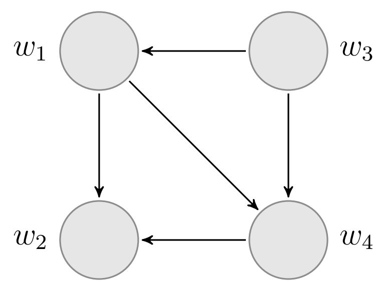
1. Consider the following relational structure (assume that there are no atomic propositions in the language):



$\mathcal{M}$

For each of the follows sets of states, find a formula that is true at precisely those sets (note that since there are no atomic propositions, the formulas will be construction using $\bot$ and $\top$): $\emptyset$, $\{w_1\}$, $\{w_2\}$, $\{w_3\}$, $\{w_4\}$, $\{w_1, w_2, w_3, w_4\}$.

Answer. We write $[\![\varphi]\!]_{\mathcal{M}}$ for the *truth set of* φ (the set of states in $\mathcal{M}$ where φ is true). Formally, $[\![\varphi]\!]_{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi\}$. Then,

$$\begin{aligned}
\emptyset &= [\![\bot]\!]_{\mathcal{M}} \\
\{w_2\} &= [\![\Box\bot]\!]_{\mathcal{M}} \\
\{w_4\} &= [\![\Diamond\top \wedge \Box\Box\bot]\!]_{\mathcal{M}} \\
\{w_1\} &= [\![\Diamond\top \wedge \Box(\Box\bot \vee \Box\Box\bot)]\!]_{\mathcal{M}} \\
\{w_3\} &= [\![\Diamond\top \wedge \Box\Diamond\Box\bot]\!]_{\mathcal{M}} \\
\{w_1, w_2, w_3, w_4\} &= [\![\top]\!]_{\mathcal{M}}
\end{aligned}$$

2. We say a frame $\langle W, R\rangle$ is **secondary reflexive** if R has the property $\forall x \forall y(xRy \rightarrow yRy)$. Prove that for all frames $\mathcal{F} = \langle W, R\rangle$, $\mathcal{F} \models \Box(\Box\varphi \rightarrow \varphi)$ iff $\mathcal{F}$ is secondary reflexive.

Proof. ($\Leftarrow$) Suppose that $\mathcal{F} = \langle W, R\rangle$ is secondary reflexive. Let $\mathcal{M} = \langle W, R, V\rangle$ be any model based on $\mathcal{F}$ and $w \in W$ any state. We must show $\mathcal{M}, w \models \Box(\Box\varphi \rightarrow \varphi)$. Let $v \in W$ be any state with wRv. We must show $\mathcal{M}, v \models \Box\varphi \rightarrow \varphi$. Suppose that $\mathcal{M}, v \models \Box\varphi$. Then for all $x \in W$, if vRx then $\mathcal{M}, x \models \varphi$. Since R is secondary reflexive and wRv, we have vRv. Therefore, $\mathcal{M}, v \models \varphi$, as desired. So, $\mathcal{M}, v \models \Box\varphi \rightarrow \varphi$; and therefore, $\mathcal{M}, w \models \Box(\Box\varphi \rightarrow \varphi)$.

($\Rightarrow$) Suppose that $\mathcal{F} = \langle W, R \rangle$ is not secondary reflexive. Then there are states $w, v \in W$ with wRv but it is not the case that vRv. Let $\mathcal{M} = \langle W, R, V \rangle$ be a models based on $\mathcal{F}$ where $x \in V(p)$ for all $x \in W$ with $x \neq v$ (i.e., $V(p) = W - \{v\}$). Then $\mathcal{M}, v \not\models p$. Furthermore, if $y \in W$ and vRy then $y \neq v$, so by construction of $\mathcal{M}$, we have $y \in V(p)$ and so $\mathcal{M}, y \models p$. Therefore, $\mathcal{M}, v \models \Box p$ and $\mathcal{M}, v \not\models \Box p \to p$. Since wRv, $\mathcal{M}, w \not\models \Box(\Box p \to p)$ which implies, $\mathcal{F} \not\models \Box(\Box\varphi \to \varphi)$. QED

3. Which one of the following two implications is valid in multiagent **S5**? Draw a counter-example for the other:

$$L_1K_2\varphi \to L_2L_1\varphi \qquad\qquad L_1K_2\varphi \to L_2K_1\varphi$$

(recall that $L_i\varphi$ is defined to be $\neg K_i \neg\varphi$)

Answer. $L_1K_2\varphi \to L_2L_1\varphi$ is valid (in multi agent **S5** and equivalently over the class of epistemic structures (Kripke structures where each relation is an equivalence relation).

We give two proofs of this fact, one semantic and one proof-theoretic. The first is to show that there is a derivation of the above formula in multiagent **S5**

Semantic Proof. We show that $L_1K_2\varphi \to L_2L_1\varphi$ is valid over the class of Kripke frames where the relations are equivalence relations (Let $\mathfrak{F}^{rat}$ denote this class of frames). Let $\mathcal{F} = \langle W, \{\sim_i\}_{i\in\mathcal{A}} \rangle$ be any Kripke frame where each $\sim_i$ is an equivalence relation. We will show that $\mathcal{F} \models L_1K_2\varphi \to L_2L_1\varphi$. Let $\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, V \rangle$ be any model based on $\mathcal{F}$ and $w \in W$. Suppose that $\mathcal{M}, w \models L_1K_2\varphi$. Then there is a $v \in W$ with $w \sim_1 v$ and $\mathcal{M}, v \models K_2\varphi$. This means that there is a $v \in W$ with $w \sim_1 v$ such that for all $x \in W$, if $v \sim_2 x$ then $\mathcal{M}, x \models \varphi$. Since $\sim_2$ is reflexive, we have $x \sim_2 x$ for each $x \in W$. Hence, since $v \sim_2 v$, we have $\mathcal{M}, v \models \varphi$. Putting everything together, we have $w \sim_2 w$ and $w \sim_2 v$ with $\mathcal{M}, v \models \varphi$. Hence, $\mathcal{M}, w \models L_2L_1\varphi$, as desired. Applying the completeness theorem for multi-agent **S5**, we conclude from the fact that $L_1K_2\varphi \to L_2L_1\varphi$ is valid on $\mathfrak{F}^{rts}$ that there must be a derivation of $L_1K_2\varphi \to L_2L_1\varphi$.

Syntactic Proof. We give a derivation in multi agent **S5** of $L_1K_2\varphi \to L_2L_1\varphi$. As a reminder, multi agent **S5** contains the following axiom schemes and rules:

tautology	All propositional tautologies
K.	$K_1(\varphi \to \psi) \to (K_i\varphi \to K_i\psi)$
T.	$K_i\varphi \to \varphi$
4.	$K_i\varphi \to K_iK_i\varphi$
5.	$\neg K_i\varphi \to K_i\neg K_i\varphi$
MP	from φ and $\varphi \to \psi$ infer ψ
Nec	from φ infer $K_i\varphi$

Giving all the details can be tedious, so I first give the key steps in the derivation:

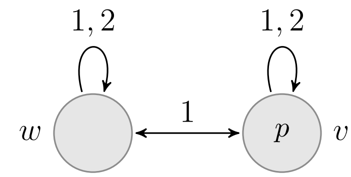
1.	$L_1\varphi \to L_2L_1\varphi$	Modal reasoning using axiom T
2.	$K_2\varphi \to \varphi$	Axiom T
3.	$L_1K_2\varphi \to L_1\varphi$	Modal reasoning
4.	$L_1K_2\varphi \to L_2L_1\varphi$	Propositional reasoning using 1 and 3

The full details of the derivation are:

1.	$K_2\varphi \to \varphi$	Instance of T
2.	$(K_2\varphi \to \varphi) \to (\neg\varphi \to \neg K_2\varphi)$	tautology
3.	$\neg\varphi \to \neg K_2\varphi$	MP 1, 2
4.	$K_1(\neg\varphi \to \neg K_2\varphi)$	Nec 3
5.	$K_1(\neg\varphi \to \neg K_2\varphi) \to (K_1\neg\varphi \to K_1\neg K_2\varphi)$	Axiom K
6.	$K_1\neg\varphi \to K_1\neg K_2\varphi$	MP 4, 5
7.	$(K_1\neg\varphi \to K_1\neg K_2\varphi) \to (\neg K_1\neg K_2\varphi \to \neg K_1\neg\varphi)$	tautology
8.	$\neg K_1\neg K_2\varphi \to \neg K_1\neg\varphi$	MP 6, 7
9.	$L_1K_2\varphi \to L_1\varphi$	Definition of L_1
10.	$K_2\neg L_1\varphi \to \neg L_1\varphi$	Axiom K
11.	$(K_2\neg L_1\varphi \to \neg L_1\varphi) \to (\neg\neg L_1\varphi \to \neg K_2\neg L_1\varphi)$	tautology
12.	$\neg\neg L_1\varphi \to \neg K_2\neg L_1\varphi$	MP 10, 11
13.	$L_1\varphi \to \neg\neg L_1\varphi$	tautology
14.	$((a \to b) \wedge (b \to c)) \to (a \to c)$	tautology with $a := L_1\varphi$, $b := \neg\neg L_1\varphi$, $c := \neg K_2\neg L_1\varphi$
15.	$(a \to (b \to (a \wedge b)))$	tautology with $a := L_1\varphi \to \neg\neg L_1\varphi$ $b := \neg\neg L_1\varphi \to \neg K_2\neg L_1\varphi$
16.	$(\neg\neg L_1\varphi \to \neg K_2\neg L_1\varphi) \to ((L_1 \to \neg\neg L_1\varphi)$ $\wedge(\neg\neg L_1\varphi \to \neg K_2\neg L_1\varphi))$	MP13, 15
17.	$(L_1 \to \neg\neg L_1\varphi) \wedge (\neg\neg L_1\varphi \to \neg K_2\neg L_1\varphi)$	MP12, 16
18.	$L_1\varphi \to \neg K_2\neg L_1\varphi$	MP14, 17
19.	$L_1\varphi \to L_2L_1\varphi$	Definition of L_2
20.	$((a \to b) \wedge (b \to c)) \to (a \to c)$	tautology with $a := L_1K_2\varphi$ $b := L_1\varphi$, $c := L_2L_1\varphi$
21.	$(a \to (b \to (a \wedge b)))$	tautology with $a := L_1K_2\varphi \to L_1\varphi$ $b := L_1\varphi \to L_2L_1\varphi$
22.	$(L_1\varphi \to L_2L_1\varphi) \to ((L_1K_2\varphi \to L_1\varphi)$ $\wedge(L_1\varphi \to L_2L_1\varphi))$	MP 21, 9
23.	$(L_1K_2\varphi \to L_1\varphi) \wedge (L_1\varphi \to L_2L_1\varphi)$	MP 22, 19
24.	$L_1K_2\varphi \to L_2L_1\varphi$	MP 20, 23

Answer. $L_1K_2\varphi \to L_2K_1\varphi$ is not valid.

Proof. To show that the above formula is not valid, it is enough to show that there is a counter-model for the following instance: $L_1K_2p \to L_2K_1p$ where p is an atomic proposition. Consider the following two world model $\mathcal{M} = \langle W, \sim_1, \sim_2, V\rangle$ with $W = \{w, v\}$, $\sim_1 = \{(w,w),(w,v),(v,w),(v,v)\}$, $\sim_2 = \{(w,w),(v,v)\}$ and $V(p) = \{v\}$. The model is pictured below:



Then $\mathcal{M}, w \models L_1K_2p \wedge \neg L_2K_1p$, as desired.

4. Read the article by Joe Halpern *Should Knowledge Entail Belief?*, Journal of Philosophical Logic (there is a link on the website). Write a short explanation in your own words summarizing Halpern's main point. (That is, explain in 1-2 paragraphs what is Halpern's main message in this article).