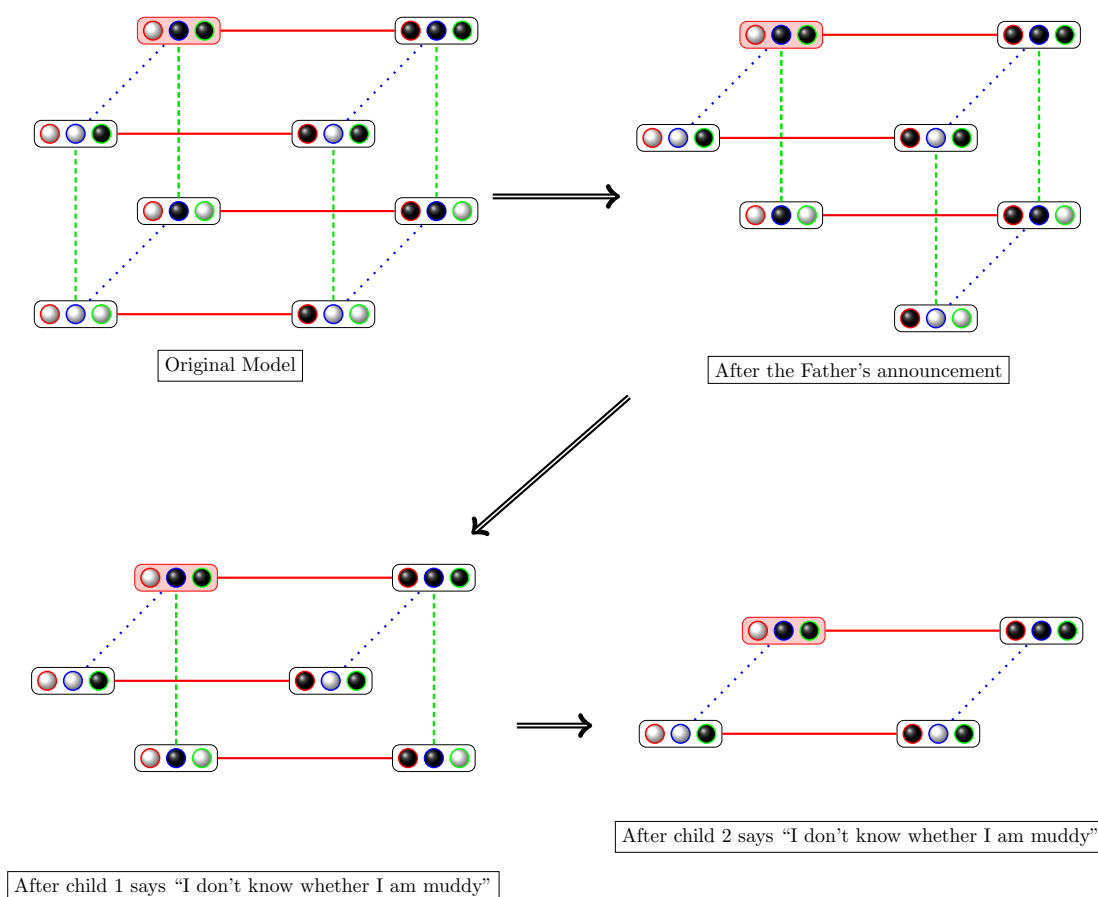


## Problem Set # 3 (Midterm Questions: Answers)

- Recall the muddy children puzzle discussed in Lecture 1: *A Primer on Epistemic Logic* (see slides 10 - 11). In the puzzle, after the father's announcement, the children's announcements are simultaneous. What happens if the children speak in turn? That is, suppose there are three children two of which have mud on their forehead (suppose it is the 2nd and 3rd child that have mud on their forehead), and the children speak in order (child 1 speaks first, then child 2, and finally child 3).

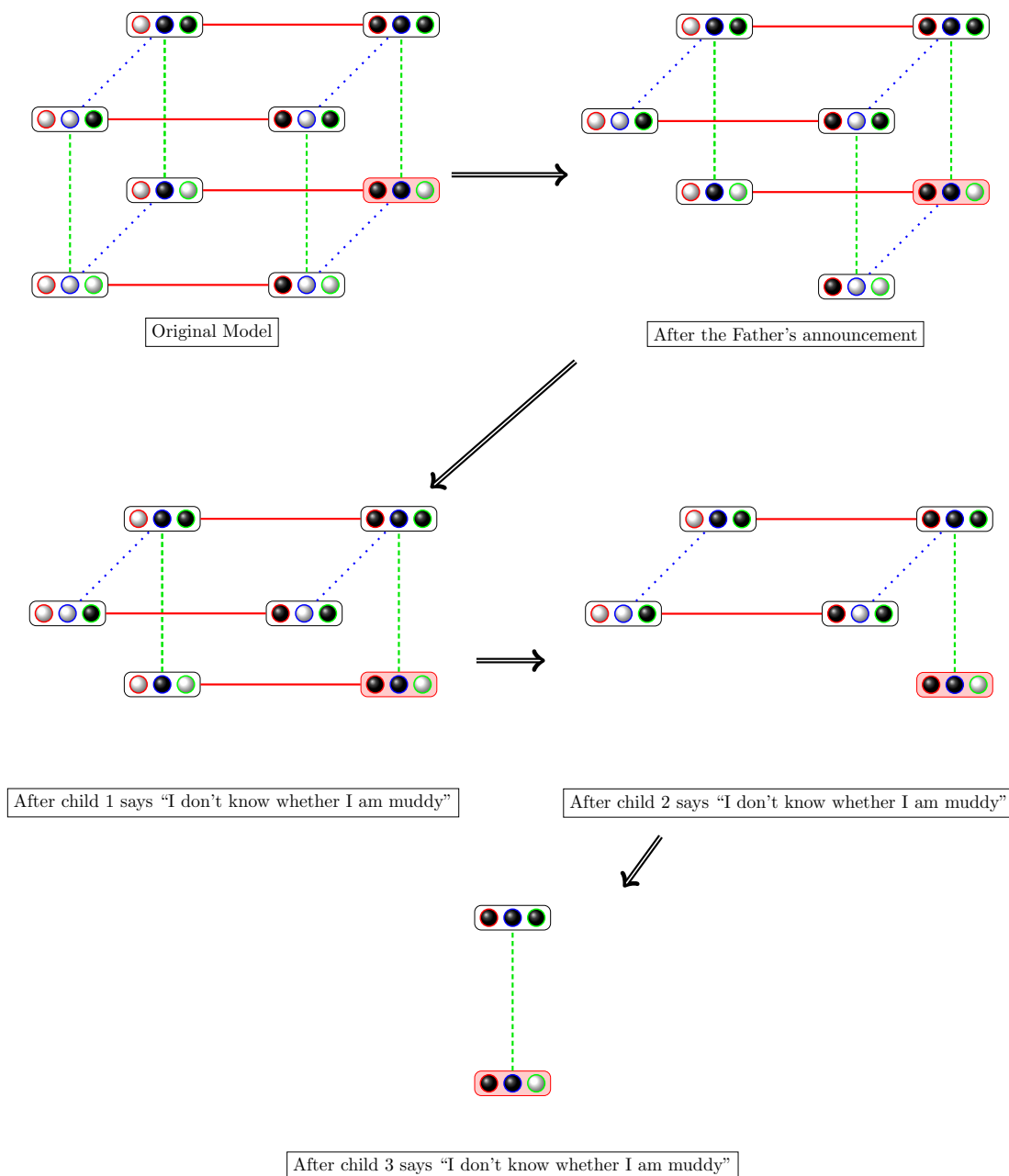
**Answer.** The successive updates after the father's and first two children's announcements are drawn below (the explanation of this diagram is provided in the slides for Lecture 1): (Child 1's relation is the solid red line, Child 2's relation is the dotted blue line and Child 3's relation is the densely dashed green line)



In the last model, child three can truthfully announce that she knows that her forehead is muddy; however, this announcement does not change the model (no states are removed).

No subsequent *truthful* announcement (child 1 says “I don’t know whether my forehead is muddy”, child 2 says “I don’t know whether my forehead is muddy” and child 3 says “my forehead is muddy”).

*Does your answer change if it is child 1 and child 2 with mud on their forehead (but the speaking order remains the same)?* The successive updates are pictured below:

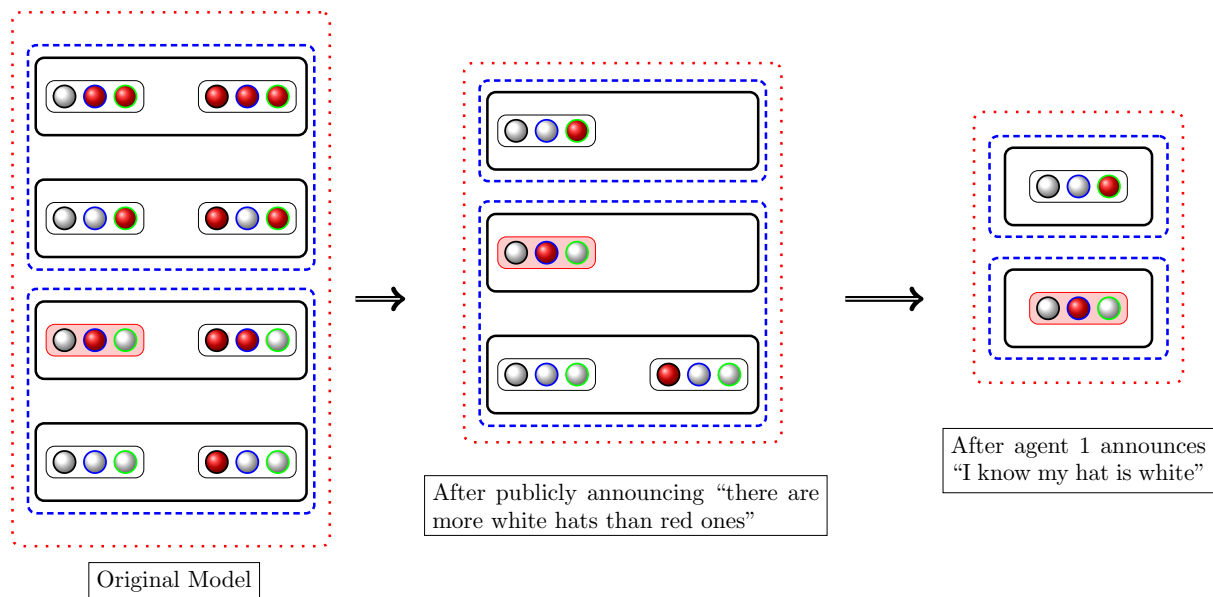


It is easy to see that no subsequent announcement (child one and two say “I know my forehead is muddy” and child three can say “I don’t know whether my forehead is muddy”)

by any of the children can change this model.

2. Three men are standing on a ladder, each wearing a hat. Each can see the colors of the hats of the people below him, but not his own or those higher up. It is common knowledge that only the colors red and white occur, and that there are more white hats than red ones. The actual order is white, red, white from top to bottom. Draw the epistemic model. The top person says: I know the color of my hat. Is that true? Draw the update. Who else knows his color now? If that person announces that he knows his color, what does the bottom person learn?

**Answer.** The top agent does, indeed, know the color of his hat. After announcing his hat, the second agent knows the color of his hat. The successive updates are given below. Assume that agent 1 (whose information cell is colored black) is at the top of the ladder, agent 2 (whose information cell is colored blue) is the second on the ladder, and agent 3 (whose information cell is colored green) is the bottom of the ladder. Each node depicts the color of the hat (with agent 1 on the left, agent 2 in the middle and agent 3 on the right). The successive updates are pictured below (with agent 1's information cell the solid black line, agent 2's information cell is the densely dashed blue line and agent 3's information cell is the dotted red line):



It is not hard to see that no subsequent truthful announcements will change this last model.

3. Recall that an **epistemic-plausibility model** is a tuple

$$\mathcal{M} = \langle W, \{\sim_i\}_{i \in \mathcal{A}}, \{\preceq_i\}_{i \in \mathcal{A}}, V \rangle$$

where  $W$  is a non-empty set of states, for each  $i \in \mathcal{A}$ ,  $\sim_i$  is an equivalence relation on  $W$ , for each  $i \in \mathcal{A}$ ,  $\preceq_i$  is reflexive, transitive, and *well-founded* (every nonempty subset  $X \subseteq W$  has a  $\preceq_i$ -minimal element), and  $V : \text{At} \rightarrow \wp(W)$  is a valuation function. In addition, the following two properties are satisfied:

- (a) *plausibility implies possibility*: if  $w \preceq_i v$  then  $w \sim_i v$ .
- (b) *locally-connected*: if  $w \sim_i v$  then either  $w \preceq_i v$  or  $v \preceq_i w$ .

Let  $\mathcal{L}_{KB}$  be the modal language defined by the following grammar:

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \psi \mid K_i\varphi \mid B_i^\varphi\psi \mid [\preceq_i]\varphi \mid B^s\varphi$$

with  $p \in \text{At}$ . Recall that  $\text{Min}_{\preceq_i}(X) = \{v \in X \mid v \preceq_i w \text{ for all } w \in X\}$  (which is always non-empty since  $\preceq_i$  is well-founded). Truth for the modal operators is defined as follows:

- $\mathcal{M}, w \models K_i\varphi$  iff for all  $v \in W$ , if  $w \sim_i v$  then  $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models B_i^\varphi\psi$  iff for all  $v \in \text{Min}_{\preceq_i}([\varphi]_{\mathcal{M}} \cap [w]_i)$ ,  $\mathcal{M}, v \models \psi$
- $\mathcal{M}, w \models [\preceq_i]\varphi$  iff for all  $v \in W$ , if  $v \preceq_i w$  then  $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models B^s\varphi$  iff  $[\varphi]_{\mathcal{M}} \cap [w]_i \neq \emptyset$  and  $[\varphi]_{\mathcal{M}} \preceq_i [\neg\varphi]_{\mathcal{M}}$

where  $[\varphi]_{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi\}$ .

- (a) Prove that the following two formulas are valid on every epistemic-plausibility model (recall that  $L_i\varphi$  is defined to be  $\neg K_i\neg\varphi$  and  $B_i\varphi$  is  $B^\top\varphi$ ):

- $B_i^\varphi\psi \leftrightarrow L_i\varphi \rightarrow L_i(\varphi \wedge [\preceq_i](\varphi \rightarrow \psi))$
- <sup>1</sup>  $B_i^s\varphi \leftrightarrow B_i\varphi \wedge K_i(\varphi \rightarrow [\preceq_i]\varphi) \wedge \neg K_i\neg\varphi$

**Claim 1**  $B_i^\varphi\psi \leftrightarrow L_i\varphi \rightarrow L_i(\varphi \wedge [\preceq_i](\varphi \rightarrow \psi))$  is valid on the class of all epistemic-plausibility models.

**Proof.** Suppose that  $\mathcal{M}, w \models B_i^\varphi\psi$ . Then,  $\text{Min}_{\preceq_i}([\varphi]_{\mathcal{M}} \cap [w]_i) \subseteq [\psi]_{\mathcal{M}}$ . We must show  $\mathcal{M}, w \models L_i\varphi \rightarrow L_i(\varphi \wedge [\preceq_i](\varphi \rightarrow \psi))$ . Suppose that  $\mathcal{M}, w \models L_i\varphi$  then there is a  $v \in W$  such that  $w \sim_i v$  and  $\mathcal{M}, v \models \varphi$ . This implies  $[\varphi]_{\mathcal{M}} \cap [w]_i \neq \emptyset$ , and since,  $\preceq_i$  is well-founded, we have  $\text{Min}_{\preceq_i}([\varphi]_{\mathcal{M}} \cap [w]_i) \neq \emptyset$ . Suppose that  $v' \in \text{Min}_{\preceq_i}([\varphi]_{\mathcal{M}} \cap [w]_i) \subseteq [\varphi]_{\mathcal{M}} \cap [w]_i$ . Then,  $v' \in [\varphi]_{\mathcal{M}} \cap [w]_i$ . We will show that  $\mathcal{M}, v' \models \varphi \wedge [\preceq_i](\varphi \rightarrow \psi)$ . It is clear that  $\mathcal{M}, v' \models \varphi$  (since  $v' \in [\varphi]_{\mathcal{M}}$ ). For the second conjunct, let  $v'' \in W$  be a state such that  $v'' \preceq_i v'$ , and suppose that  $\mathcal{M}, v'' \models \varphi$ . Then, since plausibility implies possibility ( $\preceq_i \subseteq \sim_i$ ), we have  $v'' \sim_i v'$ , and, since  $v' \in [w]_i$ , we have  $v'' \in [w]_i$ . This means that  $v'' \in [\varphi]_{\mathcal{M}} \cap [w]_i$ . Since,  $v'$  is a  $\preceq_i$ -minimal element of  $[\varphi]_{\mathcal{M}} \cap [w]_i$  and

<sup>1</sup>Note that the first conjunct ( $B_i\varphi$ ) is not actually needed here. (See if you can explain why?)

$v'' \preceq_i v'$ , we have  $v'' \in \text{Min}_{\preceq_i}(\llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i)$  (i.e.,  $v''$  must also be a  $\preceq_i$ -minimal element of  $\llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i$ ). By the first assumption,  $\text{Min}_{\preceq_i}(\llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i) \subseteq \llbracket \psi \rrbracket_{\mathcal{M}}$ , and so  $v'' \in \llbracket \psi \rrbracket_{\mathcal{M}}$ . Hence,  $\mathcal{M}, v'' \models \psi$ . Thus,  $\mathcal{M}, v'' \models \varphi \rightarrow \psi$ , and since  $v''$  was arbitrary, we have  $\mathcal{M}, v' \models [\preceq_i](\varphi \rightarrow \psi)$ . Putting everything together, we have  $\mathcal{M}, v' \models \varphi \wedge [\preceq_i](\varphi \rightarrow \psi)$ , and so,  $\mathcal{M}, w \models L_i \varphi \rightarrow L_i(\varphi \wedge [\preceq_i](\varphi \rightarrow \psi))$ .

Suppose that  $\mathcal{M}, w \models L_i \varphi \rightarrow L_i(\varphi \wedge [\preceq_i](\varphi \rightarrow \psi))$ . Then (1) there is a  $v$  such that  $w \sim_i v$  and  $\mathcal{M}, v \models \varphi$ , and (2) there is a  $v'$  such that  $w \sim_i v'$  and we have  $\mathcal{M}, v' \models \varphi \wedge [\preceq_i](\varphi \rightarrow \psi)$ . First of all, note that by (1),  $\llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i$  is non-empty, and so  $\text{Min}_{\preceq_i}(\llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i)$  is also non-empty (since  $\preceq_i$  is well-founded). We must show that  $\text{Min}_{\preceq_i}(\llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i) \subseteq \llbracket \psi \rrbracket_{\mathcal{M}}$ . Suppose that  $v'' \in \text{Min}_{\preceq_i}(\llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i)$ . Then  $v'' \in \llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i$  and  $v'' \preceq_i x$  for each  $x \in \llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i$ . By (2), we have  $v' \in [w]_i$  and  $\mathcal{M}, v' \models \varphi \wedge [\preceq_i](\varphi \rightarrow \psi)$ . Hence  $v' \in \llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i$ , and so  $v'' \preceq_i v'$ . Since (by (2)),  $\mathcal{M}, v' \models [\preceq_i](\varphi \rightarrow \psi)$ , we have  $\mathcal{M}, v'' \models \varphi \rightarrow \psi$ . Then, since  $v'' \in \llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i$ , we have  $v'' \models \varphi$ , and hence,  $\mathcal{M}, v'' \models \psi$ , as desired. Therefore,  $\text{Min}_{\preceq_i}(\llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i) \subseteq \llbracket \psi \rrbracket_{\mathcal{M}}$  and so  $\mathcal{M}, w \models B_i^\varphi \psi$ . QED

**Claim 2**  $B_i^s \varphi \leftrightarrow B_i \varphi \wedge K_i(\varphi \rightarrow [\preceq_i] \varphi) \wedge \neg K_i \neg \varphi$  is valid on the class of all epistemic-plausibility models.

**Proof.** Suppose that  $\mathcal{M}, w \models B_i^s \varphi$ . Then  $\llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i \neq \emptyset$  and  $\llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i \preceq_i \llbracket \neg \varphi \rrbracket_{\mathcal{M}} \cap [w]_i$ . The first conjunct implies that  $\mathcal{M}, w \models \neg K_i \neg \varphi$ . We must show  $\mathcal{M}, w \models B_i \varphi \wedge K_i(\varphi \rightarrow [\preceq_i] \varphi)$ . Recall that  $\llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i \preceq_i \llbracket \neg \varphi \rrbracket_{\mathcal{M}} \cap [w]_i$  provided for all  $x \in \llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i$ , for all  $y \in \llbracket \neg \varphi \rrbracket_{\mathcal{M}} \cap [w]_i$ ,  $x \preceq_i y$ . This implies  $\text{Min}_{\preceq_i}([w]_i) \subseteq \llbracket \varphi \rrbracket_{\mathcal{M}}$  (otherwise there is a  $x \in \text{Min}_{\preceq_i}$  such that  $x \notin \llbracket \varphi \rrbracket_{\mathcal{M}}$ . But then we can find a  $y \in \llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i$  such  $x \preceq_i y$  with  $x \in \llbracket \neg \varphi \rrbracket_{\mathcal{M}} \cap [w]_i$ , which contradicts the assumption that  $\llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i \preceq_i \llbracket \neg \varphi \rrbracket_{\mathcal{M}} \cap [w]_i$ ). So,  $\mathcal{M}, w \models B_i \varphi$ . For the second conjunct, let  $y \in [w]_i$  and suppose that  $\mathcal{M}, y \models \varphi$ . Hence,  $y \in \llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i$ . Let  $v \in W$  be any state such that  $v \preceq_i w$ . By the definition of an epistemic-plausibility model (assumption (a) above), this implies  $v \in [w]_i$ . If  $\mathcal{M}, v \models \neg \varphi$ , then this contradicts the assumption that  $\llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i \preceq_i \llbracket \neg \varphi \rrbracket_{\mathcal{M}} \cap [w]_i$  (this follows since we have  $y \in \llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i$ ,  $v \preceq_i y$  and  $v \in \llbracket \neg \varphi \rrbracket_{\mathcal{M}} \cap [w]_i$ ). Hence,  $\mathcal{M}, v \models \varphi$ , which means  $\mathcal{M}, y \models [\preceq_i] \varphi$ . This, in turn, means that  $\mathcal{M}, y \models \varphi \rightarrow [\preceq_i] \varphi$  and  $\mathcal{M}, w \models K_i(\varphi \rightarrow [\preceq_i] \varphi)$ , as desired.

For the converse, suppose that  $\mathcal{M}, w \models B_i \varphi \wedge K_i(\varphi \rightarrow [\preceq_i] \varphi) \wedge \neg K_i \neg \varphi$ . Since  $\mathcal{M}, w \models \neg K_i \neg \varphi$ , we have  $\llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i \neq \emptyset$ . We must show that  $\llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i \preceq_i \llbracket \neg \varphi \rrbracket_{\mathcal{M}} \cap [w]_i$ . Suppose that  $x \in \llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]_i$  and  $y \in \llbracket \neg \varphi \rrbracket_{\mathcal{M}} \cap [w]_i$ . Since,  $\preceq_i$  is locally-connected (item (b) in the above definition), either  $x \preceq_i y$  or  $y \preceq_i x$ . We will show that  $y \not\preceq_i x$ . Suppose towards a contradiction that  $y \preceq_i x$ . Since,  $\mathcal{M}, w \models K_i(\varphi \rightarrow [\preceq_i] \varphi)$  and  $x \in [w]_i$ , we have  $\mathcal{M}, x \models \varphi \rightarrow [\preceq_i] \varphi$ . In addition,  $\mathcal{M}, x \models \varphi$ , so  $\mathcal{M}, x \models [\preceq_i] \varphi$ . Since,  $y \preceq_i x$ , this means  $\mathcal{M}, y \models \varphi$ , which contradicts the assumption that  $y \in \llbracket \neg \varphi \rrbracket_{\mathcal{M}}$ . QED

- (b) Let  $\mathcal{M}$  be an epistemic-plausibility model and define  $K_w = \{\varphi \mid \mathcal{M}, w \models B\varphi\}$ . Define a revision operator  $*$  as follows:  $K_w * \psi = \{\varphi \mid \mathcal{M}, w \models B^\psi \varphi\}$ . Prove that  $*$  satisfies the AGM postulates.

**Answer.** We first need some notation. Let  $\mathcal{L}_0$  be the sublanguage consisting of only propositional formulas (the formulas do not contain any belief/knowledge modalities). Note that the previous definitions assume that the  $\psi$  and elements of  $K_w$  are restricted to the propositional language  $\mathcal{L}_0$ .

Given a set of sentences  $X \subseteq \mathcal{L}_0$  and an epistemic-plausibility model  $\mathcal{M}$ , we are interested in the *local consequences* of  $X$  at a state  $w$  in model  $\mathcal{M}$ . This is defined as follows: suppose that  $\llbracket X \rrbracket_{\mathcal{M}, w} = \bigcap_{\alpha \in X} \llbracket \alpha \rrbracket_{\mathcal{M}} \cap [w]$ , then define

$$Cn_{\mathcal{M}, w}(X) = \{\alpha \mid \llbracket X \rrbracket_{\mathcal{M}, w} \subseteq \llbracket \alpha \rrbracket_{\mathcal{M}, w}\}$$

To see the need for this local definition, suppose that  $W = \{w_1, w_2, w_3\}$  with  $w_1 \sim w_2$  (so  $[w_1] = [w_2] = \{w_1, w_2\}$ , but  $w_3$  is not in this equivalence cell). Suppose that  $V(p) = \{w_1, w_3\}$  and  $V(q) = \{w_1\}$ . Now if  $w_1$  and  $w_2$  are equally plausible, we have  $\mathcal{M}, w_1 \models B^p q$ , which implies  $q \in K_{w_1} * p$ . According to AGM, this means that  $q$  is a “consequence” of  $p$ . However, some care must be taken concerning what “consequence” means in this setting. Obviously,  $p \rightarrow q$  is not a tautology (as  $p$  and  $q$  are different atomic propositions), furthermore we do not even have  $\llbracket p \rightarrow q \rrbracket_{\mathcal{M}} = W$  (i.e.,  $p \rightarrow q$  is valid on the model). We only have a much weaker fact: the agent *knows* that  $p \rightarrow q$  is true (i.e., it is true throughout the agents information cell at  $w_1$ ). Note that this situation can also be modeled in the “standard” AGM setting: the agent is assumed to have an underlying theory which she takes as knowledge (in the previous example, the agent assumes that  $p \leftrightarrow q$  is a theorem).

Note that we make use of two similar notations:  $\llbracket \varphi \rrbracket_{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi\}$  and  $\llbracket \varphi \rrbracket_{\mathcal{M}, w} = \llbracket \varphi \rrbracket_{\mathcal{M}} \cap [w]$ .

**Fact 1.** The following basic facts will be used in the proof below.

- (a)  $\llbracket \varphi \wedge \psi \rrbracket_{\mathcal{M}} = \llbracket \varphi \rrbracket_{\mathcal{M}} \cap \llbracket \psi \rrbracket_{\mathcal{M}}$ , and so  $\llbracket \varphi \wedge \psi \rrbracket_{\mathcal{M}, w} = \llbracket \varphi \rrbracket_{\mathcal{M}, w} \cap \llbracket \psi \rrbracket_{\mathcal{M}, w}$

**Proof.** Immediate from the definitions:  $\llbracket \varphi \wedge \psi \rrbracket_{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi \wedge \psi\} = \{w \mid \mathcal{M}, w \models \varphi \text{ and } \mathcal{M}, w \models \psi\} = \{w \mid \mathcal{M}, w \models \varphi\} \cap \{w \mid \mathcal{M}, w \models \psi\} = \llbracket \varphi \rrbracket_{\mathcal{M}} \cap \llbracket \psi \rrbracket_{\mathcal{M}}$ . QED

- (b) Let  $X$  be a set of formulas and  $\varphi$  a formula, then  $\llbracket X \cup \{\varphi\} \rrbracket_{\mathcal{M}, w} = \llbracket X \rrbracket_{\mathcal{M}, w} \cap \llbracket \varphi \rrbracket_{\mathcal{M}, w}$

**Proof.** Immediate from the definitions:

$$\llbracket X \cup \{\varphi\} \rrbracket_{\mathcal{M}, w} = \bigcap_{\beta \in X \cup \{\varphi\}} \llbracket \beta \rrbracket_{\mathcal{M}, w} = \bigcap_{\beta \in X} \llbracket \beta \rrbracket_{\mathcal{M}, w} \cap \llbracket \varphi \rrbracket_{\mathcal{M}, w} = \llbracket X \rrbracket_{\mathcal{M}, w} \cap \llbracket \varphi \rrbracket_{\mathcal{M}, w}.$$

QED

(c)  $Min_{\preceq}([w] \cap [\varphi]_{\mathcal{M}}) \subseteq [K_w * \varphi]_{\mathcal{M},w}$  ( $Min_{\preceq}([w]) \subseteq [K_w]_{\mathcal{M},w}$  by letting  $\varphi$  be  $\top$ ).

**Proof.** Let  $\mathcal{M}$  be an epistemic-plausibility model and  $w$  a state in  $\mathcal{M}$ . Since for each  $\beta \in K_w * \varphi$ ,  $\mathcal{M}, w \models B^\varphi \beta$ , we have for each  $\beta \in K_w * \varphi$ ,  $Min_{\preceq}([w] \cap [\varphi]_{\mathcal{M}}) \subseteq [\beta]_{\mathcal{M}}$ . Since for all  $w$ ,  $Min_{\preceq}([w] \cap [\varphi]_{\mathcal{M}}) \subseteq [w]$ , we also have for all  $\beta \in K_w * \varphi$ ,  $Min_{\preceq}([w] \cap [\varphi]_{\mathcal{M}}) \subseteq [\beta]_{\mathcal{M}} \cap [w] = [\beta]_{\mathcal{M},w}$ . Hence,

$$Min([w] \cap [\varphi]_{\mathcal{M}}) \subseteq \bigcap_{\beta \in K_w * \varphi} [\beta]_{\mathcal{M},w} = [K_w * \varphi]_{\mathcal{M},w}$$

QED

(d) If  $Min_{\preceq}([w] \cap [\varphi]_{\mathcal{M}}) \subseteq [\alpha]_{\mathcal{M}}$ , then  $[K_w * \varphi]_{\mathcal{M},w} \subseteq [\alpha]_{\mathcal{M},w}$

**Proof.** Suppose that  $Min_{\preceq}([w] \cap [\varphi]_{\mathcal{M}}) \subseteq [\alpha]_{\mathcal{M}}$ . Then,  $\mathcal{M}, w \models B^\varphi \alpha$ . Hence,  $\alpha \in K_w * \varphi$ . This implies  $[K_w * \varphi]_{\mathcal{M},w} \subseteq [\alpha]_{\mathcal{M},w}$  (since  $\alpha$  is one of the conjuncts in  $[K_w * \varphi]_{\mathcal{M},w}$ ).

QED

(e) If  $Min_{\preceq}([w] \cap [\varphi]_{\mathcal{M}}) \cap [\psi]_{\mathcal{M}} \subseteq [\alpha]_{\mathcal{M}}$  then  $[K_w * \varphi]_{\mathcal{M},w} \cap [\psi]_{\mathcal{M}} \subseteq [\alpha]_{\mathcal{M}}$

**Proof.** We first note that  $X \cap [\alpha]_{\mathcal{M}} \subseteq [\beta]_{\mathcal{M}}$  iff  $X \subseteq [\alpha \rightarrow \beta]_{\mathcal{M}}$ . Recall that  $w \in [\alpha \rightarrow \beta]_{\mathcal{M}}$  provided, if  $w \in [\alpha]_{\mathcal{M}}$  then  $w \in [\beta]_{\mathcal{M}}$ . Suppose that  $X \cap [\alpha]_{\mathcal{M}} \subseteq [\beta]_{\mathcal{M}}$  and  $w \in X$ . If  $w \in [\alpha]_{\mathcal{M}}$ , then by assumption  $w \in X \cap [\alpha]_{\mathcal{M}}$ , and so,  $w \in [\beta]_{\mathcal{M}}$ . Suppose that  $X \subseteq [\alpha \rightarrow \beta]_{\mathcal{M}}$ . Suppose that  $w \in X \cap [\alpha]_{\mathcal{M}}$ . Then, we have  $w \in [\beta]_{\mathcal{M}}$ .

Now suppose that  $Min_{\preceq}([w] \cap [\varphi]_{\mathcal{M}}) \cap [\psi]_{\mathcal{M}} \subseteq [\alpha]_{\mathcal{M}}$ . Then,  $Min_{\preceq}([w] \cap [\varphi]_{\mathcal{M}}) \subseteq [\psi \rightarrow \alpha]_{\mathcal{M}}$ . Then by (d), we have  $[K * \varphi]_{\mathcal{M},w} \subseteq [\psi \rightarrow \alpha]_{\mathcal{M},w} \subseteq [\psi \rightarrow \alpha]_{\mathcal{M}}$ . Hence,  $[K * \varphi]_{\mathcal{M},w} \cap [\psi]_{\mathcal{M}} \subseteq [\alpha]_{\mathcal{M}}$ . QED

(f) For sets  $X \subseteq W$  and  $Y \subseteq W$ ,  $Min_{\preceq}(X) \cap Y \subseteq Min_{\preceq}(X \cap Y)$

**Proof.** If  $Min_{\preceq}(X) \cap Y = \emptyset$  then we are done. Suppose that  $v \in Min_{\preceq}(X) \cap Y$ . Then  $v \in X \cap Y$ . Let  $y \in X \cap Y$ , then  $y \in X$  and since  $v \in Min_{\preceq}(X)$ ,  $v \preceq y$ . Hence,  $v \in Min_{\preceq}(X \cap Y)$  and so  $Min_{\preceq}(X) \cap Y \subseteq Min_{\preceq}(X \cap Y)$ . QED

(g) For sets  $X \subseteq W$  and  $Y \subseteq W$ , if  $Min(X) \cap Y \neq \emptyset$ , then  $Min_{\preceq}(X \cap Y) = Min_{\preceq}(X) \cap Y$ .

**Proof.** Suppose that  $Min(X) \cap Y \neq \emptyset$ . Then, there is a  $x \in Min(X) \cap Y$ . By (1) we have  $Min_{\preceq}(X) \cap Y \subseteq Min_{\preceq}(X \cap Y)$ . We must show  $Min_{\preceq}(X \cap Y) \subseteq Min_{\preceq}(X) \cap Y$ . Suppose that  $w \in Min_{\preceq}(X \cap Y)$ . Then  $w \in X \cap Y$  and  $w \preceq y$  for each  $y \in X \cap Y$ . Then  $w \in Y$ . To see that  $w \in Min(X)$ , note that  $x \preceq z$  for each  $z \in X$  and  $x \in Y$ . Since  $w \in Min_{\preceq}(X \cap Y)$  and  $x \in X \cap Y$ , we have  $w \preceq x$ . This implies  $w \preceq z$  for each  $z \in X$ . Putting everything together, we have  $Min_{\preceq}(X \cap Y) \subseteq Min_{\preceq}(X) \cap Y$ . QED

**Proof.** Let  $\mathcal{M}$  be an epistemic-plausibility model and define  $K_w = \{\varphi \mid \mathcal{M}, w \models B\varphi\}$ , and suppose that the revision operator  $*$  is defined as follows:  $K_w * \psi = \{\varphi \mid \mathcal{M}, w \models B^\psi\varphi\}$ . We show that  $*$  satisfies the 8 AGM postulates:

*AGM 1:  $K_w * \psi$  is deductively closed:* Suppose that  $\alpha \rightarrow \beta \in K_w * \psi$  and  $\alpha \in K_w * \psi$ . Then,  $\mathcal{M}, w \models B^\psi(\alpha \rightarrow \beta)$  and  $\mathcal{M}, w \models B^\psi\alpha$ . This means that for all  $v \in \text{Min}_\preceq([w] \cap \llbracket \psi \rrbracket_{\mathcal{M}})$ ,  $\mathcal{M}, v \models \alpha \rightarrow \beta$  and  $\mathcal{M}, v \models \alpha$ . Hence, for all  $v \in \text{Min}_\preceq([w] \cap \llbracket \psi \rrbracket_{\mathcal{M}})$ ,  $\mathcal{M}, v \models \beta$ . This implies  $\mathcal{M}, w \models B^\psi\beta$ , and so,  $\beta \in K_w * \psi$ .

*AGM 2:  $\psi \in K_w * \psi$ :* Note that by definition of the  $\text{Min}_\preceq$  operator, we have for any subset of states  $U$ ,  $\text{Min}_\preceq(U) \subseteq U$ . Hence,  $\text{Min}_\preceq([w] \cap \llbracket \psi \rrbracket_{\mathcal{M}}) \subseteq [w] \cap \llbracket \psi \rrbracket_{\mathcal{M}} \subseteq \llbracket \psi \rrbracket_{\mathcal{M}}$ . This implies  $\mathcal{M}, w \models B^\psi\psi$ , and so  $\psi \in K_w * \psi$ .

*AGM 3:  $K_w * \psi \subseteq \text{Cn}_{\mathcal{M},w}(K_w \cup \{\psi\})$ :* Suppose that  $\alpha \in K_w * \psi$ . Then  $\mathcal{M}, w \models B^\psi\alpha$ . This implies  $\text{Min}_\preceq([w] \cap \llbracket \psi \rrbracket_{\mathcal{M}}) \subseteq \llbracket \alpha \rrbracket_{\mathcal{M}}$ . We must show  $\llbracket K_w \cup \{\psi\} \rrbracket_{\mathcal{M},w} \subseteq \llbracket \alpha \rrbracket_{\mathcal{M},w}$ . By Fact 1(f),

$$\text{Min}_\preceq([w]) \cap \llbracket \psi \rrbracket_{\mathcal{M}} \subseteq \text{Min}_\preceq([w] \cap \llbracket \psi \rrbracket_{\mathcal{M}}) \subseteq \llbracket \alpha \rrbracket_{\mathcal{M}}$$

By, Fact 1(e), this implies  $\llbracket K_w \rrbracket_{\mathcal{M},w} \cap \llbracket \psi \rrbracket_{\mathcal{M}} \subseteq \llbracket \alpha \rrbracket_{\mathcal{M}}$ . Hence, by Fact 1(b) and the definition of  $\llbracket \cdot \rrbracket_{\mathcal{M},w}$ ,

$$\llbracket K_w \cup \{\psi\} \rrbracket_{\mathcal{M},w} = \llbracket K_w \rrbracket_{\mathcal{M},w} \cap \llbracket \psi \rrbracket_{\mathcal{M},w} \subseteq \llbracket K_w \rrbracket_{\mathcal{M},w} \cap \llbracket \psi \rrbracket_{\mathcal{M}} \subseteq \llbracket \alpha \rrbracket_{\mathcal{M}}$$

Since,  $\llbracket K_w \cup \{\psi\} \rrbracket_{\mathcal{M},w} \subseteq [w]$ , we also have  $\llbracket K_w \cup \{\psi\} \rrbracket_{\mathcal{M},w} \subseteq \llbracket \alpha \rrbracket_{\mathcal{M},w}$ .

*AGM 4: If  $\neg\varphi \notin K_w$  then  $K_w * \psi = \text{Cn}_{\mathcal{M},w}(K_w \cup \{\psi\})$ :* By AGM 3, we need only show that if  $\neg\varphi \notin K_w$  then  $\text{Cn}_{\mathcal{M},w}(K_w \cup \{\psi\}) \subseteq K_w * \psi$ . Suppose that  $\neg\psi \notin K_w$ . Then,  $\mathcal{M}, w \not\models B\neg\psi$  and so  $\mathcal{M}, w \models \neg B\neg\psi$ . This means that  $\text{Min}_\preceq([w]) \cap \llbracket \psi \rrbracket_{\mathcal{M}} \neq \emptyset$ . Then,

$$\begin{aligned} \text{Min}_\preceq([w] \cap \llbracket \psi \rrbracket_{\mathcal{M}}) &= \text{Min}_\preceq([w] \cap \llbracket \psi \rrbracket_{\mathcal{M}}) \cap [w] && \text{(since } \text{Min}_\preceq([w] \cap \llbracket \psi \rrbracket_{\mathcal{M}}) \subseteq [w]) \\ &= \text{Min}_\preceq([w]) \cap \llbracket \psi \rrbracket_{\mathcal{M}} \cap [w] && \text{(by Fact 1(g))} \\ &\subseteq \llbracket K_w \rrbracket_{\mathcal{M},w} \cap \llbracket \psi \rrbracket_{\mathcal{M}} \cap [w] && \text{(by Fact 1(c))} \\ &= \llbracket K_w \rrbracket_{\mathcal{M},w} \cap \llbracket \psi \rrbracket_{\mathcal{M},w} && \text{(since } \llbracket \psi \rrbracket_{\mathcal{M},w} = \llbracket \psi \rrbracket_{\mathcal{M}} \cap [w]) \\ &= \llbracket K_w \cup \{\psi\} \rrbracket_{\mathcal{M},w} && \text{(by Fact 1(b))} \\ &\subseteq \llbracket \alpha \rrbracket_{\mathcal{M},w} && \text{(by assumption)} \\ &\subseteq \llbracket \alpha \rrbracket_{\mathcal{M}} && \text{(since } \llbracket \alpha \rrbracket_{\mathcal{M},w} = \llbracket \alpha \rrbracket_{\mathcal{M}} \cap [w]) \end{aligned}$$

*AGM 5:  $K_w * \psi$  is inconsistent only if  $\psi$  is inconsistent (in the sense that  $\llbracket \psi \rrbracket_{\mathcal{M}} \cap [w] = \emptyset$ ):* Suppose that  $K_w * \psi$  is inconsistent (i.e., there is a formula  $\alpha$  such that  $\alpha, \neg\alpha \in K_w * \psi$ ). Then,  $\text{Min}_\preceq([w] \cap \llbracket \psi \rrbracket_{\mathcal{M}}) = \emptyset$ . By the definition of epistemic-plausibility models, this can only happen when  $[w] \cap \llbracket \psi \rrbracket_{\mathcal{M}} = \emptyset$ .



*AGM 6: If  $\psi_1$  and  $\psi_2$  are logically equivalent (in the sense that  $\llbracket \psi_1 \rrbracket_{\mathcal{M},w} = \llbracket \psi_2 \rrbracket_{\mathcal{M},w}$ ), then  $K_w * \psi_1 = K_w * \psi_2$ :* If  $\psi_1$  and  $\psi_2$  are logically equivalent, we have  $\llbracket \psi_1 \rrbracket_{\mathcal{M},w} = \llbracket \psi_2 \rrbracket_{\mathcal{M},w}$ . Hence,

$$\text{Min}_{\preceq}([w] \cap \llbracket \psi_1 \rrbracket_{\mathcal{M}}) = \text{Min}_{\preceq}(\llbracket \psi_1 \rrbracket_{\mathcal{M},w}) = \text{Min}_{\preceq}(\llbracket \psi_2 \rrbracket_{\mathcal{M},w}) = \text{Min}_{\preceq}([w] \cap \llbracket \psi_2 \rrbracket_{\mathcal{M}})$$

And so,  $K_w * \psi_1 = K_w * \psi_2$ .

*AGM 7:  $K_w * (\varphi \wedge \psi) \subseteq \text{Cn}_{\mathcal{M},w}(K_w * \varphi \cup \{\psi\})$ :* Suppose that  $\alpha \in K_w * (\varphi \wedge \psi)$ . Then,  $\mathcal{M}, w \models B^{\varphi \wedge \psi} \alpha$  and so  $\text{Min}([w] \cap \llbracket \varphi \wedge \psi \rrbracket_{\mathcal{M}}) \subseteq \llbracket \alpha \rrbracket_{\mathcal{M}}$ . Then

$$\begin{aligned} \llbracket K_w * \varphi \cup \{\psi\} \rrbracket_{\mathcal{M},w} &= \llbracket K_w * \varphi \rrbracket_{\mathcal{M},w} \cap \llbracket \psi \rrbracket_{\mathcal{M},w} && \text{(by Fact 1(b))} \\ &= \llbracket K_w * \varphi \rrbracket_{\mathcal{M},w} \cap \llbracket \psi \rrbracket_{\mathcal{M}} \cap [w] && \text{(definition of } \llbracket \cdot \rrbracket_{\mathcal{M},w} \text{)} \\ &\subseteq \text{Min}_{\preceq}([w] \cap \llbracket \varphi \rrbracket_{\mathcal{M}}) \cap \llbracket \psi \rrbracket_{\mathcal{M}} \cap [w] && \text{(by Fact 1(c))} \\ &\subseteq \text{Min}_{\preceq}([w] \cap \llbracket \varphi \rrbracket_{\mathcal{M}} \cap \llbracket \psi \rrbracket_{\mathcal{M}}) \cap [w] && \text{(by Fact 1(f))} \\ &= \text{Min}_{\preceq}([w] \cap \llbracket \varphi \wedge \psi \rrbracket_{\mathcal{M}}) \cap [w] && \text{(by Fact 1(a))} \\ &\subseteq \llbracket \alpha \rrbracket_{\mathcal{M}} \cap [w] = \llbracket \alpha \rrbracket_{\mathcal{M},w} && \text{(by assumption)} \\ &= \llbracket \alpha \rrbracket_{\mathcal{M},w} && \text{(definition of } \llbracket \cdot \rrbracket_{\mathcal{M},w} \text{)} \end{aligned}$$

*AGM 8 if  $\neg\psi \notin K_w * \varphi$  then  $\text{Cn}(K_w * \varphi \cup \{\psi\}) \subseteq K_w * (\varphi \wedge \psi)$ :* If  $\neg\psi \notin K_w * \varphi$ , then  $\mathcal{M}, w \not\models B^{\varphi} \neg\psi$ . Hence,  $\mathcal{M}, w \models \neg B^{\varphi} \neg\psi$ . This means  $\text{Min}([w] \cap \llbracket \varphi \rrbracket_{\mathcal{M}}) \cap \llbracket \psi \rrbracket_{\mathcal{M}} \neq \emptyset$ .

$$\begin{aligned} \text{Min}_{\preceq}([w] \cap \llbracket \varphi \rrbracket_{\mathcal{M}}) \cap \llbracket \psi \rrbracket_{\mathcal{M},w} &= \text{Min}_{\preceq}([w] \cap \llbracket \varphi \rrbracket_{\mathcal{M}}) \cap \llbracket \psi \rrbracket_{\mathcal{M}} \cap [w] && \text{(def. of } \llbracket \cdot \rrbracket_{\mathcal{M},w} \text{)} \\ &= \text{Min}_{\preceq}([w] \cap \llbracket \varphi \rrbracket_{\mathcal{M}} \cap \llbracket \psi \rrbracket_{\mathcal{M}}) \cap [w] && \text{(by Fact 1(g))} \\ &= \text{Min}_{\preceq}([w] \cap \llbracket \varphi \rrbracket_{\mathcal{M}} \cap \llbracket \psi \rrbracket_{\mathcal{M}}) && (\text{Min}_{\preceq}([w] \cap X) \subseteq [w]) \\ &= \text{Min}_{\preceq}([w] \cap \llbracket \varphi \wedge \psi \rrbracket_{\mathcal{M}}) && \text{(by Fact 1(a))} \\ &\subseteq \llbracket \alpha \rrbracket_{\mathcal{M}} && \text{(by assumption)} \end{aligned}$$

By Fact 1(b & e) and the definition of  $\llbracket \cdot \rrbracket_{\mathcal{M},w}$ , this implies

$$\llbracket K * \varphi \cup \{\psi\} \rrbracket_{\mathcal{M},w} = \llbracket K * \varphi \rrbracket_{\mathcal{M},w} \cap \llbracket \psi \rrbracket_{\mathcal{M},w} \subseteq \llbracket K * \varphi \rrbracket_{\mathcal{M},w} \cap \llbracket \psi \rrbracket_{\mathcal{M}} \subseteq \llbracket \alpha \rrbracket_{\mathcal{M}}$$

Since  $\llbracket K * \varphi \cup \{\psi\} \rrbracket_{\mathcal{M},w} \subseteq [w]$ , we also have  $\llbracket K * \varphi \cup \{\psi\} \rrbracket_{\mathcal{M},w} \subseteq \llbracket \alpha \rrbracket_{\mathcal{M},w}$ , as desired.

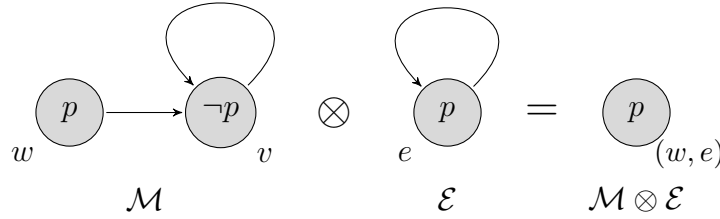
QED

4. Recall the definition of product update (slide 6 of Lecture 12: *Dynamic Logics of Information Change*). Let  $\mathcal{M} = \langle W, R, V \rangle$  be a Kripke models and  $\mathcal{E} = \langle E, S, \text{pre} \rangle$  and event model. Prove that if  $R$  and  $S$  are both transitive, then the relation in  $\mathcal{M} \otimes \mathcal{E}$  is also transitive. Is this also true if  $R$  and  $S$  are serial (a relations  $T$  is serial if for all  $x$  there is a  $y$  such that  $xTy$ )?

**Answer.** Let  $\mathcal{M} = \langle W, R, V \rangle$  be a Kripke model where  $R$  is transitive and  $\mathcal{E} = \langle E, S, \text{pre} \rangle$  and event model where  $S$  is transitive. We must show that  $\mathcal{M} \otimes \mathcal{E} =$

$\langle W', R', V' \rangle$  is a Kripke model where  $R'$  is transitive. Suppose that  $(w, e), (v, f), (x, g) \in W'$  with  $(w, e)R'(v, f)$  and  $(v, f)R'(x, g)$ . We must show that  $(w, e)R'(x, g)$ . By the definition of product update, since  $(w, e)R'(v, f)$  we have  $wRv$  and  $eSf$ . Similarly, we have  $vRx$  and  $fSg$ . Since  $R$  and  $S$  are both transitive, this implies  $wRx$  and  $eSg$ . Hence,  $(w, e)R'(x, g)$ , as desired.

The product of a Kripke models that is serial and an event model that is serial need not be a serial model as the following example shows:



Note that  $\mathcal{M} = \langle \{w, v\}, \{(w, v), (v, v)\}, V \rangle$  (with  $V(p) = \{w\}$ ) is a serial Kripke model. Also, the event model  $\mathcal{E} = \langle \{e\}, \{(e, e)\}, \text{pre} \rangle$  (with  $\text{pre}(e) = p$ ) is a serial event model. However,  $\mathcal{M} \otimes \mathcal{E} = \langle \{(w, e)\}, \emptyset, V \rangle$  is not serial.

5. Let  $W$  be a set of states,  $\mathcal{L}$  the propositional language with  $\text{At}$  as the set of atomic formulas, and  $V : \text{At} \rightarrow \wp(W)$  a valuation function. For  $\varphi \in \mathcal{L}$ , define  $\llbracket \varphi \rrbracket$  by recursion as follows:  $\llbracket p \rrbracket = V(p)$ ,  $\llbracket \neg \varphi \rrbracket = W - \llbracket \varphi \rrbracket$  and  $\llbracket \varphi \wedge \psi \rrbracket = \llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket$ . Let  $X \subseteq W$ . A system of spheres centered at  $X$  is a collection of sets  $\mathbf{S} \subseteq \wp(W)$  satisfying the following properties

- (S-a)  $\mathbf{S}$  is totally ordered by  $\subseteq$  (for  $U, V \in \mathbf{S}$ , either  $U \subseteq V$  or  $V \subseteq U$ )
- (S-b)  $X$  is the  $\subseteq$ -minimum element of  $\mathbf{S}$  ( $X \in \mathbf{S}$  and for all  $V \in \mathbf{S}$ ,  $X \subseteq V$ )
- (S-c)  $W \in \mathbf{S}$
- (S-d) For every propositional formula  $\varphi \in \mathcal{L}$  and sphere  $U \in \mathbf{S}$ , if  $U \cap \llbracket \varphi \rrbracket \neq \emptyset$  then there is a  $\subseteq$ -minimal sphere  $U_0 \in \mathbf{S}$  such that  $U_0 \cap \llbracket \varphi \rrbracket \neq \emptyset$ .

Let  $X \subseteq W$  and let  $\leq$  be a binary relation on  $W$ . We say that  $\leq$  is  $X$ -persistent if it satisfies the following properties:

- (O-a)  $\leq$  is a weak order ( $\leq$  is reflexive, transitive and complete: for all  $w, v \in W$ ,  $w \leq v$  or  $v \leq w$ )
- (O-b) <sup>2</sup>For every  $\varphi \in \mathcal{L}$ , if  $\llbracket \varphi \rrbracket \neq \emptyset$ , then  $\{v \mid v \in \llbracket \varphi \rrbracket \text{ and } v \leq w \text{ for all } w \in \llbracket \varphi \rrbracket\} \neq \emptyset$

<sup>2</sup>There was a typo in the earlier version of this midterm. The earlier version was “For every  $\varphi \in \mathcal{L}$ , if  $\llbracket \varphi \rrbracket \neq \emptyset$ , then  $\{v \mid v \in \llbracket \varphi \rrbracket \text{ and } v \leq w \text{ for all } w \in W\} \neq \emptyset$ ” which says that for each consistent  $\varphi$ ,  $\llbracket \varphi \rrbracket$  consists of  $\leq$ -minimal elements, which is not what we want.

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(O-c) For each  $w \in W$ ,  $w$  is a  $\leq$ -minima ( $w \leq v$  for all  $v \in W$ ) if and only if  $w \in X$

Let  $X \subseteq W$

- (i) Show that every system of spheres centered on  $X$  generates a  $X$ -persistent binary relation.

**Proof.** Let  $\mathbf{S}$  be a system of spheres centered on  $X$ . We define a  $X$ -persistent relation  $\leq_{\mathbf{S}} \subseteq W \times W$  as follows:

$$(*) \quad w \leq_{\mathbf{S}} v \text{ iff for each } U \in \mathbf{S}, \text{ if } v \in U \text{ then } w \in U.$$

We must show  $\leq_{\mathbf{S}}$  satisfies the above properties (O-a), (O-b) and (O-c).

**Claim O-a.**  $\leq_{\mathbf{S}}$  is a weak order.

**Proof of Claim O-a.** We first show that  $\leq_{\mathbf{S}}$  is reflexive: let  $w \in W$ , since we obviously have for all spheres  $U \in \mathbf{S}$ , we have  $w \in U$  implies  $w \in U$ ,  $w \leq_{\mathbf{S}} w$ . To show that  $\leq_{\mathbf{S}}$  is transitive, suppose that  $w, v, x \in W$  with  $w \leq_{\mathbf{S}} v$  and  $v \leq_{\mathbf{S}} x$ . We must show  $w \leq_{\mathbf{S}} x$ . Let  $U \in \mathbf{S}$  with  $x \in U$ . Since  $v \leq_{\mathbf{S}} x$ , by (\*), we have  $v \in U$ . Since,  $w \leq_{\mathbf{S}} v$ , by (\*), we have  $w \in U$ . Hence,  $w \leq_{\mathbf{S}} x$ . To show that  $\leq_{\mathbf{S}}$  is complete: suppose that  $w, v \in W$  with  $w \not\leq_{\mathbf{S}} v$ . Then there is a  $U \in \mathbf{S}$  such that  $v \in U$  but  $w \notin U$ . Let  $V \in \mathbf{S}$  be an sphere such that  $w \in V$ . Since  $\mathbf{S}$  is totally ordered by  $\subseteq$ , we must have either  $U \subseteq V$  or  $V \subseteq U$ . Since  $w \notin U$ , we cannot have  $V \subseteq U$ . Hence,  $U \subseteq V$ . Since  $v \in U$ , this implies  $v \in V$ . Hence,  $v \leq_{\mathbf{S}} w$ . QED (of Claim)

**Claim O-b.** For every  $\varphi \in \mathcal{L}$ , if  $\llbracket \varphi \rrbracket \neq \emptyset$ , then

$$\{v \mid v \in \llbracket \varphi \rrbracket \text{ and } v \leq_{\mathbf{S}} w \text{ for all } w \in \llbracket \varphi \rrbracket\} \neq \emptyset$$

**Proof of Claim O-b.** Suppose that  $\varphi \in \mathcal{L}$  and  $\llbracket \varphi \rrbracket \neq \emptyset$ . Since  $W \in \mathbf{S}$  and  $\llbracket \varphi \rrbracket \neq \emptyset$ , this implies  $W \cap \llbracket \varphi \rrbracket \neq \emptyset$ . Hence, by (S-c), there is a  $\subseteq$ -minimal  $U \in \mathbf{S}$  such that  $U \cap \llbracket \varphi \rrbracket \neq \emptyset$ . Let  $v \in U \cap \llbracket \varphi \rrbracket$ . We claim that  $v \leq_{\mathbf{S}} x$  for each  $x \in \llbracket \varphi \rrbracket$ . Suppose that  $x \in \llbracket \varphi \rrbracket$  and let  $V \in \mathbf{S}$  with  $x \in V$ . We must show that  $v \in V$ . Since  $U$  is the  $\subseteq$ -smallest subset with a nonempty intersection with  $\llbracket \varphi \rrbracket$  and  $V \cap \llbracket \varphi \rrbracket \neq \emptyset$ , we must have  $U \subseteq V$ . Since,  $v \in U$ , we have  $v \in V$ . Hence  $v \leq_{\mathbf{S}} x$ . Hence,

$$\{v \mid v \in \llbracket \varphi \rrbracket \text{ and } v \leq_{\mathbf{S}} w \text{ for all } w \in \llbracket \varphi \rrbracket\} \neq \emptyset.$$

QED (of Claim)

**Claim O-c.** For each  $w \in W$ ,  $w$  is a  $\leq_{\mathbf{S}}$ -minima ( $w \leq v$  for all  $v \in W$ ) if and only if  $w \in X$ .

**Proof of Claim O-c.** Suppose that  $w$  is a  $\leq_{\mathbf{S}}$ -minima. Then  $w \leq_{\mathbf{S}} x$  for all  $x \in W$ . This implies that for each  $U \in \mathbf{S}$ , if  $x \in U$  then  $w \in U$ . By (S-b),  $X \in \mathbf{S}$ . Note that  $X \neq \emptyset$  (this is implicitly assumed above), so there is some  $x \in X$ . Since  $w$  is a  $\leq_{\mathbf{S}}$ -minima and  $x \in X \in \mathbf{S}$ , we must have  $w \in X$ . Thus,  $w$  is a  $\leq_{\mathbf{S}}$ -minima implies  $w \in X$ . Conversely, suppose that  $w \in X$ . Let  $x \in W$  be any state and  $V \in \mathbf{S}$  any sphere with  $x \in V$ . Then, since  $X$  is  $\subseteq$ -minimal element of  $\mathbf{S}$ , we have  $X \subseteq V$ . Hence,  $w \in V$  and so  $w \leq_{\mathbf{S}} x$ . Thus,  $w \in X$  implies  $w$  is a  $\leq_{\mathbf{S}}$ -minima. QED (of Claim)

QED

- (ii) Show that every  $X$ -persistent binary relation generates a system of spheres centered on  $X$ .

**Proof.** Suppose that  $\leq$  is an  $X$ -persistent binary relation on  $W$ . For  $U \subseteq W$ , the **down set** generated by  $U$  is the set  $U\downarrow = \{v \in W \mid v \leq u \text{ for some } u \in U\}$ . Define a system of spheres as follows:

$$\mathbf{S}_{\leq} = \{V\downarrow \mid V \subseteq W\}$$

Note that if  $U \in \mathbf{S}_{\leq}$  then  $U \neq \emptyset$ . We must show  $\mathbf{S}_{\leq}$  satisfies (S-a), (S-b), (S-c), and (S-d).

**Claim S-a.**  $\mathbf{S}_{\leq}$  is totally ordered by  $\subseteq$ .

**Proof of Claim S-a.** Suppose that  $U, V \in \mathbf{S}_{\leq}$  with  $U \not\subseteq V$ . Then there is a  $u \in U$  with  $u \notin V$ . Let  $v \in V$ . We will show that  $v \in U$ . Since  $\leq$  is totally ordered, either  $u \leq v$  or  $v \leq u$ . If  $u \leq v$ , then since  $V$  is a downset ( $V = V\downarrow$ ), we must have  $u \in V$ , which is a contradiction. So,  $v \leq u$ . But this implies, since  $U$  is a downset ( $U = U\downarrow$ ) and  $u \in U$ , that  $v \in U$ , as desired. QED (of Claim)

**Claim S-b.**  $X$  is the  $\subseteq$ -minimum element of  $\mathbf{S}_{\leq}$ .

**Proof of Claim b.** First of all, note that the set of  $\leq$ -minimal elements of  $W$  is a downset (this is easy to see, since if  $w$  and  $v$  are  $\leq$ -minimal elements then  $w \leq v$  and  $v \leq w$ ). So, since by (O-c),  $X$  is the set of  $\leq$ -minimal elements, we have  $X \in \mathbf{S}_{\leq}$ . Let  $U \in \mathbf{S}_{\leq}$  be any downset. We must show that  $X \subseteq U$ . Let  $x \in X$ . Recall that  $U$  is nonempty, so there is some  $u \in U$ . By (O-c),  $x$  is a  $\leq$ -minima, so  $x \leq u$ . Hence  $x \in U$ . QED (of Claim)

**Claim S-c.**  $W \in \mathbf{S}_{\leq}$ .

**Proof of Claim S-c.** Obviously,  $W$  is a downset ( $W = W\downarrow$ ), so  $W \in \mathbf{S}_{\leq}$ .

QED (of Claim)

**Claim S-d.** For every propositional formula  $\varphi \in \mathcal{L}$  and sphere  $U \in \mathbf{S}_{\leq}$ , if  $U \cap \llbracket \varphi \rrbracket \neq \emptyset$  then there is a  $\subseteq$ -minimal sphere  $U_0 \in \mathbf{S}_{\leq}$  such that  $U_0 \cap \llbracket \varphi \rrbracket \neq \emptyset$ .

**Proof of Claim S-d.** Suppose that  $\varphi \in \mathcal{L}$  and  $U \in \mathbf{S}_{\leq}$  with  $U \cap \llbracket \varphi \rrbracket \neq \emptyset$ . Let  $U_0 = \{v \mid v \in \llbracket \varphi \rrbracket \text{ and } v \leq w \text{ for all } w \in \llbracket \varphi \rrbracket\}$ . By (O-b), since  $\llbracket \varphi \rrbracket \neq \emptyset$ ,  $U_0 \neq \emptyset$ . Then  $U_0 \downarrow \in \mathbf{S}_{\leq}$ . We claim that  $U_0 \downarrow$  is the  $\subseteq$ -smallest element of  $\mathbf{S}_{\leq}$  that overlaps  $\llbracket \varphi \rrbracket$ . Let  $V \in \mathbf{S}_{\leq}$  with  $V \cap \llbracket \varphi \rrbracket \neq \emptyset$ . Say  $y \in V \cap \llbracket \varphi \rrbracket$ . We must show  $U_0 \downarrow \subseteq V$ . Let  $x \in U_0 \downarrow$ . Then  $x \leq u$  for some  $u \in U_0$ . By construction of  $U_0$ , since  $y \in \llbracket \varphi \rrbracket$ , we have  $u \leq y$ . Since  $\leq$  is transitive,  $x \leq y$ . Since  $V$  is a downset, we have  $x \in V$ , as desired. Hence,  $U_0 \downarrow$  is the  $\subseteq$ -minimal element of  $\mathbf{S}_{\leq}$  that overlaps  $\llbracket \varphi \rrbracket$ .

QED (of Claim)

QED