ (cf. [5]) contains a conjunction over all elements of $\mathscr{E}$ reachable from e. Now if this set is infinite, then the reduction axiom will not be a *formula* of $\mathscr{L}_{\mathrm{DEL}}$ since it contains an infinite conjunction.

Example 1 (Public Announcement Logic [24, 38]) The **public announcement** of a formula $\varphi \in \mathscr{L}_{\mathrm{EL}}$ is the event model $\mathscr{E}_\varphi = \langle \{e\}, \{\longrightarrow_i\}_{i\in\mathscr{A}}, \mathsf{pre}\rangle$ where for each $i \in \mathscr{A}$, $e \longrightarrow_i e$ and $\mathsf{pre}(e) = \varphi$. As the reader is invited to verify, the product update of an epistemic model $\mathscr{M}$ with a public announcement model $\mathscr{E}_\varphi$ is the submodel of $\mathscr{M}$ containing all the states that satisfy φ. In this case, the DEL modality $\langle \mathscr{E}_\varphi, e\rangle$ will be denoted $\langle\varphi\rangle$. Let $\mathscr{L}_{\mathrm{PAL}}$ denote this language.

3 Comparing DEL and ETL

Our key observation is that by repeatedly updating an epistemic model with event models, the machinery of DEL in effect creates ETL models. However, note that an ETL model contains not only a description of how the agents' information changes over time, but also "protocol information" describing *when* each event can be performed. Thus, in rigorously comparing DEL with ETL models, the protocol information must be made explicit, constraining how the relevant conversation, observational set-up, or learning scenario can evolve.

3.1 From DEL to ETL

Let $\mathbb{E} = \{(\mathscr{E}, e) \mid \mathscr{E}$ an event model and $e \in D(\mathscr{E})\}$ be the class of all pointed event models. A **DEL protocol** is a set $\mathsf{P} \subseteq \mathbb{E}^*$ closed under the initial segment relations (cf. Definition 1).[3] Given a DEL protocol P, let σ denote an element of P (so, σ is a sequence of pointed event models). We write σ_n for the initial segment of σ of length n ($n \leq \mathsf{len}(\sigma)$) and write $\sigma_{(n)}$ for the nth component of σ. For example, if $\sigma = (\mathscr{E}_1, e_1)(\mathscr{E}_2, e_2)(\mathscr{E}_3, e_3)\cdots(\mathscr{E}_n, e_n)$, then $\sigma_3 = (\mathscr{E}_1, e_1)(\mathscr{E}_2, e_2)(\mathscr{E}_3, e_3)$ and $\sigma_{(3)} = (\mathscr{E}_3, e_3)$. Given a sequence $\sigma \in \mathbb{E}^*$, we abuse notation and write $\mathsf{pre}(\sigma_{(n)})$ for $\mathsf{pre}(e_n)$ where $\sigma_{(n)} = (\mathscr{E}_n, e_n)$. Furthermore, we write $\sigma_{(n)} \longrightarrow_i \sigma'_{(n)}$ provided $\sigma_{(n)} = (\mathscr{E}, e)$ and $\sigma'_{(n)} = (\mathscr{E}, e')$ and $e \longrightarrow_i e'$ in $\mathscr{E}$. Finally, let $Ptcl(\mathbb{E})$ be the class of all DEL protocols, i.e., $Ptcl(\mathbb{E}) = \{\mathsf{P} \mid \mathsf{P} \subseteq \mathbb{E}^*$ is closed under initial segments$\}$.

The main idea is that starting from an initial (pointed) epistemic model we construct an ETL model by repeatedly applying product update. Our most general construction will vary the DEL protocol from state-to-state:

Definition 6 (State-Dependent DEL Protocol) Let $\mathscr{M}$ be an arbitrary epistemic model. A **state-dependent DEL protocol on** $\mathscr{M}$ is any function $p : D(\mathscr{M}) \to Ptcl(\mathbb{E})$.

This is a significant generalization of the usual ETL setting where "who can say what and when" is assumed to be common knowledge (cf. [22, 37]). If a state-dependent protocol p is a constant function (i.e., for all $w \in D(\mathscr{M})$, $p(w) = \mathsf{P}$), we say p is a **uniform DEL protocol**. To ease exposition, we will denote a uniform DEL protocol by the unique DEL protocol P assigned to each state. Of course, a uniform protocol will be common knowledge among the agents (indeed, the same protocol is used at *all* states). On the other hand, state-dependent protocols represent situations where the type of conversation, experimental protocol or learning process is not known by any agent.[4] Thus, state-dependent and uniform protocols are two extreme cases with many interesting cases in between where agents have only partial knowledge of the "rules of the situation" they are in. One natural example is the assumption that all agents individually know the protocol: for each $w, v \in D(\mathscr{M})$, if wR_iv

[3]The *preconditions of DEL* also encode protocol information of a 'local' character, and so can do some of the work of global protocols, as has been pointed out in [9]. We do not pursue this division of labour here.

[4]Of course, the function p is itself common knowledge meaning it is common knowledge which observations are available at each state. One can drop this assumption by further complicating the model and working with *vectors* of state-dependent protocols (one for each agent). We do not pursue this line here.

then $p(w) = p(v)$ (so the protocol is uniform in each generated submodel but different generated submodels may be assigned different protocols). For this paper, we will restrict attention to state-dependent and uniform protocols.

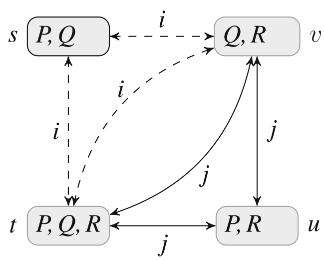
We now turn to the main construction of this paper: generating an ETL model from an initial epistemic model and a (state-dependent or uniform) DEL protocol. We start with constructing an ETL model from a uniform DEL protocol since the definition is more transparent. However, we stress that the following two definitions are special cases of the more general construction given below (cf. Definition 9 and Definition 10).

Definition 7 (σ-Generated Epistemic Model) Given an epistemic model $\mathcal{M}$ and a finite sequence of event models σ, we define the **σ-generated epistemic model**, $\mathcal{M}^\sigma$ as $\mathcal{M} \otimes \sigma_{(1)} \otimes \sigma_{(2)} \otimes \cdots \otimes \sigma_{(\mathsf{len}(\sigma))}$.

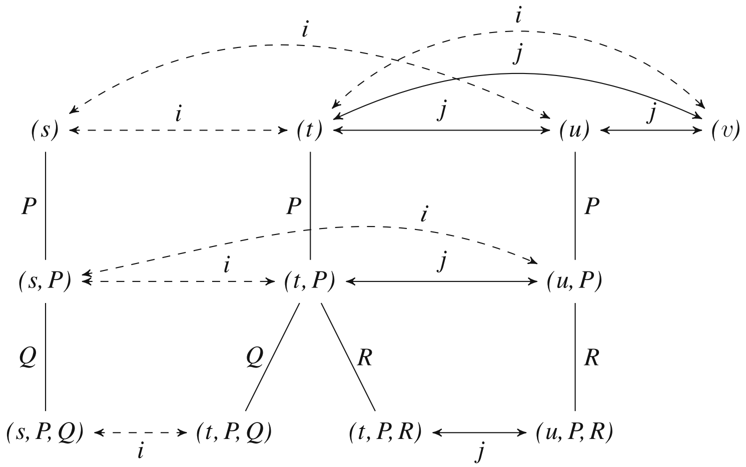
Definition 8 (ETL Model Generated from a Uniform DEL Protocol) Let $\mathcal{M}$ be a pointed epistemic model, and P a DEL protocol. The ETL model generated by $\mathcal{M}$ and P, $\mathsf{Forest}(\mathcal{M}, \mathsf{P})$, represents all possible evolutions of the system obtained by updating $\mathcal{M}$ with sequences from P. More precisely, $\mathsf{Forest}(\mathcal{M}, \mathsf{P}) = \langle \Sigma, \mathsf{H}, \{\sim_i\}_{i\in\mathcal{A}}, V\rangle$, where $\langle \mathsf{H}, \{\sim_i\}_{i\in\mathcal{A}}, V\rangle$ is the union of all models of the form $\mathcal{M}^\sigma$ with $\sigma \in \mathsf{P}$.

Since any DEL protocol P is closed under prefixes, for any epistemic model $\mathcal{M}$, $\mathsf{Forest}(\mathcal{M}, \mathsf{P})$ is indeed an ETL model. Here is a concrete illustration:

Example 2 (ETL model generated from a uniform DEL protocol) We illustrate the above construction in public announcement logic (PAL [38]) with each event model denoting an announcement or observation of some true formula. Let $\mathsf{P} = \{(P), (P, Q), (P, R)\}$ and consider the epistemic model depicted here:



Using Definition 8, we can combine $\mathscr{M}$ and P to form an ETL model $\mathsf{Forest}(\mathscr{M}, \mathsf{P})$:



Note that in this example $\mathsf{Forest}(\mathscr{M}, \mathsf{P}), (t) \models R \wedge \neg\langle R\rangle\top$. Thus even though a formula is true, it may not be "announcable" due to the underlying protocol. This reiterates the points raised in the Introduction and will be discussed in more detail in Section 4.1.

The ETL model $\mathsf{Forest}(\mathscr{M}, \mathsf{P})$ in Example 2 satisfies a strong *uniformity condition*: if $(\mathscr{E}, e)$ is allowable according to the protocol P, then for all histories h, the epistemic action $(\mathscr{E}, e)$ can be executed at h iff $\mathsf{pre}(e)$ is true at h. This implies that the protocol P is common knowledge.[5] Of course, this condition will not be satisfied in ETL models generated from state-dependent protocols.

Definition 9 (p-Generated Model) Let $\mathscr{M} = \langle W, \{R_i\}_{i\in\mathcal{A}}, V\rangle$ be an epistemic model and p, a state-dependent DEL-protocol on $\mathscr{M}$. The **p-generated model at level** n, $\mathscr{M}^{n,p} = \langle W^{n,p}, \{R_i^{n,p}\}_{i\in\mathcal{A}}, V^{n,p}\rangle$, is defined by induction on n:

1. $W^{0,p} = W$, for each $i \in \mathscr{A}$, $R_i^{0,p} = R_i$ and $V^{0,p} = V$.
2. $w\sigma \in W^{n+1,p}$ iff (1) $w \in D(\mathscr{M})$, (2) $\mathsf{len}(\sigma) = n+1$, (3) $w\sigma_n \in W^{n,p}$, (4) $\sigma \in p(w)$, and (5) $\mathscr{M}^{n,p}, w\sigma_n \models \mathsf{pre}(\sigma_{(n)})$.
3. For $w\sigma, v\sigma' \in W^{n+1,p}$, $w\sigma R_i^{n+1,p} v\sigma'$ iff $w\sigma_n R_i^{n,p} v\sigma'_n$ and $\sigma_{(n+1)} \longrightarrow_i \sigma'_{(n+1)}$.
4. For each $P \in \mathsf{At}$, $V^{n+1,p}(P) = \{w\sigma \in W^{n+1,p} \mid w \in V(P)\}$.

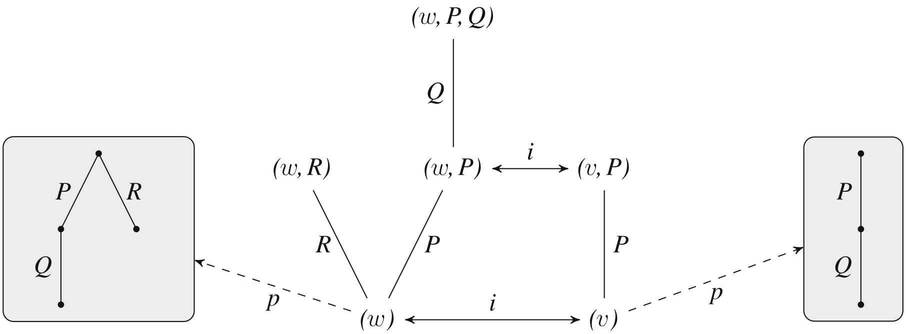
[5] In fact, it implies the stronger fact that, if $(\mathscr{E}, e)$ can be executed, it can be executed *anywhere* in the current model (not just in the reachable states) provided $\mathsf{pre}(e)$ is true.

Definition 10 (Generated ETL Model) Let $\mathcal{M} = \langle W, \{R_i\}_{i\in\mathcal{A}}, V\rangle$ be an epistemic model and p a state-dependent DEL protocol on $\mathcal{M}$. An ETL model $\mathsf{Forest}(\mathcal{M}, p) = \langle \mathsf{H}, \{\sim_i\}_{i\in\mathcal{A}}, V'\rangle$ is defined as follows:

1. $\mathsf{H} \;=\; \{h \mid \text{there is a } w \in W, \sigma \in \bigcup_{w\in W} p(w) \text{ with } h = w\sigma \in W^{\mathsf{len}(\sigma),p}\}$.
2. For all $h, h' \in \mathsf{H}$ with $h = w\sigma$ and $h' = v\sigma'$, $h \sim_i h'$ iff $\mathsf{len}(\sigma) = \mathsf{len}(\sigma')$ and $w\sigma R_i^{\mathsf{len}(\sigma),p} v\sigma'$.
3. For each $P \in \mathsf{At}$ and $h = w\sigma \in \mathsf{H}$, $h \in V'(P)$ iff $h \in V^{\mathsf{len}(\sigma),p}(P)$.

Since each DEL protocol P is closed under prefixes, so is the domain of $\mathsf{Forest}(\mathcal{M}, p)$. Hence, Definition 10 indeed describes an ETL model. It is not difficult to see that Definition 7 and Definition 8 are special cases of Definition 9 and Definition 10, respectively, when we restrict attention to uniform protocols (the details are left to the reader). We illustrate this construction with another example.

Example 3 (From state-dependent DEL protocols to ETL models) Let $\mathcal{M}$ consist of two worlds, w and v which are indistinguishable for agent i (the only one here). Furthermore, let the valuation make P and R true at both worlds and Q true only at w. Let p be a state-dependent DEL protocol defined as follows: $p(w) = \{(P), (PQ), (R)\}$ and $p(v) = \{(P), (PQ)\}$. Using Definition 10, we can combine $\mathcal{M}$ and p to form $\mathsf{Forest}(\mathcal{M}, p)$:



where the horizontal lines represent the indistinguishability relation and the dashed lines represent the state-dependent protocol function p. Note that this model does not satisfy the uniformity condition mentioned above. In fact, we have $\mathsf{Forest}(\mathcal{M}, p), (w) \models \langle R\rangle\top$, but $\mathsf{Forest}(\mathcal{M}, p), (v) \models R \wedge \neg\langle R\rangle\top$.

Remark 2 (Public announcement protocols) Restricting to public announcement simplifies many of the definitions in this Section. Since we identify a public announcement event model with a formula (cf. Example 1), a **PAL protocol** is a set of sequences of formulas of $\mathcal{L}_{\mathrm{EL}}$ closed under the initial segment relation. More formally, define

$$Ptcl(\mathcal{L}_{\mathrm{EL}}) = \{\mathsf{P} \mid \mathsf{P} \subseteq \mathcal{L}_{\mathrm{EL}}^{*} \text{ where } \mathsf{P} \text{ is closed under initial segments}\}.$$

Given an epistemic model $\mathcal{M}$, a **state-dependent PAL protocol on $\mathcal{M}$** is a function $p : D(\mathcal{M}) \to Ptcl(\mathcal{L}_{\mathrm{EL}})$.

When p is a PAL state-dependent PAL protocol, histories in $\mathsf{Forest}(\mathcal{M}, p)$ are (states followed by) sequences of formula. This leads to simpler versions of Definition 9 and Definition 10. First of all, note that two different sequences of formulas σ and σ' both of length n must lead to epistemically disjoint submodels[6] of $\mathcal{M}^{n,p}$. We can use this observation to simplify Definition 9: given an epistemic model $\mathcal{M} = \langle W, \{R_i\}_{i\in\mathcal{A}}, V\rangle$, a state-dependent PAL protocol p, each sequence of formulas σ defines a model $\mathcal{M}^{\sigma,p} = \langle W^{\sigma,p}, \{R_i^{\sigma,p}\}_{i\in\mathcal{A}}, V^{\sigma,p}\rangle$ by induction on the length of σ:

1_{pal}. $W^{\sigma_0,p} = W$, for each $i \in \mathcal{A}$, $R_i^{\sigma_0,p} = R_i$ and $V^{\sigma_0,p} = V$.
2_{pal}. $w\sigma_{m+1} \in W^{\sigma_{m+1},p}$ iff (1) $w \in W$, (2) $\mathcal{M}^{\sigma_m,p}, w\sigma_m \models \sigma_{(m+1)}$, and (3) $\sigma_{m+1} \in p(w)$.
3_{pal}. For each $w\sigma_{m+1}, v\sigma_{m+1} \in W^{\sigma_{m+1},p}$, $w\sigma_{m+1} R_i^{\sigma_{m+1},p} v\sigma_{m+1}$ iff wR_iv.
4_{pal}. For each $P \in \mathsf{At}$, $V^{\sigma_{m+1},p}(P) = \{w\sigma_{m+1} \in W^{\sigma_{m+1},p} \mid w \in V(P)\}$.

We can also simplify Definition 10: let $\mathcal{M} = \langle W, \{R_i\}_{i\in\mathcal{A}}, V\rangle$ be an epistemic model and p a state-dependent PAL protocol on $\mathcal{M}$, define $\mathsf{Forest}(\mathcal{M}, p) = \langle \mathsf{H}, \{\sim_i\}_{i\in\mathcal{A}}, V'\rangle$ as follows:

1_{pal}. $\mathsf{H} = \{h \mid h \in W^{\sigma,p}$ for some $\sigma \in \bigcup_{w\in W} p(w)\}$.
2_{pal}. For all $h, h' \in \mathsf{H}$ with $h = w\sigma$ and $h' = v\sigma$ for some $\sigma \in \bigcup_{w\in W} p(w)$, $h \sim_i h'$ iff $hR_i^{\sigma,p}h'$.
3_{pal}. For each $P \in \mathsf{At}$, $h \in V'(p)$ iff $h \in V^{\sigma,p}(p)$, where $h = w\sigma$ for some $\sigma \in \bigcup_{w\in W} p(w)$.

Our subsequent analysis will focus on two classes of structures. Given a class of state-dependent (or uniform) DEL protocols $\mathbf{X}$, let

$$\mathbb{F}(\mathbf{X}) = \{\mathsf{Forest}(\mathcal{M}, p) \mid \mathcal{M} \text{ an epistemic model and } p \in \mathbf{X}\}$$

(respectively $\mathbb{F}(\mathbf{X}) = \{\mathsf{Forest}(\mathcal{M}, \mathsf{P}) \mid \mathcal{M}$ an epistemic model and $\mathsf{P} \in \mathbf{X}\}$, when $\mathbf{X}$ is a set of uniform protocols). If $\mathbf{X} = \{p\}$ (respectively $\mathbf{X} = \{\mathsf{P}\}$) then we write $\mathbb{F}(p)$ (respectively $\mathbb{F}(\mathsf{P})$) instead of $\mathbb{F}(\{p\})$ (respectively $\mathbb{F}(\{\mathsf{P}\})$).

Our first observation is that under mild assumptions we can think of the languages $\mathcal{L}_{\mathrm{DEL}}$ and $\mathcal{L}_{\mathrm{ETL}}$ (when based on the same set of events Σ) as the *same formal language*. That is, the above model transformation allows use to reinterpret the DEL dynamic modality $\langle \mathcal{E}, e\rangle$ as a labeled temporal modality. Of course, to recover a DEL modality from an ETL temporal modality $\langle e\rangle$ we must know to which event model e belong. The main point is that, since a primitive event e may occur in different event models, a formula of $\mathcal{L}_{\mathrm{ETL}}$ does

[6]That is, we cannot have $w\sigma R_i^{n,p} v\sigma'$ if $\sigma \neq \sigma'$. This is not true for general DEL protocols where agents may not be able to distinguish between different primitive events.

not contain enough information to determine which event model different occurrences of the same primitive event e belongs. However, once an ETL model is fixed this information can be extracted (see the proof of Theorem 1). Thus $\mathcal{L}_{\mathrm{ETL}}$ and $\mathcal{L}_{\mathrm{DEL}}$ are the same formal language under the mild assumption that it can always be determined which event models different occurrences of the same primitive event belongs.

An easy induction shows that this model transformation preserves truth in the following sense. Let $\mathsf{ProtocolDEL}$ be the protocol of *all* finite sequences of DEL event models and $\mathcal{M}$ an epistemic model with $w \in D(\mathcal{M})$ (and hence (w) is a history in $\mathsf{Forest}(\mathcal{M}, \mathsf{ProtocolDEL})$):

Proposition 1 *For any formula $\varphi \in \mathcal{L}_{DEL}$,*

$$\mathcal{M}, w \models \varphi \textit{ iff } \mathsf{Forest}(\mathcal{M}, \mathsf{ProtocolDEL}), (w) \models \varphi.$$

Proof The proof is by induction on the structure of φ. The only interesting case is the (labeled) temporal modality. Suppose φ is of the form $\langle \mathcal{E}, e \rangle \psi$. The key observation is that for all epistemic models $\mathcal{M}$ with $w \in D(\mathcal{E})$ and event models $\mathcal{E}$ with $e \in D(\mathcal{E})$, for any formula $\chi \in \mathcal{L}_{\mathrm{DEL}}$,

$$(**) \qquad \mathsf{Forest}(\mathcal{M} \otimes \mathcal{E}, \mathsf{ProtocolDEL}), (w, e) \models \chi \text{ iff } \mathsf{Forest}(\mathcal{M}, \mathsf{ProtocolDEL}), (w, e) \models \chi$$

The proof of $(**)$ follows easily from the definitions. Note that the right to left direction uses the fact that $\mathsf{ProtocolDEL}$ contains all pointed event models. Using this observation, the proof of the modal case is straightforward:

$$\begin{aligned} \mathcal{M}, w \models \langle \mathcal{E}, e \rangle \psi &\text{ iff } \mathcal{M}, w \models \mathsf{pre}(e) \text{ and } \mathcal{M} \otimes \mathcal{E}, (w, e) \models \psi && \textit{(Definition 5)} \\ &\text{ iff } \mathsf{Forest}(\mathcal{M} \otimes \mathcal{E}, \mathsf{ProtocolDEL}), (w, e) \models \psi && \textit{(I.H.)} \\ &\text{ iff } \mathsf{Forest}(\mathcal{M}, \mathsf{ProtocolDEL}), (w, e) \models \psi && \textit{(**)} \\ &\text{ iff } \mathsf{Forest}(\mathcal{M}, \mathsf{ProtocolDEL}), (w) \models \langle \mathcal{E}, e \rangle \psi && \textit{(Definition 3)} \end{aligned}$$

□

Proposition 1 explains a common intuition about linking DEL to ETL. But there is more to come! Indeed, varying the parameters in Proposition 1 opens the door to a number of new questions. For example, we can extend the DEL language with temporal operators, or vary the protocol to create new DEL and ETL-style logics: much more on this will be found in Section 4 below.

3.2 From ETL to DEL

Not all ETL models can be generated by a DEL protocol. Indeed, such generated ETL models have a number of special properties. In this section we study precisely which properties these are. The main result (Theorem 1) of this section is a characterization of the ETL models that are generated by some (uniform) DEL protocol. This is an improvement of an existing characterization result found in [9] and provides a precise comparison between the DEL and ETL frameworks.

We start with the properties identified by van Benthem [9] needed to characterize the ETL models resulting from consecutive updates with one single event model. These properties come from the definition of product update (Definition 5).

Definition 11 (Synchronicity, Perfect Recall, Uniform No Miracles) Let $\mathscr{H} = \langle \Sigma, H, \{\sim_i\}_{i \in \mathscr{A}}, V \rangle$ be an ETL model. $\mathscr{H}$ satisfies:

- **Sychronicity** iff for all $h, h' \in \mathsf{H}$, if $h \sim_i h'$ then $\mathsf{len}(h) = \mathsf{len}(h')$ ($\mathsf{len}(h)$ is the number of events in h).
- **Perfect Recall** iff for all $h, h' \in \mathsf{H}$, $e, e' \in \Sigma$ with $he, h'e' \in \mathsf{H}$, if $he \sim_i h'e'$, then $h \sim_i h'$
- **Uniform No Miracles** iff for all $h, h' \in \mathsf{H}$, $e, e' \in \Sigma$ with $he, h'e' \in \mathsf{H}$, if there are $h'', h''' \in \mathsf{H}$ with $h''e, h'''e' \in \mathsf{H}$ such that $h''e \sim_i h'''e'$ and $h \sim_i h'$, then $he \sim_i h'e'$.

Remark 3 (Alternative Definition of Perfect Recall) van Benthem gives an alternative definition of Perfect Recall in [9]:

$$\text{if } he \sim_i h' \text{ then there is an event } f \text{ with } h' = h''f \text{ and } h \sim_i h''.$$

This property is equivalent over the class of ETL models to the above definition of Perfect Recall and synchronicity. We use the above formulation of Perfect Recall in order to stay closer to the computer science literature on verifying multiagent systems (cf. [22]) and the game theory literature (cf. [19]).

Note that Definition 11 are properties of ETL *frames*. Already with these properties we can say something about how to relate the two frameworks. Suppose that $\mathscr{H}$ is an ETL frame satisfying the properties in Definition 11. We can easily read off an epistemic *frame* (i.e., a set of states W and relations R_i for each agent $i \in \mathscr{A}$ on W) to serve as the initial model (let the histories of length 1 be the states and simply copy the uncertainty relations). Furthermore, we can define a "DEL-like" protocol $\mathsf{P}_{\mathscr{H}}$ (the construction is given below in the proof of Theorem 1) consisting of sequences of event models where the precondition function assigns to the primitive events *sets of finite histories*. Intuitively, if e is a primitive event (i.e., a state in an event model), then $\mathsf{pre}(e)$ is the set of histories where e can "be performed". Thus, we have a comparison of the two frameworks at the level of frames provided we work with a modified definition of an event model.

However, our main theorem is stated in terms of models, so we need additional properties. In particular, at each level of the ETL model we will need to specify a *formula* of $\mathscr{L}_{\mathrm{EL}}$ as a pre-condition for each primitive event e

(cf. Definition 5). As usual, this requires that the set of histories preceding an event e be *bisimulation-closed*:

Definition 12 (Epistemic Bisimulation) Let $\mathscr{H} = \langle \mathsf{H}, \{\sim_i\}_{i\in\mathscr{A}}, V\rangle$ and $\mathscr{H}' = \langle \mathsf{H}', \{\sim'_i\}_{i\in\mathscr{A}}, V\rangle$ be two ETL models. A relation $Z \subseteq \mathsf{H} \times \mathsf{H}'$ is an **epistemic bisimulation** provided that, for all $h \in \mathsf{H}$ and $h' \in \mathsf{H}'$, if hZh', then

(prop) h and h' satisfy the same propositional formulas,
(forth) for every $g \in \mathsf{H}$, if $h \sim_i g$ then there exists $g' \in \mathsf{H}'$ with $h' \sim_i g'$ and gZg'
(back) for every $g' \in \mathsf{H}'$, if $h' \sim'_i g'$ then there exists $g \in \mathsf{H}$ with $h \sim_i g$ and gZg'.

If Z is an epistemic bisimulation and hZh' then we say h **and** h' **are epistemically bisimilar**. An ETL model $\mathscr{H}$ satisfies **epistemic bisimulation invariance** iff for all epistemically bisimilar histories $h, h' \in \mathsf{H}$, if $he \in \mathsf{H}$ then $h'e \in \mathsf{H}$.

However, as is well-known, bisimulation-invariance alone is typically not enough to guarantee the existence of such a formula. More specifically, there are examples of *infinite* sets that are bisimulation closed but not definable by any formula of $\mathscr{L}_{EL}$ (however, it will be definable by a formula of epistemic logic with *infinitary* conjunctions—see [18] for a discussion). Thus, if the set of histories at some level in which an event e can be executed is infinite, there may not be a formula of $\mathscr{L}_{EL}$ that defines this set to be used as a pre-condition for e. Such a formula will exist under an appropriate **finiteness assumption**: at each level there are only finitely many histories in which e can be executed,[7] i.e., for each n, the set $\{h \mid he \in \mathsf{H} \text{ and } \mathsf{len}(h) = n\}$ is finite.

Note that both no miracles and bisimulation invariance are "global properties". It turns out that we only need "local" versions of these properties:

Definition 13 (Local No Miracles, Local Bisimulation Invariance) Let $\mathscr{H} = \langle \Sigma, \mathsf{H}, \{\sim_i\}_{i\in\mathscr{A}}, V\rangle$ be an ETL model. $\mathscr{H}$ satisfies:

- *Local No Miracles* iff for all $h_1, h_2, h, h' \in \mathsf{H}$, $e, e' \in \Sigma$ with $h_1e, h_2e' \in \mathsf{H}$, if $h_1e \sim_i h_2e'$ and $h \sim_i h'$ and $h_1 \sim^* h$, then $he \sim_i h'e'$ (provided $he, h'e' \in \mathsf{H}$) (Here, $\sim^*$ is the reflexive transitive closure of the union of the $\sim_i$ relations.)
- *Local Bisimulation Invariance* iff for all $h, h' \in \mathsf{H}$, if $h \sim^* h'$ and h and h' are epistemically bisimilar, and $he \in \mathsf{H}$, then $h'e \in \mathsf{H}$

One final assumption is needed since we are assuming that product update does not change the ground facts. An ETL model $\mathscr{H}$ satisfies **propositional stability** provided for all histories h in $\mathscr{H}$, events e with he in $\mathscr{H}$ and all propositional variables P, if P is true at h then P is true at he. We remark that

[7] Note that this property may be violated even in an ETL model generated from only finitely many events.

this property is not crucial for the results in this section and can be dropped provided we allow product update to change the ground facts (cf. [12]).

Theorem 1 (Main Representation Theorem) *Let $\mathbf{X}^{uni}_{DEL}$ be the class of uniform DEL protocols. If an ETL model is in $\mathbb{F}(\mathbf{X}^{uni}_{DEL})$ then it satisfies propositional stability, synchronicity, perfect recall, local no miracles, as well as local bisimulation invariance.*

If an ETL model satisfies the finiteness assumption, propositional stability, synchronicity, perfect recall, local no miracles, and local bisimulation invariance, then it is in $\mathbb{F}(\mathbf{X}^{uni}_{DEL})$.

Proof Suppose that $\mathcal{H} = \langle \Sigma, \mathsf{H}, \{\sim_i\}_{i\in\mathcal{A}}, V\rangle \in \mathbb{F}(\mathbf{X}^{\text{uni}}_{\text{DEL}})$. Then $\mathcal{H} = \mathsf{Forest}(\mathcal{M}, \mathsf{P})$ for some initial epistemic model $\mathcal{M}$ and DEL protocol P. We show that $\mathcal{H}$ satisfies local bisimulation invariance, and leave it to the reader to check that $\mathcal{H}$ satisfies the remaining properties. Suppose that $h, h' \in \mathsf{H}$ with $h \sim^* h'$, h and h' are epistemically bisimilar, and $he \in \mathsf{H}$ for some event $e \in \Sigma$ $(= D(\mathsf{P}))$. We must show $h'e \in \mathsf{H}$. By construction (Definition 8), $h = se_1e_2\cdots e_ne \in D(\mathcal{M} \otimes \mathcal{E}_1 \otimes \cdots \mathcal{E}_n \otimes \mathcal{E})$ where $(\mathcal{E}_1, e_1)(\mathcal{E}_2, e_2)\cdots(\mathcal{E}_n, e_n)(\mathcal{E}, e) \in \mathsf{P}$, $s \in D(\mathcal{M})$, for each $i = 1, \ldots, n$, $e_i \in D(\mathcal{E}_i)$ and $e \in D(\mathcal{E})$. In order to prove $h'e \in \mathsf{H}$, it is enough to show $h'e \in D(\mathcal{M} \otimes \mathcal{E}_1 \otimes \cdots \mathcal{E}_n \otimes \mathcal{E})$. This follows from two facts: (1) $h' \in D(M \otimes \mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_n)$ and (2) $h' \models \mathsf{pre}(e)$. (2) follows from the fact that h and h' are epistemically bisimilar and $\mathsf{pre}(e)$ is assumed to be a formula of $\mathcal{L}_{\text{EL}}$. (1) follows from the assumption that $h \sim^* h'$.

Suppose $\mathcal{H} = \langle \Sigma, \mathsf{H}, \{\sim_i\}_{i\in\mathcal{A}}, V\rangle$ is an ETL model satisfying the above properties. We must show there is an epistemic model $\mathcal{M}_{\mathcal{H}}$ and a DEL protocol $\mathsf{P}_{\mathcal{H}}$ such that $\mathcal{H} = \mathsf{Forest}(\mathcal{M}_{\mathcal{H}}, \mathsf{P}_{\mathcal{H}})$. For the initial epistemic model, let $\mathcal{M}_{\mathcal{H}} = \langle W, \{R_i\}_{i\in\mathcal{A}}, V'\rangle$ with $W = \{h \mid \mathsf{len}(h) = 1\}$, for $h, h' \in W$, define hR_ih' provided $h \sim_i h'$, and for each $P \in \mathsf{At}$, $V'(P) = V(P) \cap W$.

Call a history $h \in \mathsf{H}$ **maximal** if there is no $h' \in \mathsf{H}$ such that $h \prec h'$. Now, for each maximal history $h \in \mathsf{H}$, define the closure of h, denoted $C(h)$, to be the smallest set that contains all finite prefixes of h, and if $h' \in C(h)$ and $h' \sim^* h''$, then also $h'' \in C(h)$. Note that by perfect recall, $C(h)$ is closed under finite prefixes and is completely connected with respect to the $\sim^*$ relation. It is easy to see that According to Definition 1, H only contains *finite* histories. This restriction is not crucial, however, and our result remains true without it. $\mathsf{H} = \bigcup\{C(h) \mid h \text{ is a maximal history}\}$.

We define, for each maximal history $h \in \mathsf{H}$ and $j = 1, \ldots, \mathsf{len}(h)$, an event model $\mathcal{E}^h_j = \langle S^h_j, \{\longrightarrow_i\}_{i\in\mathcal{A}}, \mathsf{pre}\rangle$ as follows:

1. $S^h_j = \{e \in \Sigma \mid$ there is a history h of length j in H with $h = h' \cdot e\}$.
2. For each $e, e' \in S^h_j$, define $e \longrightarrow_i e'$ provided there are histories h and h' of length j ending in e and e' respectively, such that $h \sim_i h'$.
3. For each $e \in S^h_j$, let $\mathsf{pre}(e)$ be the formula that characterizes the set $\{h \mid he \in \mathsf{H}$ and $\mathsf{len}(h) = j\}$. Such a formula does exist, due to local bisimulation invariance and the finiteness assumption.

Finally, let $\mathsf{P}_{\mathscr{H}} = \{(\mathscr{E})^h_j \mid h$ is a maximal history in H and $j \leq \mathsf{len}(h)\}$. Clearly, $\mathsf{P}_{\mathscr{H}}$ is a DEL protocol and so is an element of $\mathbf{X}^{\text{uni}}_{\text{DEL}}$. It is easy to see that $\mathsf{Forest}(\mathscr{M}_{\mathscr{H}}, \mathsf{P}_{\mathscr{H}})$ and $\mathscr{H}$ have the same set of histories. All that remains is to prove that the epistemic relations are the same in $\mathscr{H}$ and $\mathsf{Forest}(\mathscr{M}_{\mathscr{H}}, \mathsf{P}_{\mathscr{H}})$

Claim For each $h_1, h_2 \in \mathsf{H}$, $h_1 \sim_i h_2$ in $\mathscr{H}$ iff $h_1 \sim_i h_2$ in $\mathsf{Forest}(\mathscr{M}_{\mathscr{H}}, \mathsf{P}_{\mathscr{H}})$.

Proof of Claim The proof is by induction on the length of h and h' (which is the same by synchronicity). If $\mathsf{len}(h) = 1$, the claim is immediate by the definition of $\mathscr{M}_{\mathscr{H}}$.

For the induction step, let $h_1 = h \cdot e$ and $h_2 = h' \cdot e'$. Suppose $h_1 \sim_i h_2$ in $\mathscr{H}$. Then by perfect recall, $h \sim_i h'$ in $\mathscr{H}$. So, by the induction hypothesis, $h \sim_i h'$ in $\mathsf{Forest}(\mathscr{M}_{\mathscr{H}}, \mathsf{P}_{\mathscr{H}})$ as well. By the definition given above, $e \longrightarrow_i e'$ in the appropriate event model $\mathscr{E}^{h_m}_j$ for a maximal history h_m and $j = \mathsf{len}(h_1)$. It follows by the definition of product update that $h_1 \sim_i h_2$ in $\mathsf{Forest}(\mathscr{M}_{\mathscr{H}}, \mathsf{P}_{\mathscr{H}})$.

For the other direction, assume $h_1 \sim_i h_2$ in $\mathsf{Forest}(\mathscr{M}_{\mathscr{H}}, \mathsf{P}_{\mathscr{H}})$. Then, by definition of product update, $h \sim_i h'$ in $\mathsf{Forest}(\mathscr{M}_{\mathscr{H}}, \mathsf{P}_{\mathscr{H}})$ and $e \longrightarrow_i e'$ in the appropriate event model. By the way the event model is defined, there must be some x and x' with $x \cdot e \sim_i x' \cdot e'$ in $\mathscr{H}$, and therefore, by local no miracles, also $h \cdot e \sim_i h' \cdot e'$ in $\mathscr{H}$. □

An immediate consequence is that $\mathscr{H}$ and $\mathsf{Forest}(\mathscr{M}_{\mathscr{H}}, \mathsf{P}_{\mathscr{H}})$ are the same model. □

Note that the finiteness assumption can be dropped at the expense of allowing preconditions to come from a more expressive language (specifically, infinitary epistemic logic). Alternatively, as remarked above, we can define the preconditions to be *sets* of histories (instead of formulas of some logical language). A possible compromise is to work with state-dependent protocols instead of uniform protocols. More precisely, in the above proof, we set the precondition of $e \in S^h_j$ to be $\top$, and define a local DEL protocol p so that, for all $w \in W$, $p(w) = \{(\mathscr{E})^h_j \mid h$ is a maximal history in H and $j \geq \mathsf{len}(h)\}$. Using this observation, we can argue in the same style as above to show the following representation theorem for state-dependent DEL protocols.

Theorem 2 *Let $\mathbf{X}_{DEL}$ be the class of all state-dependent DEL-protocols. Then, an ETL model is in $\mathbb{F}(\mathbf{X}_{DEL})$ iff it satisfies propositional stability, synchronicity, perfect recall, and local no miracles.*

The above theorems identify the *minimal properties* that any DEL generated model must satisfy, and thus describe exactly what type of agent is presupposed in the DEL framework. The proof generalizes the one in van Benthem & Liu [15], which is an immediate special case: let $\mathscr{E}$ be a fixed event model and $\mathsf{P}_{\mathscr{E}}$

be the protocol that consists of all finite sequences of repetitions of $\mathcal{E}$. That is, $\mathsf{P}_{\mathcal{E}} = (\{(\mathcal{E}, e) \mid e \in D(\mathcal{E})\})^* - \{\lambda\}$, where λ is the empty string.

Proposition 2 ([9, 15]) *An ETL model* $\mathcal{H} \in \mathbb{F}(\{\mathsf{P}_{\mathcal{E}}\})$ *for some event model* $\mathcal{E}$ *iff* $\mathcal{H}$ *satisfies propositional stability, synchronicity, perfect recall, uniform no miracles, as well as epistemic bisimulation invariance.*

We do not repeat the proof from [9] here since it is a specific case of our main representation theorem (Theorem 1). But there are many further DEL protocols of interest[8] and additional properties of the ETL models are needed depending on the class of DEL protocols considered. For example, let $\mathbf{X}^{\text{uni}}_{\text{PAL}}$ be the class of all uniform PAL protocols (a PAL protocol is a DEL protocol where each event model is a public announcement event model) and recall that $\mathbb{F}(\mathbf{X}^{\text{uni}}_{\text{PAL}}) = \{\mathsf{Forest}(\mathcal{M}, \mathsf{P}) \mid \mathcal{M}$ an epistemic model and P a PAL protocol$\}$.

Proposition 3 (PAL-generated models) *Let* $\mathcal{H} = \langle \Sigma, \mathsf{H}, \{\sim_i\}_{i \in \mathcal{A}}, V \rangle$. *Then* $\mathcal{H} \in \mathbb{F}(\mathbf{X}^{uni}_{PAL})$ *iff* $\mathcal{H}$ *satisfies the minimal properties of Theorem* 1*, and:*

- *for all* $h, h', he, h'e \in \mathsf{H}$*, if* $h \sim_i h'$*, then* $he \sim_i h'e$ *(all events are reflexive)*
- *for all* $h, h' \in \mathsf{H}$*, if* $he \sim_i h'e'$*, then* $e = e'$ *(no different events are linked).*

Again, the proof will be an easy variant of Theorem 1. The details are left to the reader.

4 Constrained Public Announcement Logic

The representation theorems in Section 3.2 are one way of comparing and contrasting the DEL and ETL paradigms. In this Section we turn to our second objective of this paper: to illustrate some new issues that arise when DEL and ETL are *merged* as a model of multi-agent interactive communication and learning. To that end, we study the logics of ETL models generated by *PAL protocols* (DEL protocols consisting only of *public announcements*).

More precisely, let $\mathbf{X}_{\text{PAL}}$ and $\mathbf{X}^{\text{uni}}_{\text{PAL}}$ be the set of state-dependent PAL protocols and uniform PAL protocols respectively. We will present the logics of the classes $\mathbb{F}(\mathbf{X}_{\text{PAL}})$ and $\mathbb{F}(\mathbf{X}^{\text{uni}}_{\text{PAL}})$. These classes can be thought of as representing the space of all "conversation scenarios" or "learning procedures". We first axiomatize the class $\mathbb{F}(\mathbf{X}_{\text{PAL}})$, and then turn to the uniform case $\mathbb{F}(\mathbf{X}^{\text{uni}}_{\text{PAL}})$ which will require us to extend our language somewhat. Recall that in the restricted setting of public announcements, we can simplify many of the definitions from Section 3.1 (see Remark 2 for a discussion).

[8] van Benthem and Liu [15] suggest that iterating one large disjoint union of event models involving suitable preconditions can 'mimic' ETL style evolution for more complex protocols with varying event models. We do not pursue this claim here.

4.1 The Logic of $\mathbb{F}(\mathbf{X}_{\mathrm{PAL}})$

We work with the language $\mathscr{L}_{\mathrm{PAL}}$ (containing epistemic modalities $[i]$ and announcement modalities $\langle A\rangle$ with A a formula of $\mathscr{L}_{\mathrm{EL}}$). One distinguishing feature of the current setting is that the truth of A is no longer equivalent to the availability of A for assertion. This means that the usual reduction axioms of PAL are no longer valid. Thus, the standard axiomatization of PAL does not work and we have to redo the work. We denote this new logical framework by TPAL ("temporal public announcement logic").

Definition 14 (**TPAL**-Axioms) Let **TPAL** be the smallest set of formulas of $\mathscr{L}_{\mathrm{PAL}}$ that contains the following axiom schemes.

PC. Propositional validities
K_i. $[i](\varphi \to \psi) \to ([i]\varphi \to [i]\psi)$
K_A. $[A](\varphi \to \psi) \to ([A]\varphi \to [A]\psi)$
A1. $\langle A\rangle\top \to A$
R1. $\langle A\rangle P \leftrightarrow \langle A\rangle\top \wedge P$
R2. $\langle A\rangle\neg\varphi \leftrightarrow \langle A\rangle\top \wedge \neg\langle A\rangle\varphi$
R3. $\langle A\rangle[i]\varphi \leftrightarrow \langle A\rangle\top \wedge [i](\langle A\rangle\top \to \langle A\rangle\varphi)$

Furthermore, **TPAL** is closed under $[i]$- and $[A]$- necessitation and modus ponens. We write $\vdash \varphi$ if $\varphi \in$ **TPAL**.

Remark 4 (Failure of uniform substitution) Notice that **TPAL** does not satisfy uniform substitution. For one thing, axiom $R1$ only applies to atomic propositions $P \in \mathsf{At}$. Furthermore, only formulas of $\mathscr{L}_{\mathrm{EL}}$ can be announced. So, for example, $\langle\langle B\rangle\top\rangle P \leftrightarrow \langle\langle B\rangle\top\rangle\top \wedge P$ is *not* an instance of axiom $R1$. We could actually lift this restriction somewhat without endangering our results, but will not do so here.

These axioms illustrate the mixture of factual and procedural truth, which drives conversations or processes of observation.[9] In TPAL, $\langle A\rangle\top$ means that A is *announceable*. More precisely, $\langle A\rangle\top$ represents one temporal step in a generated ETL model $\mathsf{Forest}(\mathscr{M}, p)$ for some initial model $\mathscr{M}$ and state-dependent protocol p. So, axiom A2 represents the procedural information that "only true formulas can be announced". The converse (which is derivable in PAL) is valid only on a *specific protocol* following the rule "if A is true then it can be announced".

[9]Similarly to how we set up TPAL, Lorini and Castelfranchi [32] re-define PAL as "either take the restricted model, or let it be undefined", in effect defining a PAL protocol. They do not give a completeness result, but do formulate laws similar to our TPAL axioms.

Before turning to the main result of this Section, we consider axiom R3 in more detail. Consider the following three variations of R3:

1. $\langle A\rangle[i]P \leftrightarrow A \wedge [i]\langle A\rangle P$
2. $\langle A\rangle[i]P \leftrightarrow \langle A\rangle\top \wedge [i](A \rightarrow \langle A\rangle P)$
3. $\langle A\rangle[i]P \leftrightarrow \langle A\rangle\top \wedge [i](\langle A\rangle\top \rightarrow \langle A\rangle P)$

Each of these axioms represent a different assumption about the underlying protocol and how that affects the agents' knowledge. The first is the usual PAL reduction axiom and assumes a specific protocol (which is common knowledge) where all true formulas are always available for announcement. The second (weaker) axiom is valid when there is a fixed protocol that is common knowledge (cf. Section 5.2). Finally, the third is an instance of R3 which adds a requirement that the agents must know which formulas are currently available for announcement.

Our goal in this Section is to prove the following Theorem:

Theorem 3 **TPAL** *is sound and strongly complete with respect to the class of ETL models* $\mathbb{F}(\mathbf{X}_{\mathrm{PAL}})$.

The proof is a variant of the standard Henkin construction.[10] We construct the canonical ETL model from the set of **TPAL** maximal consistent sets (mcs). The main idea is that each mcs defines sequences of 'legal' public announcements which we use to define a canonical state-dependent protocol. We start by defining the set of legal histories and a function λ_n that assigns maximally consistent sets to each node on a history.

Definition 15 (Legal Histories) Let W_0 be the set of all **TPAL** maximal consistent sets. We define λ_n and H_n ($0 \leq n \leq d(\Sigma)$) as follows:

- Set $H_0 = W_0$, and for each $w \in H_0$, $\lambda_0(w) = w$.
- Let $H_{n+1} = \{hA \mid h \in H_n \text{ and } \langle A\rangle\top \in \lambda_n(h)\}$. For each $h = h'A \in H_{n+1}$, define $\lambda_{n+1}(h) = \{\varphi \mid \langle A\rangle\varphi \in \lambda_n(h')\}$.

We first confirm that each map λ_n is well-defined.

Lemma 1 *For each* $n \geq 0$*, for each* $\sigma \in H_n$*,* $\lambda_n(\sigma)$ *is maximally consistent.*

Proof The proof is by induction on n. The case $n = 0$ is by definition. Suppose that the statement holds for H_n and λ_n. Suppose $\sigma \in H_{n+1}$ with $\sigma = \sigma' A$. By the induction hypothesis, $\lambda_n(\sigma')$ is a maximally consistent set. Furthermore,

[10] The usual completeness proofs for PAL and DEL reduce the DEL expressions to standard modal logic. This device is no longer available to us here in a straightforward manner, though TPAL does allow for some 'normal form reduction'. Accordingly, the completeness proof for TPAL in this Section uses a Henkin-style model—a method also used for DEL in [24].

by the construction of H_{n+1}, $\langle A\rangle\top \in \lambda_n(\sigma)$. Therefore, $\lambda_{n+1}(\sigma) \neq \emptyset$. Let $\varphi \in \mathscr{L}_{TPAL}$. Since $\lambda_n(\sigma')$ is a maximally consistent set, either $\langle A\rangle\varphi \in \lambda_n(\sigma')$ or $\neg\langle A\rangle\varphi \in \lambda_n(\sigma')$. If $\langle A\rangle\varphi \in \lambda_n(\sigma')$, $\varphi \in \lambda_{n+1}(\sigma)$ by construction. If $\neg\langle A\rangle\varphi \in \lambda_n(\sigma')$, by axiom R2, we have $\langle A\rangle\neg\varphi \in \lambda_n(\sigma')$. Thus, by construction, $\neg\varphi \in \lambda_{n+1}(\sigma)$. Hence, for all $\varphi \in \mathscr{L}_{TPAL}$, either $\varphi \in \lambda_{n+1}(\sigma)$ or $\neg\varphi \in \lambda_{n+1}(\sigma)$.

To show that λ_{n+1} is consistent, assume toward contradiction that there are formulas $\varphi_1, ..., \varphi_m \in \lambda_{n+1}(\sigma)$ such that $\vdash \bigwedge_{i=1}^{m} \varphi \to \bot$. Using standard modal reasoning, $\vdash \langle A\rangle\top \to \bigvee_{i=1}^{m}\langle A\rangle\neg\varphi_i$. Since $\langle A\rangle\top \in \lambda_n(\sigma')$, we have $\bigvee_{i=1}^{m}\langle A\rangle\neg\varphi \in \lambda_n(\sigma')$. And so, since $\lambda_n(\sigma')$ is a maximally consistent set, there is some j with $1 \leq j \leq m$ and $\langle A\rangle\neg\varphi_j \in \lambda_n(\sigma')$. Using axioms $R2$, we have $\neg\langle A\rangle\varphi_j \in \lambda_n(\sigma')$. By construction of $\lambda_{n+1}(\sigma)$ we have for each $i = 1, \ldots, m$, $\langle A\rangle\varphi_i \in \lambda_n(\sigma')$. This contradicts the fact that $\lambda_n(\sigma')$ is consistent. □

We now define a canonical ETL model $\mathscr{H}^{\text{can}}$. We start by defining $\mathscr{H}_0^{\text{can}} = \langle H_0, \{\sim_i^0\}_{i\in\mathscr{A}}, V^0\rangle$. For this, we use the usual definitions:

- For $w, v \in H_0$, let $w \sim_i^0 v$ iff $\{\varphi \mid [i]\varphi \in w\} \subseteq v$.
- For each $P \in \mathsf{At}$ and $w \in H_0$, $w \in V^0(P)$ iff $P \in w$.

Definition 16 (Canonical Model) The canonical model $\mathscr{H}^{\text{can}} = \langle \mathsf{H}^{\text{can}}, \{\sim_i^{\text{can}}\}_{i\in\mathscr{A}}, V^{\text{can}}\rangle$ is defined as follows:

- $\mathsf{H}^{\text{can}} = \bigcup_{i=0}^{\infty} H_i$.
- For each $h, h' \in \mathsf{H}^{\text{can}}$ with $h = w\sigma$ and $h' = w'\sigma'$, let $h \sim_i^{\text{can}} h'$ iff (1) $\sigma = \sigma'$ and (2) $w \sim_i^0 v$.
- For every $P \in \mathsf{At}$ and $h = w\sigma \in \mathsf{H}^{\text{can}}$, $w\sigma \in V^{\text{can}}(P)$ iff $w \in V^0(P)$.

Given $h \in \mathsf{H}^{\text{can}}$ with $h = wA_1 \cdots A_n$, we write $\lambda(h)$ for $\lambda_n(h)$. We now show that the canonical model $\mathscr{H}^{\text{can}}$ works as intended:

Lemma 2 (Truth Lemma) *For every $\varphi \in \mathscr{L}_{PAL}$, for each $h \in \mathsf{H}^{\text{can}}$,*

$$\varphi \in \lambda(h) \quad \textit{iff} \quad \mathscr{H}^{\text{can}}, h \models \varphi.$$

Proof We show by induction on the structure of $\varphi \in \mathscr{L}_{\text{PAL}}$ that for each $h \in \mathsf{H}^{\text{can}}$, $\varphi \in \lambda(h)$ iff $\mathscr{H}^{\text{can}}, h \models \varphi$. The base and the boolean cases are straightforward. For the knowledge modality, let $h \in \mathsf{H}^{\text{can}}$ with $h = wA_1 \cdots A_n$ and assume $[i]\psi \in \lambda(h)$. Suppose $h' \in \mathsf{H}^{\text{can}}$ with $h \sim_i h'$. By construction of the canonical model, we know that $h' = vA_1 \cdots A_n$ for some $v \in H_0$ with $w \sim_i^0 v$. By Definition 15, since $[i]\psi \in \lambda(wA_1 \cdots A_n)$, we have $\langle A_n\rangle[i]\psi \in \lambda(wA_1 \cdots A_{n-1})$. Using Axiom $R3$, we have $[i](\langle A_n\rangle\top \to \langle A_n\rangle\psi) \in \lambda(wA_1 \cdots A_{n-1})$. Continuing this way, we have

$$[i](\langle A_1\rangle\top \to \langle A_1\rangle(\langle A_2\rangle\top \to \langle A_2\rangle(\cdots\langle A_{n-1}\rangle(\langle A_n\rangle\top \to \langle A_n\rangle\psi)\cdots)) \in w.$$

By Definition 16, since $h \sim_i^{\text{can}} h'$, we have $w \sim_i^0 v$. Hence,

$$\langle A_1\rangle\top \rightarrow \langle A_1\rangle(\langle A_2\rangle\top \rightarrow \langle A_2\rangle(\cdots\langle A_{n-1}\rangle(\langle A_n\rangle\top \rightarrow \langle A_n\rangle\psi)\cdots) \in v.$$

Now note that

$$\langle A_1\rangle\top \in \lambda(w), \langle A_2\rangle\top \in \lambda(wA_1), \ldots, \langle A_n\rangle\top \in \lambda(wA_1 \ldots A_{n-1}).$$

Thus, we have

$$\langle A_2\rangle\top \rightarrow \langle A_2\rangle(\cdots\langle A_{n-1}\rangle(\langle A_n\rangle\top \rightarrow \langle A_n\rangle\psi)\cdots) \in \lambda(vA_1)$$

$$\langle A_3\rangle\top \rightarrow \langle A_3\rangle(\cdots\langle A_{n-1}\rangle(\langle A_n\rangle\top \rightarrow \langle A_n\rangle\psi)\cdots) \in \lambda(vA_1A_2)$$

$$\vdots$$

$$\langle A_n\rangle\psi \in \lambda(vA_1\cdots A_{n-1})$$

Therefore, $\psi \in \lambda(vA_1\cdots A_n) = \lambda(h')$. By the induction hypothesis, $\mathscr{H}^{\text{can}}, h' \models \psi$. Therefore, $\mathscr{H}^{\text{can}}, h \models [i]\psi$, as desired.

For the other direction, let $h \in \mathsf{H}^{\text{can}}$ and assume $[i]\psi \notin \lambda(h)$. For simplicity, we let $h = wA$ with $w \in W_0$ and $A \in \mathscr{L}_{\text{EL}}$. The argument can easily be generalized to deal with the general case along the lines of the argument above. Since $\lambda(h)$ is a maximally consistent set, we have $\neg[i]\psi \in \lambda(h)$. Thus, by Definition 15, $\langle A\rangle\neg[i]\psi \in \lambda(w)$. Using axiom $R2$, $\neg\langle A\rangle[i]\psi \in \lambda(w)$; and so, by Axiom $R3$, $\neg\langle A\rangle\top \vee \neg[i](\langle A\rangle\top \rightarrow \langle A\rangle\psi) \in \lambda(w)$. Since $\langle A\rangle\top \in \lambda(w)$ by construction, it follows that $\neg[i](\langle A\rangle\top \rightarrow \langle A\rangle\psi) \in \lambda(w)$. Now consider the set $v_0 = \{\theta \mid [i]\theta \in \lambda(w)\} \cup \{\neg(\langle A\rangle\top \rightarrow \langle A\rangle\psi)\}$. We claim that this set is consistent. Suppose not. Then, there are formulas $\theta_1, \ldots, \theta_m$ such that $\vdash \bigwedge_{j=1}^m \theta_j \rightarrow \langle A\rangle\top \rightarrow \langle A\rangle\psi$ and for $j = 1, \ldots, m$, $[i]\theta_j \in \lambda(w)$. By standard modal reasoning, $\vdash \bigwedge_{j=1}^m [i]\theta_j \rightarrow [i](\langle A\rangle\top \rightarrow \langle A\rangle\psi)$. This implies that $[i](\langle A\rangle\top \rightarrow \langle A\rangle\psi) \in \lambda(w)$. However, this contradicts the fact that $\neg[i](\langle A\rangle\top \rightarrow \langle A\rangle\psi) \in \lambda(w)$, since $\lambda(w)$ is a maximally consistent set. Now using standard arguments (Lindenbaum's lemma), there exists a maximally consistent set v with $v_0 \subseteq v$. By the construction of v, we must have $w \sim_i^0 v$ and thus $wA \sim_i^{\text{can}} vA$. Also, since $\neg(\langle A\rangle\top \rightarrow \langle A\rangle\psi) \in v$, we have $\langle A\rangle\top \in \lambda(v)$ and $\neg\langle A\rangle\psi \in \lambda(v)$. Therefore, by axiom $R2$, $\langle A\rangle\neg\psi \in \lambda(v)$. Hence $\neg\psi \in \lambda(vA)$ and therefore $\psi \notin \lambda(vA)$. By the induction hypothesis, $\mathscr{H}^{\text{can}}, vA \not\models \psi$. This implies $\mathscr{H}^{\text{can}}, wA \not\models [i]\psi$, as desired.

For the public announcement operator, assume that $\langle A\rangle\psi \in \lambda(h)$. Since $\langle A\rangle\top \in \lambda(h)$ (for $\neg\langle A\rangle\top \in \lambda(h)$ makes $\lambda(h)$ inconsistent), $\psi \in \lambda(hA)$. By the induction hypothesis, we have $\mathscr{H}^{\text{can}}, hA \models \psi$, which implies $\mathscr{H}^{\text{can}}, h \models \langle A\rangle\psi$. For the other direction, assume $\mathscr{H}^{\text{can}}, h \models \langle A\rangle\psi$. Then, $\mathscr{H}^{\text{can}}, hA \models \psi$. By the induction hypothesis, we have $\psi \in \lambda(hA)$ and thus $\langle A\rangle\psi \in \lambda(h)$. □

All that remains is to show that canonical model $\mathscr{H}^{\text{can}}$ is in the class of intended models: i.e., it is an element of $\mathbb{F}(\mathbf{X}_{\text{PAL}})$.

Lemma 3 *$\mathscr{H}^{can}$ is in $\mathbb{F}(\mathbf{X}_{PAL})$. That is, there is an epistemic model $\mathscr{M}$ and state-dependent protocol p on $\mathscr{M}$ such that $\mathscr{H}^{can} = \mathsf{Forest}(\mathscr{M}, p)$.*

Proof Let $\mathscr{M}_{\text{can}} = \langle W_0, \{\sim_i^0\}_{i \in \mathscr{A}}, V^0 \rangle$ and define $p_{\text{can}} : W_0 \to \mathscr{L}_{\text{EL}}^*$ so that $p_{\text{can}}(w) = \{\sigma \mid w\sigma \in \mathsf{H}^{\text{can}}\}$. Suppose that $\mathscr{H}^{p_{\text{can}}} = \mathsf{Forest}(\mathscr{M}_{\text{can}}, p_{\text{can}})$. We claim that $\mathscr{H}^{\text{can}}$ and $\mathscr{H}^{p_{\text{can}}}$ are the same model. For this, it suffices to show that for all $w \in W_0$ and $\sigma \in (\mathscr{L}_{\text{EL}})^*$ we have $w\sigma \in \mathsf{H}^{\text{can}}$ iff $w\sigma \in W^{\sigma, p_{\text{can}}}$ (cf. Definition 9 and Remark 2). For this implies $\mathsf{H}^{\text{can}} = \mathsf{H}^{p_{\text{can}}}$, where $\mathsf{H}^{p_{\text{can}}}$ is the domain of $\mathscr{H}^{p_{\text{can}}}$. Then, by inspecting Definition 10 (and Remark 2) and Definition 16, we see that $\mathscr{H}^{\text{can}}$ and $\mathscr{H}^{p_{\text{can}}}$ are the same model.

We show by induction on the length of $\sigma \in \mathscr{L}_{\text{EL}}^*$ that for any $w \in W_0$, $w\sigma \in \mathsf{H}^{\text{can}}$ iff $w\sigma \in W^{\sigma, p_{\text{can}}}$. The base case ($\mathsf{len}(\sigma) = 0$) is clear. Assume that the claim holds for all σ with $\mathsf{len}(\sigma) = n$.

Given any $\sigma \in \mathscr{L}_{\text{EL}}^*$ with $\mathsf{len}(\sigma) = n$, we first show by subinduction (on the structure of A) that, for all $A \in \mathscr{L}_{\text{EL}}$, $\mathscr{H}^{\text{can}}, w\sigma \models A$ iff $\mathscr{M}^{\sigma, p_{\text{can}}}, w\sigma \models A$. The base and boolean cases are straightforward. Suppose that $\mathscr{H}^{\text{can}}, w\sigma \models [i]B$. We must show $\mathscr{M}^{\sigma, p_{\text{can}}}, w\sigma \models [i]B$. Let $v\sigma \in W^{\sigma, p_{\text{can}}}$ with $w\sigma \sim_i^{\sigma, p} v\sigma$. By the main induction hypothesis, we have both $v\sigma \in \mathsf{H}^{\text{can}}$ and $w\sigma \in W^{\sigma, p_{\text{can}}}$. By Definition 9 and Remark 2, since $w\sigma \sim_i^{\sigma, p_{\text{can}}} v\sigma$, we have $w \sim_i^0 v$. Thus by Definition 16, $w\sigma \sim_i^{\text{can}} v\sigma$. Hence, $\mathscr{H}^{\text{can}}, v\sigma \models B$. By the subinduction hypothesis, $\mathscr{M}^{\sigma, p_{\text{can}}}, v\sigma \models B$. Therefore, $\mathscr{M}^{\sigma, p_{\text{can}}}, w\sigma \models [i]B$.

Coming back to the main induction, assume $w\sigma A \in \mathsf{H}_{\text{can}}$. This implies that $\langle A \rangle \top \in \lambda(w\sigma)$. By the Truth Lemma, we have $\mathscr{H}^{\text{can}}, w\sigma \models \langle A \rangle \top$. This, together with axiom $A2$, implies $\mathscr{H}^{\text{can}}, w\sigma \models A$. From the above subinduction, it follows that $\mathscr{M}^{\sigma, p_{\text{can}}}, w\sigma \models A$ (recall that $A \in \mathscr{L}_{\text{EL}}$ by definition). Thus, by the construction of p_{can}, we have $w\sigma A \in W^{\sigma A, p_{\text{can}}}$. This shows that if $w\sigma A \in \mathsf{H}^{\text{can}}$ then $w\sigma A \in W^{\sigma A, p_{\text{can}}}$. The other direction is similar. This completes the proof. □

The proof of the completeness theorem (Theorem 3) follows from Lemma 2 and Lemma 3 using a standard argument. The details are left to the reader.

5 Exploring TPAL Further

Completeness for **TPAL** is just the beginning of exploring its logical properties. In this Section, we briefly consider a few more, referring to the extended on-line version of this paper [13] and the forthcoming dissertation by Tomohiro Hoshi [30] for details.

5.1 Decidability

As is the case for public announcement logic, the satisfiability problem for **TPAL** is decidable. As usual, we can show this by constructing a *finite* model for a given satisfiable formula φ, but the precise implementation takes some care. We state a few highlights without proof.

First of all, formulas $\varphi \in \mathscr{L}_{\text{PAL}}$ can describe, at most, what is true after a sequence of announcements bounded in length by the *depth* of φ.

Definition 17 (Depth of a Formula) Suppose $\varphi \in \mathscr{L}_{\mathrm{PAL}}$. The **depth** of φ, denoted $d(\varphi)$, is defined as follows:

- $d(P) = 0$ with $P \in \mathsf{At}$
- $d(\neg\varphi) = d(\varphi)$
- $d(\varphi \wedge \psi) = max(d(\varphi), d(\psi))$
- $d(K_i\varphi) = d(\varphi)$
- $d(\langle A\rangle\varphi) = 1 + d(\varphi)$

This definition is lifted to a set $X \subseteq \mathscr{L}_{\mathrm{PAL}}$ as follows: $d(X) = \max\{d(\varphi) \mid \varphi \in X\}$.

Given a protocol p on $\mathscr{M}$ and a sequence $\sigma \in (\mathscr{L}_{\mathrm{EL}})^*$ with $\sigma \in p(w)$ for some $w \in D(\mathscr{M})$, we define a protocol $p_k^{\sigma<}$ on $\mathscr{M}^{\sigma,p}$ so that $p_k^{\sigma<}(w\sigma) = \{\tau \mid \sigma\tau \in p(w) \text{ and } \mathsf{len}(\tau) \le k\}$ for all $w\sigma \in D(\mathscr{M}^{\sigma,p})$. This family represents which sequences of formulas of length k or less are announcable after σ.

Observation 1 Let $\mathscr{M}$ be an epistemic model, p a state-dependent protocol on $\mathscr{M}$. For all $w \in D(\mathscr{M})$ and $\sigma \in \bigcup_{w\in D(\mathscr{M})} p(w)$,

$$\mathsf{Forest}(\mathscr{M}, p), w\sigma \models \varphi \text{ iff } \mathsf{Forest}(\mathscr{M}^{\sigma,p}, p_{d(\varphi)}^{\sigma<}), w\sigma \models \varphi.$$

Next, the histories relevant to evaluate a given formula $\varphi \in \mathscr{L}_{\mathrm{PAL}}$ are the ones that contain its subformulas. Let $\mathrm{sub}^a(\varphi)$ be the set of subformulas of φ that are in $\mathscr{L}_{\mathrm{EL}}$. Given a state-dependent protocol p on a model $\mathscr{M}$, for $w \in D(\mathscr{M})$ define $(p(w))_{\mathrm{sub}^a(\varphi)}$ as follows:

$$(p(w))_{\mathrm{sub}^a(\varphi)} = \{\sigma \in p(w) \mid \text{ for each } A \text{ in } \sigma,\ A \in \mathrm{sub}^a(\varphi)\}.$$

These are the admissible φ-restricted announcement sequences at w.

Observation 2 Let $\mathscr{M}$ be an epistemic model, f, g two protocols on $\mathscr{M}$. Suppose we have $(f(v))_{\mathrm{sub}^a(\varphi)} = (g(v))_{\mathrm{sub}^a(\varphi)}$ for all $v \in D(\mathscr{M})$. Then for all $w \in D(\mathscr{M})$,

$$\mathsf{Forest}(\mathscr{M}, f), w \models \varphi \text{ iff } \mathsf{Forest}(\mathscr{M}, g), w \models \varphi.$$

Finally we state the analogue of Proposition 1. Given a formula $\varphi \in \mathscr{L}_{\mathrm{PAL}}$ and an epistemic model $\mathscr{M}$, define f_φ so that, for all $w \in D(\mathscr{M})$, $f_\varphi(w) = \{A_1 \cdots A_k \mid A_i \in \mathrm{sub}^a(\varphi)\ (1 \le i \le k) \text{ for some } k\}$.

Observation 3 Let $\varphi \in \mathscr{L}_{\mathrm{TPAL}}$. Then

$$\mathscr{M}, w \models \varphi \text{ iff } \mathsf{Forest}(\mathscr{M}, f_\varphi), w \models \varphi.$$

One can now follow the completeness proof from Section 4.1 and construct a canonical model from a finite set of formulas satisfying some closure conditions whose combinatorial details we omit here. The eventual result is the following two finitized correctness statements (the superscript fin in the statements below signifies that the object is *finite*):

Lemma 4 (Finite Truth Lemma) *Let Σ be a finite set of formulas satisfying certain closure conditions (cf.* [13]*) and $\varphi \in \Sigma$. For all $h \in H^{fin}$ such that* $\mathsf{len}(h) \leq d(\Sigma) - d(\varphi) + 1$,

$$\varphi \in \lambda^{fin}(h) \text{ iff } \mathscr{H}^{fin}, h \models \varphi.$$

Lemma 5 *$\mathscr{H}^{fin}$ is an ETL model generated from an epistemic model and a PAL protocol.*

Putting everything together in the usual manner, we have that:

Theorem 4 (Decidability of TPAL) *The satisfiability problem for* **TPAL** *is decidable.*

What this analysis does not supply is the precise *computational complexity* of **TPAL**, which remains an open problem. For **PAL**, the complexity of satisfiability is Pspace-complete, but we do not know if the reduction technique of [33] for that result lifts to our setting.

5.2 The Logic of $\mathbb{F}(\mathbf{X}^{\mathrm{uni}}_{\mathrm{PAL}})$

Next consider the issue of special models on top of our general semantics, in particular, the move from state-dependent to uniform protocols. Formally, let $\mathscr{H}$ be an ETL model generated by an epistemic model $\mathscr{M} = \langle W, \{R_i\}_{i \in \mathscr{A}}, V \rangle$ and a (state-dependent or uniform) PAL protocol. Let $w \in W$ and σ be a sequence of announcements with $w\sigma \in D(\mathscr{H})$. We interpret an additional existential modality $E\varphi$ as accessibility at the same tree level:

$$\mathscr{H}, w\sigma \models E\varphi \text{ iff } \exists v \in W \text{ such that } v\sigma \in D(\mathscr{H}) \text{ and } \mathscr{H}, v\sigma \models \varphi.$$

This operator functions as an existential modality at each 'stage' of successive public announcements. The dual U is a universal modality in the same sense. We denote the extension of $\mathscr{L}_{\mathrm{PAL}}$ with the existential modality by $\mathscr{L}^{E}_{\mathrm{PAL}}$ (similarly, $\mathscr{L}^{E}_{\mathrm{EL}}$ is the language extending $\mathscr{L}_{\mathrm{EL}}$ with the existential modality E).

Moving to a larger language gives us more formulas to announce: the definitions of a PAL event model and PAL (state-dependent) protocols should be amended to allow formulas of $\mathscr{L}^{E}_{\mathrm{EL}}$ to be announced. In this new setting, we extend the logic **TPAL** with the following set of axioms:

E1. $E(\varphi \to \psi) \to (E\varphi \to E\psi)$
E2. $\varphi \to E\varphi$
E3. $\varphi \to UE\varphi$

E4. $EE\varphi \to E\varphi$
E5. $U\varphi \to [i]\varphi$
R4. $\langle A\rangle E\varphi \leftrightarrow \langle A\rangle\top \wedge E\langle A\rangle\varphi$

We also include U-necessitation. The resulting logic will be denoted by **TPAL**^{E}. Axioms E1-5 are the standard axiomatization of the existential modality (cf. [18, pg. 417]). It is not hard to see that **TPAL**^{E} is a conservative extension of **TPAL** and that analogues of Lemma 2 and Lemma 3 can be proven. However, we are interested in axiomatizing the following class of ETL models:

$$\mathbb{F}(\mathbf{X}^{\mathrm{uni}}_{\mathrm{PAL}}) = \{\mathsf{Forest}(\mathcal{M}, \mathsf{P}) \mid \mathcal{M} \text{ an epistemic model and } \mathsf{P} \text{ a PAL}^E \text{ protocol}\}$$

For this, we add the following axiom to **TPAL**^{E}:

Uni. $\langle A\rangle\top \to U(A \to \langle A\rangle\top)$.

This formula *characterizes* uniform protocols in the following sense. Say a state-dependent protocol p on an epistemic model $\mathcal{M}$ **generates a uniform ETL model** if $\mathsf{Forest}(\mathcal{M}, p) = \mathsf{Forest}(\mathcal{M}, \mathsf{P})$ for some uniform PAL protocol P. We first make a simple observation:[11]

Observation 4 Let $\mathcal{H} = \mathsf{Forest}(\mathcal{M}, p)$. For $\varphi \in \mathcal{L}^E_{\mathrm{EL}}$, histories h in $\mathsf{Forest}(\mathcal{M}, p)$ with $h = w\sigma$ where $w \in D(\mathcal{M})$ and $\sigma \in (\mathcal{L}^E_{\mathrm{EL}})^*$, $\mathcal{H}, h\sigma \models \varphi$ iff $\mathcal{M}^{\sigma,p}, w\sigma \models \varphi$.

Proposition 4 *The axiom Uni is valid on a frame of* $\mathsf{Forest}(\mathcal{M}, p)$ *iff* p *generates a uniform ETL model.*

Proof ($\Leftarrow$) Assume that p generates a uniform ETL model $\mathcal{H} = \mathsf{Forest}(\mathcal{M}, p)$. Then there is some uniform protocol P such that $\mathcal{H} = \mathsf{Forest}(\mathcal{M}, \mathsf{P})$. Now suppose that $w \in D(\mathcal{M})$ and $\sigma \in (\mathcal{L}^E_{\mathrm{EL}})^*$. Assume that $\mathcal{H}, w\sigma \models \langle A\rangle\top$. Then, we have $w\sigma A \in D(\mathcal{H})$. This means that $\sigma A \in p(w)$. Since p is uniform, there is some PAL protocol P such that $\mathcal{H} = \mathsf{Forest}(\mathcal{M}, \mathsf{P})$. Therefore $\sigma A \in \mathsf{P}$. Now, let v be an arbitrary state in $\mathcal{M}$. If $\mathcal{H}, v\sigma \models A$, then, since $\sigma A \in \mathsf{P}$, we have $v\sigma A \in D(\mathcal{H})$. Hence $\mathcal{H}, v\sigma \models \langle A\rangle\top$. Since v was arbitrary, we have $\mathcal{H}, w\sigma \models U(A \to \langle A\rangle\top)$.

($\Rightarrow$) Let *Uni* be valid on an ETL model $\mathcal{H}^p = \mathsf{Forest}(\mathcal{M}, p)$. Construct a protocol $\mathsf{P} = \{\sigma \mid w\sigma \text{ is in } \mathcal{H}^p \text{ for some } w \in D(\mathcal{M})\}$. Clearly, P is closed under prefixes, so it is a PAL protocol. We need to show that $\mathcal{H}^p = \mathsf{Forest}(\mathcal{M}, \mathsf{P})$. For this, it suffices that, for all σ, $\mathcal{M}^{\sigma,p} = \mathcal{M}^{\sigma,\mathsf{P}}$, equivalently (via Definition 9 and Remark 2) $W^{\sigma,p} = W^{\sigma,\mathsf{P}}$. The left-to-right inclusion is clear by the construction of P. For the converse, we use induction on the length of σ. For the base case, σ is the empty sequence; and so, the inclusion clearly

[11] We remark that an analogous result is true in the more general setting of arbitrary DEL protocols—truth of epistemic formulas only depends on the model at the “current level”.

holds as $W^{\sigma,p} = W^{\sigma,\mathsf{P}} = D(\mathscr{M})$. For the inductive step, assume that $w\sigma A \in W^{\sigma A,\mathsf{P}}$. Then we have $\mathscr{M}^{\sigma,\mathsf{P}}, w\sigma \models A$. By the induction hypothesis, we have $\mathscr{M}^{\sigma,p}, w\sigma \models A$. Since $A \in \mathscr{L}^{E}_{\mathrm{EL}}$, it follows from Observation 4 that $\mathscr{H}^{p}, w\sigma \models A$. Now by the construction of P, there must be some $v \in D(\mathscr{M})$ such that $v\sigma A \in W^{\sigma,p}$. This implies that $\mathscr{H}^{p}, v\sigma \models \langle A\rangle\top$. Here, since Uni is valid in $\mathscr{H}^{p}$, we have $\mathscr{H}^{p}, v\sigma \models U(A \to \langle A\rangle\top)$. Thus, it follows that $\mathscr{H}^{p}, w\sigma \models A \to \langle A\rangle\top$. From the fact that $\mathscr{H}^{p}, w\sigma \models A$, we then have $\mathscr{H}^{p}, w\sigma \models \langle A\rangle\top$, which is equivalent to $w\sigma A \in D(\mathscr{H}^{p})$, i.e., $w\sigma A \in W^{\sigma A,p}$, as desired. □

Let $\mathbf{TPAL}^{Uni}$ be the extension of $\mathbf{TPAL}^{E}$ with the axiom *Uni*. The following is an immediate consequence of a suitable Truth Lemma (cf. Lemma 2) and the above proposition:

Corollary 1 $\mathbf{TPAL}^{Uni}$ *is sound and strongly complete with respect to* $\mathbb{F}(\mathbf{X}^{uni}_{\mathrm{PAL}})$.

Proof The proof is similar to the one outlined in Section 4.1, where Proposition 4 shows that the canonical model is generated by a uniform protocol. □

5.3 Embedding PAL in TPAL

Finally, here is a perhaps surprising issue. The relation between the original public announcement logic (**PAL**, cf. [24, 38]) and our new **TPAL** is not completely straightforward. Clearly, all principles of **TPAL** are valid in **PAL**. Indeed, the inclusion seems proper, as standard public announcement logic is about special "full" protocols. But is it really *stronger* than **TPAL**? Using the existential modality of the previous Section, we can answer this question almost in the negative by providing an effective semantic *translation* from **PAL** into $\mathbf{TPAL}^{E}$:

We write $\models_{\mathrm{PAL}^E} \varphi$ if φ is valid on all epistemic models $\mathscr{M}$ where truth is defined as in Definition 5 (all event models are public announcements) and the existential modality is interpreted as above. We write $\models_{TPAL^E} \varphi$ if φ is valid on all models of the form $\mathsf{Forest}(\mathscr{M}, p)$ where p is a state-dependent protocol (and the existential modality is defined as above). Given a formula $\varphi \in \mathscr{L}^{E}_{\mathrm{PAL}}$, let $Ptcl(\varphi)$ be the set of formulas of the form:

$$U(A_1 \to \langle A_1\rangle(A_2 \to \langle A_2\rangle(\cdots\langle A_k\rangle(A_k \to \langle A_k\rangle\top)\cdots)))$$

where $A_i \in \mathrm{sub}^{a}(\varphi)$ $(1 \le i \le k)$ and $1 \le k \le d(\varphi)$. The formulas in $Ptcl(\varphi)$ state that the public announcements relevant to the truth value of φ are all announceable at any node of a given ETL-model.

Theorem 5 *For any formula* $\varphi \in \mathscr{L}_{PAL}$,

$$\models_{PAL^E} \varphi \;\textit{ iff }\; \models_{TPAL^E} \bigwedge Ptcl(\varphi) \to \varphi.$$

Proof The proof revolves around the following result (cf. Observation 3):

Claim If h' is in $\mathsf{Forest}(\mathscr{M}^{\sigma,p}, p_\varphi)$, then h' is in $\mathsf{Forest}(\mathscr{M}, p)$.

By the claim, $\mathsf{Forest}(\mathscr{M},p)$ includes $\mathsf{Forest}(\mathscr{M}^{\sigma,p},p_\varphi)$. Hence, $\mathsf{Forest}(\mathscr{M}^{\sigma,p}, p_\varphi), w\sigma \models \varphi$, and by Observations 1 and 2, $\mathsf{Forest}(\mathscr{M}, p) \models \varphi$. □

We do not know if we can do this reduction without the existential modalitiy. Also, we have not solved the opposite question, whether TPAL can be *faithfully embedded* into PAL, though we think the answer is negative.

6 A Glimpse of Broader Issues

Epistemic temporal logic and dynamic epistemic logic are two important but interestingly different ways of describing knowledge-based agent interaction over time. We have shown how the two can be linked in two ways: using structural representation theorems (cf. Theorem 1 and related results in Section 3.2) and new sorts of axiomatic completeness theorems for epistemic temporal model classes generated by DEL protocols (cf. Theorem 3 and other results in Section 4). Our results suggest a more systematic 'logic of protocols' using ideas from DEL to add fine-structure to ETL. In this final Section, we sketch some general issues suggested by this perspective, referring again to [13] and [30] for details.

More specifically, using the construction discussed in Section 3.1, each set of DEL protocols induces a class of ETL models: those generated by an initial model and a protocol from the given set. Recall that if $\mathbf{X}$ is a set of DEL protocols, we define $\mathbb{F}(\mathbf{X}) = \{\mathsf{Forest}(\mathscr{M}, \mathsf{P}) \mid \mathscr{M}$ an epistemic model and $\mathsf{P} \in \mathbf{X}\}$. This construction suggests the following natural questions:

- Which DEL protocols generate interesting ETL models?
- Which modal languages are most suitable to describe these models?
- Can we axiomatize interesting classes of DEL-generated ETL models?

For some combinations of model classes and languages we already know answers. For example, recall that $\mathsf{ProtocolDEL}$ is the set of *all* finite sequences of DEL event models. Then

$$\mathbb{F}(\{\mathsf{ProtocolDEL}\}) = \{\mathsf{Forest}(\mathscr{M}, \mathsf{ProtocolDEL}) \mid \mathscr{M} \text{ an epistemic model}\}$$

is the set consisting of all DEL generated ETL models. Its logic (with respect to the language $\mathscr{L}_{\mathrm{DEL}}$) can be axiomatized using the well-known reduction axioms: indeed this is the standard completeness theorem for DEL [5].

6.1 Language Extensions

$\mathscr{L}_{\text{ETL}}$ is only one of many languages for reasoning about DEL generated ETL models, and there are many other temporal and epistemic operators of interest. Typical examples include a common knowledge operator $C\varphi$ and a "backwards-looking" operator $\langle e^{-}\rangle\varphi$ meaning that φ was true before event e happened (and e happened just before). Another obvious extension is with temporal operators of the form $F\varphi$: "φ is true some time in the future" or $\langle e^{*}\rangle\varphi$: "$\varphi$ is true after a finite sequence of e events" expressing the temporal future of the current process.

Group Knowledge Operators Several notions of group knowledge are relevant to our representation results, as well as understanding protocols—and we have not analyzed what these do in the setting of our logic **TPAL**. Let $\mathscr{H} = \langle \Sigma, \mathsf{H}, \{\sim_i\}_{i\in\mathscr{A}}, V\rangle$ be an ETL model with $\sim^{*}$ the reflexive transitive closure of the relation $\bigcup_{i\in\mathscr{A}} \sim_i$. Common knowledge of φ (denoted $C\varphi$) is defined as follows:

$$\mathscr{H}, h \models C\varphi \text{ iff for each } h' \in \mathsf{H}, \text{ if } h \sim^{*} h' \text{ then } h' \models \varphi$$

van Benthem et al. [12] discuss the technical issues that arise when axiomatizing Public Announcement Logic in languages with common knowledge. They introduce a new "relativized common knowledge" operator $C(\psi|\varphi)$ saying that all ψ-paths end in a states satisfying φ. More formally, with $[\![\varphi[\!]$ the set of histories satisfying φ, and X^{+} is the transitive closure of any set X:

$$\mathscr{H}, h \models C(\psi|\varphi) \text{ iff for each } h' \in \mathsf{H}, \text{ if } (h, h') \in (\textstyle\bigcup_{i\in\mathscr{A}} \sim_i \, \cap \, (\mathsf{H} \times [\![\psi]\!]))^{+}, \text{ then } \mathscr{H}, h' \models \varphi$$

The usual common knowledge operator $C\varphi$ can be defined as $C(\top|\varphi)$. van Benthem et al. axiomatizes public announcement logic with this additional operator, and shows how similar ideas work for DEL as a whole [12]. We conjecture that the class $\mathbb{F}(\mathbf{X}_{\text{PAL}})$ can be axiomatized by adding the following axiom to **TPAL**:

$$\langle A\rangle C(\psi|\varphi) \leftrightarrow \langle A\rangle T \wedge C(\langle A\rangle\psi|\langle A\rangle\varphi).$$

Another important notion of group knowledge is *distributed knowledge* (denoted $D\varphi$), defined using an intersection of accessibilities:

$$\mathscr{H}, h \models D\varphi \text{ iff for each } h', \text{ if } h \sim_i h' \text{ for each } i \in \mathscr{A}, \text{ then } \mathscr{H}, h' \models \varphi.$$

This notion is not bisimulation invariant, and complexity of validity tends to go up. Still, distributed knowledge is essential to understanding what a group comes to know if agents publicly share everything they currently know. We will need an extension of $\mathscr{L}_{\text{ETL}}$ with distributed knowledge on communication protocols (cf. Section 6.3). For now, we conjecture that **TPAL** plus the usual rules for the modal operator D and the following axiom scheme:

$$\langle A\rangle\overline{D}\varphi \leftrightarrow (\langle A\rangle\top \wedge \overline{D}\langle A\rangle\varphi)$$

is sound and complete for the class $\mathbb{F}(\mathbf{X}_{\mathrm{PAL}})$, where $\overline{D}\varphi$ is $\neg D\neg\varphi$.

Temporal Operators To understand where we find ourselves with this paper, we can profit from existing work on languages for reasoning about arbitrary ETL models, not just those generated by DEL protocols. Key results of Halpern and Vardi show that both imposing agent idealizations (such as Perfect Recall or No Miracles) and language extensions (arbitrary future plus common knowledge) lead to high undecidability results [27]. van Benthem and Pacuit [17] provide an overview of these results and related ones from other areas of computer science. We summarize a few essential notions and results.

Let $\mathscr{H} = \langle \Sigma, \mathsf{H}, \{\sim_i\}_{i\in\mathscr{A}}, V\rangle$ is an ETL model, where $h \in \mathsf{H}$. If $e \in \Sigma$ and n a natural number, then e^n is the sequence of $ee\cdots e$ of length n. We define:

- $\mathscr{H}, h \models F\varphi$ iff there exists $h' \in \mathsf{H}$, $h \preceq h'$ and $\mathscr{H}, h' \models \varphi$.
- $\mathscr{H}, h \models \langle e^*\rangle\varphi$ iff there is an $h' \in \mathsf{H}$ with $h' = he^n$ for some n and $\mathscr{H}, h' \models \varphi$
- $\mathscr{H}, h \models N\varphi$ iff there is an $h' \in \mathsf{H}$ with $h' = he$ for some $e \in \Sigma$ and $\mathscr{H}$, $h' \models \varphi$
- $\mathscr{H}, h \models \langle e^-\rangle\varphi$ iff there exists $h' \in \mathsf{H}$ such that $h' \prec_e h$ and $\mathscr{H}, h' \models \varphi$
- $\mathscr{H}, h \models P\varphi$ iff there is some $h' \preceq h$ such that $\mathscr{H}, h' \models \varphi$

We write $G\varphi$ for $\neg F\neg\varphi$ (i.e., $G\varphi$ means φ is true in all future extensions). Much work in this area has already been done, including logics with a next-time operator N or a Kleene star on events $\langle e^*\rangle$.

Let $\mathsf{ProtocolPAL}$ be the set of all finite sequences of public announcement event models (i.e., the full tree of all possible sequences of PAL event models). Similarly, let $\mathsf{ProtocolDEL}$ be the set of all finite sequences of event models. The usual axiomatization of public announcement logic works for the class $\mathbb{F}(\{\mathsf{ProtocolPAL}\})$ Similarly, the standard axiomatization of DEL [5, 12] is the logic of $\mathbb{F}(\{\mathsf{ProtocolDEL}\})$. The following table summarizes what we know about complete logics for such languages with forward looking modalities (F.A. stands for 'Finitely Axiomatizable' and $EPDL$ stands for *epistemic propositional dynamic logic*. See [12] for details.):

Language	$\mathbb{F}(\{\mathsf{ProtocolPAL}\})$	$\mathbb{F}(\{\mathsf{ProtocolDEL}\})$
$[i]$, $\langle e\rangle$	F.A. [38]	F.A. [5]
$[i]$, $\langle e\rangle$, C	F.A. [5]	F.A. [5]
$EPDL$, $\langle e\rangle$	F.A. [12]	F.A. [12]
$[i]$, $\langle e\rangle$, N	F.A. [2]	Open
$[i]$, $\langle e\rangle$, $\langle e^*\rangle$	Not F.A. [34]	Open
$[i]$, $\langle e^*\rangle$	Open	Open
$[i]$, $\langle e\rangle$, C, $\langle e^*\rangle$	Not F.A. [34]	Open

Miller and Moss [34] show that $\mathbb{F}^{\infty}_{\mathscr{E}_0} = \{\mathsf{Forest}(\mathscr{M}, \mathscr{E}_0) \mid \mathscr{M}$ infinite $\}$ where $\mathscr{E}_0 = \{\langle i \rangle \top\}^*$ is not even axiomatizable for languages that contain knowledge modalities and arbitrary future modalities. There is some recent work axiomatizing the above classes with various backwards-looking modalities (see [40] and [39] for details). Even so, open problems abound here (cf. [17] and [26]).

Arbitrary Announcement Operators Finally, we mention a recent paper by Balbiani et al. [2]. Their main operator $\Diamond\varphi$ means "after any sequence of public announcements, φ is true". This revolves around the following notion:

$$\mathscr{M}, w \models \Diamond\varphi \text{ iff there is a formula } \psi \in \mathscr{L}_{\text{EL}}\ \mathscr{M}, w \models \langle \psi \rangle \varphi.$$

Among other interesting results, a complete axiomatization with respect to the language containing epistemic and public announcement modalities plus this new "arbitrary announcement" modality is provided for the class $\mathbb{F}(\{\mathsf{ProtocolPAL}\})$. These ideas merge naturally with ours. In particular, Hoshi [29] incorporates arbitrary announcement into the TPAL setting (allowing the underlying protocol to vary), in both one-step and finite-step versions.

6.2 Towards a Correspondence Theory

Our representation theorems suggest a more general correspondence theory[12] relating special properties of ETL frames to axioms in suitable modal languages. Just to show how this works, we focus on three properties (note that whenever we write *he*, it is assumed that *he* is actually in the ETL frame). The ETL frames $\mathscr{H}$ in our theorems satisfied (cf. Section 3.2):

1. **Synchronicity**: if $h \sim_i h'$, then $\mathsf{len}(h) = \mathsf{len}(h')$
2. **Perfect Recall**: if $he \sim_i h'f$, then $h \sim_i h'$
3. **Local No Miracles**: if $h_1 e \sim_i h_2 f$, $h_1 \sim^* h$ and $h \sim_i h'$, then $he \sim_i h'f$.

In correspondence terms, these express ETL versions of the two sides of the crucial DEL reduction axiom:

$$\langle \mathscr{E}, e \rangle \langle i \rangle \varphi \leftrightarrow \mathsf{pre}(e) \wedge \langle i \rangle \bigvee_{\substack{e \longrightarrow_i f \\ \text{in } \mathscr{E}}} \langle \mathscr{E}, f \rangle \varphi$$

which permutes the order of the dynamic and epistemic modalities. Note that in this Section we assume that the set of primitive events is finite to ensure that formulas such as the one above are in our language (cf. Remark 1).

Each of the facts below can be proven using a standard Sahlqvist argument (see [18], Section 3.6, for details). In our statements, purely for convenience, we use (a) that all events e are deterministic ($\langle e \rangle \varphi \to [e]\varphi$ is valid), (b) looking backwards, there is at most one event, and (c) epistemic accessibility is an equivalence relation.

[12] van Benthem [7] discusses related correspondence issues but without our connection to DEL protocols.

Perfect Recall This property suggests extending the language with a temporal "past" modality. Let $\mathscr{H} = \langle \Sigma, \mathsf{H}, \{\sim_i\}_{i\in\mathscr{A}}, V\rangle$ be an ETL model, then define

$$\mathscr{H}, h \models \langle e^- \rangle\varphi \text{ iff there is a } h' \in \mathsf{H} \text{ with } h'e = h \text{ and } \mathscr{H}, h' \models \varphi$$

Observation 5 An ETL frame $\mathscr{H}$ satisfies Perfect Recall iff the following is valid on $\mathscr{H}$

$$\langle e\rangle\langle i\rangle(\varphi \wedge \langle f^-\rangle\top) \to \langle i\rangle\langle f\rangle\varphi$$

Remark 3 states a somewhat stronger version of Perfect Recall (equivalent over ETL models to Synchronicity plus the above Perfect Recall version):

$$\text{if } he \sim_i h', \text{ then there is an event } f \text{ with } h' = h''f \text{ and } h \sim_i h''$$

Again, it is not hard to see that the above property corresponds to the following:

$$\langle e\rangle\langle i\rangle\varphi \to \langle i\rangle \bigvee_{f \text{ any event}} \langle f\rangle\varphi$$

Local No Miracles This property involves quantification over the common-knowledge relation $\sim^*$. Following the recent extension of correspondence to modal fixed-point languages [8], we have:

Observation 6 An ETL frame $\mathscr{H}$ satisfies Local No Miracles iff

$$(\langle e\rangle\langle i\rangle\langle f^-\rangle\top \wedge \overline{C}(\varphi \wedge \langle e\rangle\top \wedge \langle i\rangle\langle f\rangle\psi)) \to \overline{C}(\varphi \wedge \langle e\rangle\langle i\rangle\psi)$$

is valid on $\mathscr{H}$, where $\overline{C}\varphi$ is $\neg C\neg\varphi$..

6.3 Logics of Specific Protocols

Now that we have the general correspondence and axiomatization results of this paper, one next area of investigation should be the more detailed study of specific types of protocol, and the logical validities which they induce. We want to put our languages to use in bringing out special structures of reasoning. What follows are just two samples of such a logical *protocol theory*, to show its viability. These come from [23], which studies simple conversational-style restrictions, and [9] on algorithms for 'maximal communication' eventually turning distributed knowledge into common knowledge.[13]

Honest communication and the communicative core A minimal requirement for an "honest" public announcement of φ is that the speaker believes what

[13]Similar procedures have been proposed by now for creating common beliefs out of individual beliefs, or social choice-style shared group preferences out of preferences of single agents.

he announces. This can be represented as a public announcement with a precondition $\varphi \wedge [i]\varphi$ for some $i \in \mathscr{A}$. The matching protocol ProtocolHonest uses all and only public announcements with preconditions of this form. "Runs" of this protocol satisfy a "safety property": all information announced is already known by one of the agents, and hence the knowledge present in the whole system should neither increase nor decrease. What such runs achieve is increasing the shared group knowledge of such facts (modulo some complications having to do with epistemic non-factual assertions).

To make this precise, we represent "the knowledge present in a system" as the following variant of the earlier distributed group knowledge. Let $\mathscr{M} = \langle W, \{R_i\}_{i\in\mathscr{A}}, V\rangle$ be an epistemic model with the R_i all equivalence relations, and $w \in W$. The "communicative core" at w is the submodel $\mathscr{M}|^I_w$ of $\mathscr{M}$ whose states are only those worlds w' with wR_iw' for each agent i. Here is a corresponding modal operator:

– $\mathscr{M}, w \models I\varphi$ iff for each $w' \in D(\mathscr{M}|^I_w)$ with wR_iw', $\mathscr{M}|^I_w, w' \models \varphi$

Gerbrandy [23] provides a complete axiomatization for $\mathscr{L}_{\mathrm{EL}}$ this operator. Here is our first epistemic-temporal validity expressing a significant property of protocols:

Proposition 5 *For all $\mathscr{M}$ in which all R_i are equivalence relations, and each φ that is purely epistemic (that is, it has no temporal operators):*

$$\mathsf{Forest}(\mathscr{M}, \mathsf{ProtocolHonest}) \models I\varphi \leftrightarrow GI\varphi$$

This leaves open the issue whether agents can actually reach the communicative core by communicating. van Benthem [9] shows how, in finite models with two agents, the communicative core can be reached by the agents telling 'all they know', though 'bisimulation contractions' may be needed along the way. But [25] shows how this may break down with more than two agents. Intuitively, the following principle should be valid, with 'F' a future operator:

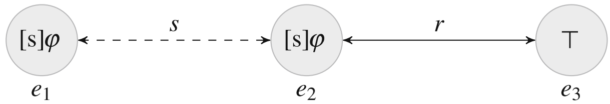
$$D\varphi \rightarrow FC\varphi.$$

But given what we just said, the precise set of protocols supporting this conversion from distributed into common knowledge remains to be determined.

Communication over an insecure channel More realistic interactive protocols involve partial observation by agents, and insecure channels of communication. This needs the full apparatus of DEL rather than PAL, to which our approach can be lifted [30]. DEL protocols formalize well-known phenomena (cf. [9, 36]) such as the classic "coordinated attack" problem [22] where messages are not guaranteed to arrive.

Sending a message φ that need not arrive can be represented by a DEL event model with three events: e_1 where the message was sent and received, e_2

where the message was sent but not received, and e_3 in which no message was sent at all. Events e_1 and e_2 have as their precondition that the message φ that was sent is believed to be true by the sender, i.e. $[s]\varphi$; while e_3 has just a trivial precondition $\top$. The sender of the message cannot distinguish e_1 and e_2, the receiver cannot distinguish e_2 from e_3 (even if she knows no message arrived, for her, this may mean that no message was sent in the first place, or it got lost on the way):



Now, let ProtocolInsecure be the DEL protocol that contains all and only sequences of this type of message. This time, we get conclusions opposite to our previous setting: agents may learn new facts, but *common knowledge* will never grow! For all propositional formulas, or, more generally, any formula φ in which knowledge operators only occur under an even number of negations, either φ is common knowledge throughout each run of the protocol, or it never is, was or will be. However, common knowledge may 'change'. Even with no message sent at all, it becomes common knowledge that the message φ *might* have been sent. Here is the logical fact behind these observations:

Proposition 6 *In all models $\mathcal{M}$ where the accessibility relations are equivalence relations, the following holds for all formulas φ in which epistemic operators occur only positively:*

$$\mathsf{Forest}(\mathcal{M}, \mathsf{ProtocolInsecure}) \models C\varphi \leftrightarrow GC\varphi$$

This differs from a celebrated result by [22] that common knowledge cannot change when communication is not reliable. The latter result depends on the assumption that the system is *asynchronous*—agents that do not receive a message also do not know that a message might have been sent.

7 Conclusion

We have shown how DEL and ETL as major approaches to modeling interacting agents over time complement each other: both in the way that models are constructed ("globally" in the ETL approach, "locally" from an initial model in the DEL approach), and in the *kind* of models constructed. As we have seen, the ETL models generated by DEL protocols are a proper subclass of the full set—for example, asynchronous systems cannot be described by DEL updates. On the other hand, in the model constructions in [22] and [37], the assumption that epistemic relations are equivalence relations is more or less built in, while DEL also handles cases where information may be false. But our major conclusion has been that the approaches are similar enough to admit significant merges, as in our TPAL-based theory of protocols.

Our framework raises many new open problems. We mention issues of:

1. *System Comparison:* Can TPAL be embedded faithfully into PAL?
2. *Complexity:* Theorem 4 shows that the satisfiability problem for TPAL is decidable. What is its precise computational complexity?
3. *Language Extensions:* Section 6.1 extends the language $\mathscr{L}_{\mathrm{PAL}}$ with common and distributed group knowledge. What about complete axiomatizations? Also, our version of TPAL assumes that the statements that can be announced come only from the epistemic base language. What if we lift this restriction?
4. *Partial Observation:* The TPAL framework in this paper generalizes to DEL, with simple twists to axioms. What are the answers to the preceding questions then? Also, when can protocol information be encoded completely 'locally' in DEL preconditions, as was the case for 'honest communication'?
5. *Protocol Logic:* What is the complete logic of specific protocol classes? For example, what is the complete logic of "honest announcements" $\mathbb{F}(\mathsf{ProtocolHonest})$?[14] A number of recent papers has raised similar issues (see, for example, Baltag [3] and van Eijck and Wang[21]).
6. *From Knowledge to Belief:* How to extend the analysis in this paper to doxastic logic and agents' changing beliefs over time, using doxastic-temporal logics and recent versions of DEL for belief change [10] to deal with protocols?[15]
7. *Learning Theory:* Connect our logic of protocols with learning theory [31].
8. *Process Theories:* A broader challenge is using DEL, with its explicit account of model construction inside the logic, as an intermediate between ETL-style frameworks which describe properties of states and histories inside given models, and paradigms like *process algebra* or *game semantics*, with their explicit construction of dynamic processes.

In summary, we hope to have shown that interfacing DEL and ETL, as major views of informative processes, is significant, productive, and well-worth exploring further.

[14] We conjecture that the logic of $\mathbb{F}(\mathsf{ProtocolHonest})$ (in the language $\mathscr{L}_{\mathrm{PAL}}$) is **TPAL** with the axioms $\langle !\varphi \wedge [i]\varphi\rangle\top$ for all formulas φ, plus axioms of the form $\neg\langle\psi\rangle\top$ for all formulas ψ that are not of the form $\varphi \wedge [i]\varphi$.

[15] An analogue of our main representation theorem in terms of 'Priority Update' of plausibility models and modal correspondence results for doxastic temporal models has just been given in [11].

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