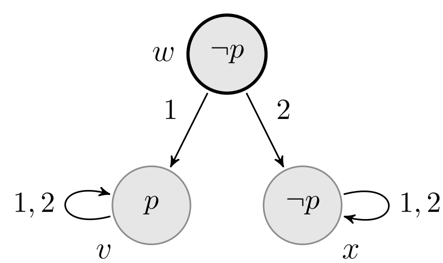
1. In this problem we consider a possible definition of common belief, analogous to the definition of common knowledge. Suppose there are two agents and a belief model $\langle W, \{R_1, R_2\}, V\rangle$ where R_1 and R_2 are serial, transitive and Euclidean relations. Let $R_B = (R_1 \cup R_2)^+$, where R^+ is the transitive closure of R (the smallest transitive relation containing R). Define the common belief operator C^B as follows:

$$\mathcal{M}, w \models C^B\varphi \text{ iff for each } v \in W, \text{ if } wR_Bv \text{ then } \mathcal{M}, v \models \varphi$$

   (a) Provide a **KD45** model $\mathcal{M} = \langle W, \{R_1, R_2\}, V\rangle$ and a state $w \in W$ where $\mathcal{M}, w \models B_1(C^Bp)$ but $\mathcal{M}, w \models \neg C^Bp$ (i.e., a state where agent 1 believes that p is commonly believed, but p is, in fact, not commonly believed).

   (b) Provide an example that shows that negative introspection for common belief ($\neg C^B\varphi \to C^B\neg C^B\varphi$) is not valid

   Answer. The same model works for both 1. and 2. Let $W = \{w, v, x\}$ and $R_1 = \{(w, v), (v, v), (x, x)\}$ and $R_2 = \{(w, x), (x, x), (v, v)\}$ and $V(p) = \{w\}$. This model can be pictured as follows:



   Then, $(R_1 \cup R_2)^+ = \{(w, v), (v, v), (w, x), (x, x)\}$. This means we have the following:

   (a) $\mathcal{M}, v \models C^Bp$ and so $\mathcal{M}, w \models B_1C^Bp$

   (b) $\mathcal{M}, w \models \neg B_2p$ and so $\mathcal{M}, w \models \neg C^Bp$

   (c) Since $\mathcal{M}, v \models C^Bp$ and $(w, v) \in (R_1 \cup R_2)^+$, we have $\mathcal{M}, w \models \neg C^B\neg C^Bp$.

2. We have argued that $K_i\varphi \to K_j\varphi$ is valid on a frame $\langle W, \{R_i\}_{i\in\mathcal{A}}\rangle$ iff for each $i, j \in \mathcal{A}$, $R_j \subseteq R_i$. Find a property on frames $\langle W, \{R_i\}_{i\in\mathcal{A}}\rangle$ that guarantees that $K_i\varphi \to K_iK_j\varphi$ is valid.

   Answer. The required property is a generalization of transitivity:

   (ij-Transitive) For all $w, v, x \in W$, if wR_iv and vR_jx then wR_ix.

Suppose that $\mathcal{F}$ is a frame with relations R_i and R_j.

Claim 1 *$K_i\varphi \to K_iK_j\varphi$ is valid on $\mathcal{F}$ iff $\mathcal{F}$ satisfies ij-Transitivity.*

Proof. ($\Leftarrow$) Suppose that $\mathcal{F} = \langle W, \{R_i\}_{i\in\mathcal{A}}\rangle$ is a frame with the ij-transitivity property. Let $\mathcal{M} = \langle W, \{R_i\}_{i\in\mathcal{A}}, V\rangle$ be any model based on $\mathcal{F}$. Suppose that $w \in W$. We will show $\mathcal{M}, w \models K_i\varphi \to K_iK_j\varphi$. Suppose that $\mathcal{M}, w \models K_i\varphi$, then for each $v \in W$, if wR_iv then $\mathcal{M}, v \models \varphi$. Suppose that $y, x \in W$ with wR_iy and yR_jx. We must show $\mathcal{M}, x \models \varphi$. Since wR_iy and wR_jx, by ij-transitivity we have wR_ix, which implies $\mathcal{M}, x \models \varphi$. Since x and y are arbitrary, we have $\mathcal{M}, w \models K_iK_j\varphi$ and so $\mathcal{M}, w \models K_i\varphi \to K_iK_j\varphi$, as desired.

($\Rightarrow$) Suppose that a frame $\mathcal{F} = \langle W, \{R_i\}_{i\in\mathcal{A}}\rangle$ is not ij-transitive. Then there are $w, x, y \in W$ with wR_ix and xR_jy but it is not the case that wR_iy. Let p be a proposition with $V(p) = W - \{y\}$. Then $\mathcal{M}, w \models K_ip$, since $\{v \mid wR_iv\} \subseteq V(p)$, but we have $\mathcal{M}, x \not\models K_jp$ and so $\mathcal{M}, w \not\models K_iK_jp$. QED

3. For a Bayesian model with a common prior $\langle W, \{\sim_i\}_{i\in\mathcal{A}}, \pi\rangle$, prove that for each $i \in \mathcal{A}$, $\pi(E \mid B_i^p(E)) \geq p$.

Proof. Recall the following two facts:

(a) For each $w \in B_i^p(E)$, $\pi(E \mid [w]_i) = \frac{\pi(E\cap[w]_i)}{\pi([w]_i)} \geq p$, which implies for each $w \in B_i^p(E)$, $\pi(E \cap [w]_i) \geq p\pi([w]_i)$; and

(b) $B_i^p(E) = \bigcup_{w\in B_i^p(E)}[w]_i$.

Then,

$$\pi(E \mid B_i^p(E)) = \frac{\pi(E\cap B_i^p(E))}{\pi(B_i^p(E))} = \frac{\pi(\bigcup_{w\in B_i^p(E)}(E\cap[w]_i))}{\pi(\bigcup_{w\in B_i^p(E)}([w]_i))} = \frac{\sum_{w\in B_i^p(E)}\pi(E\cap[w]_i)}{\sum_{w\in B_i^p(E)}\pi([w]_i)}$$

$$\geq \frac{\sum_{w\in B_i^p(E)}p\pi([w]_i)}{\sum_{w\in B_i^p(E)}\pi([w]_i)} = \frac{p\cdot\sum_{w\in B_i^p(E)}\pi([w]_i)}{\sum_{w\in B_i^p(E)}\pi([w]_i)} = p$$

QED

4. Explain why Aumann's original agreeing to disagree theorem (Theorem 7 in the handout for lecture 8) follows from Samet's generalized agreeing to disagree theorem (Theorem 4 in the handout for lecture 8). *Hint: fix an event $E \subseteq W$ and for each agent i, let the decision function $\mathbf{d}_i$ be defined as follows: $\mathbf{d}_i(w) = \pi(E \mid [w]_i)$ (the posterior probability of E for agent i at state w). Prove that $\mathbf{d}$ satisfies the ISTP.*

Answer. Let $\langle W, \{\sim_i\}_{i\in\mathcal{A}}, \pi\rangle$ be a Bayesian model. We must show that for any event $E \subseteq W$ and any set $\{r_1, \ldots, r_n\}$ of real numbers that are not identical, $C(\bigcap_{i\in\mathcal{A}} E_{i,r_i}) = \emptyset$. Fix an event E and define a decision function $\mathbf{d}_i : W \to [0,1]$ as follows $\mathbf{d}_i(w) = \pi(E \mid [w]_i)$.

Claim 2 *The above decision function* $\mathbf{d}$ *has the ISTP in* $\langle W, \{\sim_i\}_{i\in\mathcal{A}}$*:*

$$\text{(ISTP)} \qquad i, j \in \mathcal{A},\ K_j([i \succeq j] \cap [d_i = r]) \subseteq [d_j = r].$$

Proof. First, note the following two facts:

(a) For any $w \in W$, $w \in K_j([i \succeq j])$ iff $[w]_j = \bigcup_{v\in[w]_j} [v]_i$.

First of all, recall that $w \in [i \succeq j]$ iff $[w]_i \subseteq [w]_j$. Suppose that $w \in K_j([i \succeq j])$. Then, we have $[w]_j \subseteq [i \succeq j]$. This means that for all $v \in [w]_j$, $v \in [i \succeq j]$ which implies for all $v \in [w]_j$, $[v]_i \subseteq [v]_j = [w]_j$.

Suppose that $[w]_j = \bigcup_{v\in[w]_j} [v]_i$. We must show that $[w]_j \subseteq [i \succeq j]$. Suppose that $x \in [w]_j$. Then $x \in [v]_i$ for some $v \in [w]_j$. Let $E \subseteq W$ be any event and suppose that $x \in K_j(E)$. Then $[x]_j \subseteq E$. Then $[x]_i = [v]_i \subseteq [w]_j = [x]_j \subseteq E$. Hence $v \in K_i(E)$ and so $[w]_j \subseteq [i \succeq j]$, as desired.

(b) $w \in K_j([d_i = r])$ implies that for all $v \in [w]_j$, $v \in [d_i = r]$. So for all $v \in [w]_j$, $d_i(v) = \pi(E \mid [v]_i) = \frac{\pi(E\cap[v]_i)}{\pi([v]_i)} = r$. So for all $v \in [w]_j$, $\pi(E \cap [v]_i) = r\pi([v]_i)$.

Suppose that $w \in K_j([i \succeq j] \cap [d_i = r])$. Then,

$$\mathbf{d}_j(w) = \pi(E \mid [w]_j) = \frac{\pi(E \cap [w]_j)}{\pi([w]_j)} = \frac{\pi(\bigcup_{v\in[w]_j}(E \cap [v]_i))}{\pi(\bigcup_{v\in[w]_j}([v]_i))} = \frac{\sum_{v\in[w]_j} \pi(E \cap [v]_i)}{\sum_{v\in[w]_j} \pi([v]_i)}$$

$$= \frac{\sum_{v\in[w]_j} r\pi([v]_i)}{\sum_{v\in[w]_j} \pi([v]_i)} = \frac{r \cdot \sum_{v\in[w]_j} \pi([v]_i)}{\sum_{v\in[w]_j} \pi([v]_i)} = r$$

And so, $w \in [\mathbf{d}_j = r]$, as desired. Finally, it is clear that if $\sim_{n+1}$ is an epistemic dummy and $\mathbf{d}_{n+1} : W \to [0,1]$ defined by $\mathbf{d}_{n+1}(w) = \pi(E \mid [w]_{n+1})$, then $\mathbf{d}' = (\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n, \mathbf{d}_{n+1})$ has the ISTP in $\langle W, \{\sim_i\}_{i\in\mathcal{A}} \cup \{\sim_{n+1}\}\rangle$ (the above argument works even if there is an epistemic dummy among the agents). QED