1. ***Linear time models***: A linear time model is a tuple $\mathcal{M} = \langle T, <, V\rangle$ where T is a set of **time points** (or **moments**), $< \subseteq T \times T$ is the **precedence relation**: $s < t$ ("time point occurs earlier than t") is irreflexive and transitive, and $V : \mathsf{At} \to \wp(T)$ is a valuation function (describing when the atomic propositions are true). The linear time language is given by the following grammar:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid G\varphi \mid H\varphi$$

where $p \in \mathsf{At}$ (a countable set of atomic propositions). Truth is defined as follows:

- $\mathcal{M}, t \models p$ iff $t \in V(p)$
- $\mathcal{M}, t \models \neg\varphi$ iff $\mathcal{M}, t \not\models \varphi$
- $\mathcal{M}, t \models \varphi \wedge \psi$ iff $\mathcal{M}, t \models \varphi$ and $\mathcal{M}, t \models \psi$
- $\mathcal{M}, t \models G\varphi$ iff for all $s \in T$, if $t < s$ then $\mathcal{M}, s \models \varphi$
- $\mathcal{M}, t \models H\varphi$ iff for all $s \in T$, if $s < t$ then$\mathcal{M}, s \models \varphi$

We define $F\varphi := \neg G \neg\varphi$ and $P\varphi := \neg H \neg \varphi$, so truth for these operators is:

- $\mathcal{M}, t \models F\varphi$ iff there is $s \in T$ such that $t < s$ and $\mathcal{M}, s \models \varphi$
- $\mathcal{M}, t \models P\varphi$ iff there is $s \in T$ such that $s < t$ and $\mathcal{M}, s \models \varphi$

We say φ is valid on a temporal model $\mathcal{M} = \langle T, <, V\rangle$ provided $\mathcal{M}, t \models \varphi$ for all $t \in T$, and φ is valid on a temporal frame $\langle T, <\rangle$, provided φ is valid on every model based on $\langle T, <\rangle$ (these are standard definitions — see the notes on modal logic).

(a) A temporal frame $\langle T, <\rangle$ is **past-linear** provided for all $s, x, y \in T$, if $x < s$ and $y < s$, then either $x < y$ or $x = y$ or $y < x$.

Claim 1 *$FP\varphi \to (F\varphi \vee \varphi \vee P\varphi)$ is valid on $\langle T, <\rangle$ iff $\langle T, <\rangle$ is past-linear.*

Proof. Suppose that $\mathcal{T} = \langle T, <\rangle$ is past-linear and $\mathcal{M} = \langle T, <, V\rangle$ is a model based on $\mathcal{T}$. We must show $FP\varphi \to (F\varphi \vee \varphi \vee P\varphi)$ is valid on $\mathcal{M}$. Let $t \in T$ be any moment and suppose that $\mathcal{M}, t \models FP\varphi$. Then, there is a $s \in T$ such that $t < s$ and $\mathcal{M}, s \models P\varphi$. This implies there is a s' such that $s' < s$ with $\mathcal{M}, s' \models \varphi$. Since $\mathcal{T}$ is past-linear and $t < s$ and $s' < s$ we have three cases: either $t < s'$ or $t = s'$ or $s' < t$. In the first case $\mathcal{M}, t \models F\varphi$, in the second case $\mathcal{M}, t \models \varphi$ and in the third case $\mathcal{M}, t \models P\varphi$. Hence, $\mathcal{M}, t \models F\varphi \vee \varphi \vee P\varphi$, as desired.

Suppose that $\mathcal{T} = \langle T, <\rangle$ is not past-linear. Then, there are moments s, s', and t such that $s < t$, $s' < t$ but $s \neq s'$, $s \not< s'$ and $s' \not< s$. Let $\mathcal{M} = \langle T, <, V\rangle$ be a model based on T where $V(p) = \{s'\}$. Since, $s' < t$ and $\mathcal{M}, s' \models p$, we have $\mathcal{M}, t \models Pp$. Then, since $s < t$, we have $\mathcal{M}, s \models FPp$. Note that $\mathcal{M}, s \models \neg Pp \wedge p \wedge \neg Fp$ (this follows since the only state satisfying p is s' and s' is incomparable with s). Hence, $\mathcal{M}, s \not\models FPp \to (Pp \vee p \vee Fp)$. QED

2. ***Branching-time temporal models***: Given a temporal model $\langle T, <, V\rangle$ a **branch** b is a maximal linearly ordered set of moments. We say $s \in T$ is **on a branch b of** T provided $s \in b$ (we also say "b is a branch going through t"). The branching time language is given by the following grammar:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid G\varphi \mid H\varphi \mid \Box\varphi$$

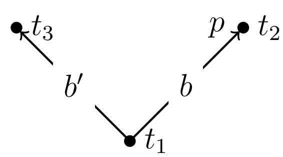
where $p \in \mathsf{At}$. Truth is defined at pairs t/b where t is a moment on branch b:

- $\mathcal{M}, t/b \models p$ iff $t/b \in V(p)$
- $\mathcal{M}, t/b \models \neg\varphi$ iff $\mathcal{M}, t/b \not\models \varphi$
- $\mathcal{M}, t/b \models \varphi \wedge \psi$ iff $\mathcal{M}, t/b \models \varphi$ and $\mathcal{M}, t/b \models \psi$
- $\mathcal{M}, t/b \models G\varphi$ iff for all $s \in T$, if s is on b and $t < s$ then $\mathcal{M}, s/b \models \varphi$
- $\mathcal{M}, t/b \models H\varphi$ iff for all $s \in T$, if s is on b and $s < t$ then $\mathcal{M}, s/b \models \varphi$
- $\mathcal{M}, t/b \models \Box\varphi$ iff for all branches c through t, $\mathcal{M}, s/c \models \varphi$

For each of the following formulas, determine which are valid on all temporal frames (for those that are not valid, provide counterexamples):

(a) $\Diamond F\varphi \to F\Diamond\varphi$ is not valid.

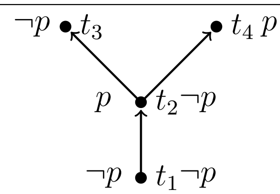
Proof. Let $T = \{t_1, t_2, t_3\}$ with $t_1 < t_2$ and $t_1 < t_3$, so there are two branches $b = \{t_1, t_2\}$ and $b' = \{t_1, t_3\}$. Let $V(p) = \{t_2/b\}$. Then, $\mathcal{M}, t_1/b \models Fp$ and so $\mathcal{M}, t_1/b' \models \Diamond Fp$. However, since b' is the only branch going through t_3 and $\mathcal{M}, t_3/b' \not\models p$, we have $\mathcal{M}, t_3/b' \not\models \Diamond p$. Furthermore, since t_3 is the only moment on b' such that $t_1 < t_3$, we have $\mathcal{M}, t_1/b' \not\models F\Diamond p$. Hence, $\Diamond Fp \to F\Diamond p$ is not valid. This model is pictured below:



QED

(b) $\Box F\varphi \to F\Box\varphi$ is not valid.

Proof. Suppose that $T = \{t_1, t_2, t_3, t_4\}$ with $t_1 < t_2 < t_3$ and $t_1 < t_2 < t_4$. There are two branches: $b_1 = \{t_1, t_2, t_3\}$ and $b_2 = \{t_1, t_2, t_4\}$. Suppose that $V(p) = \{t_2/b_1, t_4/b_2\}$. Then, since $t_1 < t_2$ and $t_1 < t_4$, we have $\mathcal{M}, t_1/b_1 \models Fp$ and $\mathcal{M}, t_1/b_2 \models Fp$. Hence, $\mathcal{M}, t_1/b_1 \models \Box Fp$. However, since $\mathcal{M}, t_3/b_1 \not\models \Box p$ (this follows from the fact that $\mathcal{M}, t_3/b_1 \not\models p$ and b_1 is the only branch through t_3) and $\mathcal{M}, t_2/b_1 \not\models \Box p$ (this follows since $\mathcal{M}, t_2/b_2 \not\models p$), we have $\mathcal{M}, t_1/b_1 \not\models F\Box p$. Therefore, $\Box F\varphi \to F\Box\varphi$ is not valid. This model is pictured below:



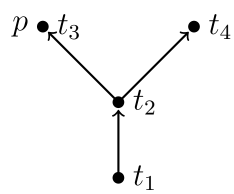
QED

(c) $F\Diamond\varphi \to \Diamond F\varphi$ is valid.

Proof. Suppose that $\mathcal{M}, t/b \models F\Diamond\varphi$. Then there is a $t' \in b$ such that $t < t'$ and $\mathcal{M}, t'/b \models \Diamond\varphi$. This implies there is a branch c going through t' such that $\mathcal{M}, t'/c \models \varphi$. Since t' is $t < t'$, any branching going through t' must also go through t (recall that branches are *maximal* sets of linearly ordered moments), so c is a branching going through t. Since $\mathcal{M}, t'/c \models \varphi$ and $t < t'$, we have $\mathcal{M}, t/c \models F\varphi$. Since both c and b go through t, we have $\mathcal{M}, t/b \models \Diamond F\varphi$. Hence, $F\Diamond\varphi \to \Diamond F\varphi$ is valid. QED

(d) $F\Box\varphi \to \Box F\varphi$ is not valid.

Proof. Suppose that $T = \{t_1, t_2, t_3, t_4\}$ with $t_1 < t_2 < t_3$ and $t_1 < t_2 < t_4$. There are two branches: $b_1 = \{t_1, t_2, t_3\}$ and $b_2 = \{t_1, t_2, t_4\}$. Suppose that $V(p) = \{t_3/b_1\}$. Since $\mathcal{M}, t_3/b_1 \models p$ and b_1 is the only branch through t_3, we have $\mathcal{M}, t_3/b_1 \models \Box p$. Hence, $\mathcal{M}, t_1/b_1 \models F\Box p$. However, since $\mathcal{M}, t_4/b_2 \not\models p$ and $\mathcal{M}, t_2/b_2 \not\models p$, we have $\mathcal{M}, t_1/b_2 \not\models Fp$ and so $\mathcal{M}, t_1/b_1 \not\models \Box Fp$. This model is pictured below:



QED

3. ***Logics of Ability***: The logics of ability models of Brown are tuples $\langle W, R, V\rangle$ where $R \subseteq W \times \wp(W)$ is a relation between states and subsets of W (which Brown calls "clusters") and $V : \mathsf{At} \to \wp(W)$ a valuation function. The ability language is generated by the following grammar:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid [\![\ ]\!]\varphi \mid \langle\!\langle\ \rangle\!\rangle\varphi$$

where $p \in \mathsf{At}$. The intended meaning is that $[\![\ ]\!]\varphi$ expresses "the agent is able to bring about a state where φ is true" and $\langle\!\langle\ \rangle\!\rangle\varphi$ is the weaker claim that "the agent is able to do something consistent with φ". Truth is defined as follows:

- $\mathcal{M}, w \models p$ iff $w \in V(p)$
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, t \models \llbracket\ \rrbracket\varphi$ iff there is a $X \subseteq W$ such that wRX and for all $v \in X$, $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, t \models \langle\langle\ \rangle\rangle\varphi$ iff there is a $X \subseteq W$ such that wRX and there is a $v \in X$ such that $\mathcal{M}, v \models \varphi$

Answer the following questions:

(a) Give a counter-model to $\llbracket\ \rrbracket(\varphi \vee \psi) \rightarrow (\llbracket\ \rrbracket\varphi \vee \llbracket\ \rrbracket\psi)$.

Answer. Let $W = \{w_1, w_2\}$ and suppose that $V(p) = \{w_1\}$ and $V(q) = \{w_3\}$. Let $R \subseteq W \times \wp(W)$ be such that $w_1 R\{w_1, w_2\}$. Then we have $\mathcal{M}, w_1 \models \llbracket\ \rrbracket(p \vee q)$ since $wR\{w_1, w_2\}$ and $\{w_1, w_2\} \subseteq \llbracket p \vee q\rrbracket_{\mathcal{M}} = \llbracket p\rrbracket_{\mathcal{M}} \cup \llbracket q\rrbracket_{\mathcal{M}} = \{w_1\} \cup \{w_2\} = \{w_1, w_2\}$. However, $\mathcal{M}, w_1 \not\models \llbracket\ \rrbracket p$ since $\{w_1, w_2\} \not\subseteq \llbracket p\rrbracket_{\mathcal{M}} = \{w_1\}$, and similarly $\mathcal{M}, w_1 \not\models \llbracket\ \rrbracket q$. Hence, $\mathcal{M}, w_1 \not\models \llbracket\ \rrbracket(p \vee q) \rightarrow (\llbracket\ \rrbracket p \vee \llbracket\ \rrbracket q)$.

(b) Prove that $\llbracket\ \rrbracket(\varphi \vee \psi) \rightarrow (\langle\langle\ \rangle\rangle\varphi \vee \llbracket\ \rrbracket\psi)$ is valid.

Proof. Suppose that $\mathcal{M}, w \models \llbracket\ \rrbracket(\varphi \vee \psi)$ then there is a $X \subseteq W$ such that wRX and $X \subseteq \llbracket\varphi \vee \psi\rrbracket_{\mathcal{M}} = \llbracket\varphi\rrbracket_{\mathcal{M}} \cup \llbracket\psi\rrbracket_{\mathcal{M}}$. Note that either $X \cap \llbracket\varphi\rrbracket_{\mathcal{M}} \neq \emptyset$ or $X \cap \llbracket\varphi\rrbracket_{\mathcal{M}} = \emptyset$. In the first case, $\mathcal{M}, w \models \langle\langle\ \rangle\rangle\varphi$. In the second case, since $X \cap \llbracket\varphi\rrbracket_{\mathcal{M}} = \emptyset$ and $X \subseteq \llbracket\varphi\rrbracket_{\mathcal{M}} \cup \llbracket\psi\rrbracket_{\mathcal{M}}$, we have $X \subseteq \llbracket\psi\rrbracket_{\mathcal{M}}$. Hence, $\mathcal{M}, w \models \llbracket\ \rrbracket\psi$ Thus, in either case, $\mathcal{M}, w \models \langle\langle\ \rangle\rangle\varphi \vee \llbracket\ \rrbracket\psi$. And so, $\mathcal{M}, w \models \llbracket\ \rrbracket(\varphi \vee \psi) \rightarrow (\langle\langle\ \rangle\rangle\varphi \vee \llbracket\ \rrbracket\psi)$. QED

(c) Is $\llbracket\ \rrbracket\varphi \rightarrow \langle\langle\ \rangle\rangle\varphi$ valid? If it is, give a proof, and if it is not valid, give a property that would make it valid.

Answer. No, $\llbracket\ \rrbracket\varphi \rightarrow \langle\langle\ \rangle\rangle\varphi$ is not valid. Let $\mathcal{M} = \langle W, R, V\rangle$ be a model where there is a state w with $wR\emptyset$. Then for any formula φ, we have $\mathcal{M}, w \models \llbracket\ \rrbracket\varphi$, but $\mathcal{M}, w \not\models \langle\langle\ \rangle\rangle\varphi$. It is not hard to see that if we assume that for all w we do not have $wR\emptyset$, then $\llbracket\ \rrbracket(\varphi \vee \psi) \rightarrow (\langle\langle\ \rangle\rangle\varphi \vee \llbracket\ \rrbracket\psi)$ is valid.

4. ***STIT models***: A **stit model** is a tuple $\mathcal{M} = \langle T, <, Choice, V\rangle$ where $\langle T, <, V\rangle$ is a temporal model (defined as above), and $Choice : \mathcal{A} \times T \rightarrow \wp(\wp(H_t))$ is a function mapping each agent to a partition of H_t (H_t is the set of branches going through t) satisfying the following conditions (we write $Choice_i^t$ for $Choice(i, t)$:

- $Choice_i^t \neq \emptyset$
- $K \neq \emptyset$ for each $K \in Choice_i^t$
- For all t and mappings $s_t : \mathcal{A} \rightarrow \wp(H_t)$ such that $s_t(i) \in Choice_i^t$, we have $\bigcap_{i \in \mathcal{A}} s_t(i) \neq \emptyset$

The STIT language is defined according to the following grammar:

$$\varphi \; = \; p \mid \neg\varphi \mid \varphi \wedge \psi \mid [i \text{ stit}]\varphi \mid \Box\varphi$$

where $p \in \mathsf{At}$. Truth is defined as follows:

- $\mathcal{M}, t/h \models p$ iff $t/h \in V(p)$
- $\mathcal{M}, t/h \models \neg\varphi$ iff $\mathcal{M}, t/h \not\models \varphi$
- $\mathcal{M}, t/h \models \varphi \wedge \psi$ iff $\mathcal{M}, t/h \models \varphi$ and $\mathcal{M}, t/h \models \psi$
- $\mathcal{M}, t/h \models \Box\varphi$ iff $\mathcal{M}, t/h' \models \varphi$ for all $h' \in H_t$
- $\mathcal{M}, t/h \models [i \text{ stit}]\varphi$ iff $\mathcal{M}, t/h' \models \varphi$ for all $h' \in Choice_i^t(h)$ ($Choice_i^t(h)$ is the partition cell of $Choice_i^t$ containing h)

Define $\langle i \text{ stit}\rangle\varphi$ to be $\neg[i \text{ stit}]\neg\varphi$ and $\Diamond\varphi$ to be $\neg\Box\neg\varphi$. Answer the following two questions: Suppose that there are only two agents $\mathcal{A} = \{1, 2\}$, then

(a) Prove that $\Diamond\varphi \to \langle 1 \text{ stit}\rangle\langle 2 \text{ stit}\rangle\varphi$ is valid.

Proof. Suppose that $\mathcal{M}, t/h \models \Diamond\varphi$ then there is a $h' \in H_t$ such that $\mathcal{M}, t/h' \models \varphi$. Consider the selection $s_t(1) = Choice_t^1(h)$ (agent 1's choice at h/t) and $s_t(2) = Choice_t^2(h')$ (agent 2's choice at t/h'). Then by the independence property, $s_t(1) \cap s_t(2) \neq \emptyset$. So, there is a history $h'' \in s_t(1) \cap s_t(2) = Choice_t^1(h) \cap Choice_t^2(h')$. Then, since $h' \in Choice_t^2(h'')$ (recall, $Choice_t^2$ is a partition) and $\mathcal{M}, t/h' \models \varphi$, we have $\mathcal{M}, t/h'' \models \langle 2 \text{ stit}\rangle\varphi$. Since $h'' \in Choice_t^1(h)$, we have $\mathcal{M}, t/h \models \langle 1 \text{ stit}\rangle\langle 2 \text{ stit}\rangle\varphi$. QED

(b) Conclude that $\Box\varphi$ is definable as $[1 \text{ stit}][2 \text{ stit}]\varphi$ (argue that $\Box\varphi \leftrightarrow [1 \text{ stit}][2 \text{ stit}]\varphi$ can be derived from the above axiom using the **S5** axioms for $\Box$ and $[i \text{ stit}]$, and the axiom $\Box\varphi \to [i \text{ stit}]\varphi$).

Proof. We derive $\Box\varphi \leftrightarrow [1 \text{ stit}][2 \text{ stit}]\varphi$ using the STIT axioms:

Prop: all instances of propositional tautologies

S5 for $\Box$	**S5 for $[i \text{ stit}]$**
$K_\Box$: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	K_{stit}: $[i\ stit](\varphi \to \psi) \to ([i\ stit]\varphi \to [i\ stit]\psi)$
$T_\Box$: $\Box\varphi \to \varphi$	T_{stit}: $[i\ stit]\varphi \to \varphi$
$4_\Box$: $\Box\varphi \to \Box\Box\varphi$	4_{stit}: $[i\ stit]\varphi \to [i\ stit][i\ stit]\varphi$
$5_\Box$: $\neg\Box\varphi \to \Box\neg\Box\varphi$	5_{stit}: $\neg[i\ stit]\varphi \to [i\ stit]\neg[i\ stit]\varphi$
$Nec_\Box$: for φ, infer $\Box\varphi$	Nec_{stit}: for φ, infer $[i \text{ stit}]\varphi$

$\Box\!\to\![i \text{ stit}]$: $\Box\varphi \to [i\ stit]\varphi$

Ind: $(\bigwedge_{i\in\mathcal{A}} \Diamond[i\ stit]\varphi_i) \to \Diamond(\bigwedge_{i\in\mathcal{A}}[i\ stit]\varphi_i)$

We make use of the following rules of propositional logic:

Prop Reasoning: Trans

$$\frac{\begin{array}{l} A \rightarrow B \\ B \rightarrow C \end{array}}{A \rightarrow C}$$

Prop Reasoning: Equiv

$$\frac{A \leftrightarrow B}{\varphi[C/A] \leftrightarrow \varphi[C/B]}$$ ($\varphi[C/A]$ is φ with all occurrences of C replaced with A)

Below is a derivation of $\Box\varphi \rightarrow [1 \text{ stit}][2 \text{ stit}]\varphi$:

1.	$\Box\varphi \rightarrow [2 \text{ stit}]\varphi$	Axiom $\Box \rightarrow [2 \text{ stit}]$
2.	$\Box(\Box\varphi \rightarrow [2 \text{ stit}]\varphi)$	$Nec_{\Box}$ 1.
3.	$\Box(\Box\varphi \rightarrow [2 \text{ stit}]\varphi) \rightarrow (\Box\Box\varphi \rightarrow \Box[2 \text{ stit}]\varphi)$	Axiom $K_{\Box}$
4.	$\Box\Box\varphi \rightarrow \Box[2 \text{ stit}]\varphi$	MP 2,3
5.	$\Box\varphi \rightarrow \Box\Box\varphi$	Axiom $4_{\Box}$
6.	$\Box\varphi \rightarrow \Box[2 \text{ stit}]\varphi$	Prop Reasoning: Trans 4, 5
7.	$\Box[2 \text{ stit}]\varphi \rightarrow [1 \text{ stit}][2 \text{ stit}]\varphi$	Axiom $\Box \rightarrow [1 \text{ stit}]$
8.	$\Box\varphi \rightarrow [1 \text{ stit}][2 \text{ stit}]\varphi$	Prop Reasoning: Trans 6, 7

Below is a derivation of $[1 \text{ stit}][2 \text{ stit}]\varphi \rightarrow \Box\varphi$:

1.	$\Diamond\neg\varphi \rightarrow \langle 1 \text{ stit}\rangle\langle 2 \text{ stit}\rangle\neg\varphi$	Axiom
2.	$\neg\langle 1 \text{ stit}\rangle\langle 2 \text{ stit}\rangle\neg\varphi \rightarrow \neg\Diamond\neg\varphi$	Prop reasoning
3.	$\neg\neg[1 \text{ stit}]\neg\neg[2 \text{ stit}]\neg\neg\varphi \rightarrow \neg\Diamond\neg\varphi$	$[i \text{ stit}]$-dual
4.	$[1 \text{ stit}][2 \text{ stit}]\varphi \rightarrow \neg\Diamond\neg\varphi$	Prop reasoning ($\neg\neg\varphi \leftrightarrow \varphi$)
5.	$[1 \text{ stit}][2 \text{ stit}]\varphi \rightarrow \Box\varphi$	$\Box$-dual

QED

The homework is DUE Tuesday, November 22 (put you answers in my mailbox).