1. Prove that the following axiom of ceteris paribus logic is valid (see slide 22 of lecture 21 on 11/16):
$$(\alpha \wedge \langle\Gamma\rangle^{\leq}(\alpha \wedge \varphi)) \to \langle\Gamma \cup \{\alpha\}\rangle^{\leq}\varphi$$

Proof. Let $\mathcal{M}$ be a preference model and w a state in $\mathcal{M}$. Suppose that $\mathcal{M}, w \models \alpha \wedge \langle\Gamma\rangle^{\leq}(\alpha \wedge \varphi)$. Then, $\mathcal{M}, w \models \alpha$ and there is a v such that $w \equiv_{\Gamma} v$ and $w \leq v$ and $\mathcal{M}, v \models \alpha \wedge \varphi$. Now we have for all $\varphi \in \Gamma$, $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}, v \models \varphi$ and $\mathcal{M}, w \models \alpha$ and $\mathcal{M}, v \models \alpha$. Hence, $w \equiv_{\Gamma\cup\{\alpha\}} v$. Since $w \leq v$, we have $\mathcal{M}, w \models \langle\Gamma \cup \{\alpha\}\rangle^{\leq}\varphi$. Since $\mathcal{M}$ and w were arbitrary, $(\alpha \wedge \langle\Gamma\rangle^{\leq}(\alpha \wedge \varphi)) \to \langle\Gamma \cup \{\alpha\}\rangle^{\leq}\varphi$ is valid. QED

2. Let X, Y be subsets of W and suppose that $\leq$ is a reflexive, connected and transitive order over W. Say $X \leq_{\forall\forall} Y$ provided for all $x \in X$ and for all $y \in Y$, we have $x \leq y$. Assume that $\leq$ is reflexive, transitive and complete, is $\leq_{\forall\forall}$ also reflexive, transitive, and complete? If so, prove it and if not, give a counterexample.

Proof. Suppose that $\leq \subseteq W \times W$ is reflexive, transitive and connected. We show that $\leq_{\forall\forall}$ is transitive but not reflexive nor connected.

$\leq_{\forall\forall}$ *is not reflexive*: Suppose that $W = \{1, 2, 3, 4\}$ with $1 < 2 < 3 < 4$ (where $i < j$ means $i \leq j$ but $j \not\leq i$). Consider $X = \{1, 2\}$, then $X \not\leq_{\forall\forall} X$ since $2 \not\leq 1$.

$\leq_{\forall\forall}$ *is transitive* for all nonempty sets: First of all, not that for any sets X and Y, $X \leq_{\forall\forall} \emptyset$ and $\emptyset \leq_{\forall\forall} Y$. Transitivity would imply $X \leq_{\forall\forall} Y$, but it is easy to find counterexamples to this. Suppose that X, Y, and Z are nonempty. Suppose that $X \leq_{\forall\forall} Y$ and $Y \leq_{\forall\forall} Z$, we must show that $X \leq_{\forall\forall} Z$. Let $x \in X$ and $z \in Z$. Since Y is nonempty there is an element $y \in Y$. Since $X \leq_{\forall\forall} Y$, we have $x \leq y$. Since, $Y \leq_{\forall\forall} Z$, we have $y \leq z$. Since $\leq$ is transitive, we have $x \leq z$. Since x and z were arbitrary elements of X and Z, respectively, we have $X \leq_{\forall\forall} Z$.

$\leq_{\forall\forall}$ *is not connected*: Suppose that $W = \{1, 2, 3, 4\}$ with $1 < 2 < 3 < 4$ (where $i < j$ means $i \leq j$ but $j \not\leq i$). Let $X = \{1, 3\}$ and $Y = \{2, 4\}$ then $X \not\leq_{\forall} Y$ and $Y \not\leq_{\forall\forall} X$.

QED

Can you think of any other interesting principles that $\leq_{\forall\forall}$ satisfies? One interesting set of principles are downward and upwards monotonicity:

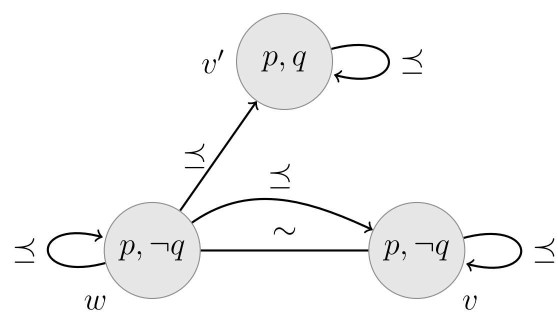
- If $X \leq_{\forall\forall} Y$ and $Z \subseteq X$, then $Z \leq_{\forall\forall} Y$.
- If $X \leq_{\forall\forall} Y$ and $Z \subseteq X$, then $Z \leq_{\forall\forall} Z$.

3. Recall the model of knowledge and preference from Lecture 22 (on 11/21): $\mathcal{M} = \langle W, \sim, \preceq, V\rangle$ where $\sim$ is an equivalence relation and $\preceq$ is a reflexive, transitive and total preference relation. Truth is defined as follows:

   - $\mathcal{M}, w \models K\varphi$ iff for all $v \in W$, if $w \sim v$ then $\mathcal{M}, v \models \varphi$
   - $\mathcal{M}, w \models \langle\preceq\rangle\varphi$ iff there is a $v \in W$ with $w \preceq v$ and $\mathcal{M}, v \models \varphi$
   - $\mathcal{M}, w \models A\varphi$ iff for all $v \in W$, $\mathcal{M}, v \models \varphi$
   - $\mathcal{M}, w \models \langle\sim \cap \preceq\rangle\varphi$ iff there is a $v \in W$ such that $w \sim v$ and $w \preceq v$ with $\mathcal{M}, v \models \varphi$

   Given an example to show that $K(\psi \to \langle\preceq\rangle\varphi)$ and $K(\psi \to \langle\sim \cap \preceq\rangle\varphi)$ or not equivalent (i.e., find a model and state where one of the formulas is true, but the other is not true). It is easy to see that $A(\psi \to \langle\preceq\rangle\varphi) \to K(\psi \to \langle\preceq\rangle\varphi)$ is valid (this is an instance of the validity $A\varphi \to K\varphi$), but what is the relationship between $A(\psi \to \langle\preceq\rangle\varphi)$ and $K(\psi \to \langle\sim \cap \preceq\rangle\varphi$ (does one imply the other or are the two formulas independent)?

   Answer. We can construct a model where $K(p \to \langle\preceq\rangle q)$ is true but $K(p \to \langle\preceq \cap \sim\rangle q)$ is false. The model is drawn below (with the undirected lines denoting the information relation $\sim$ and the arrows denoting the preference relation where an arrow from w to v means $w \preceq v$).



   Then, $\mathcal{M}, w \models K(p \to \langle\preceq\rangle q)$, but $\mathcal{M}, w \not\models K(p \to \langle\preceq \cap \sim\rangle q)$

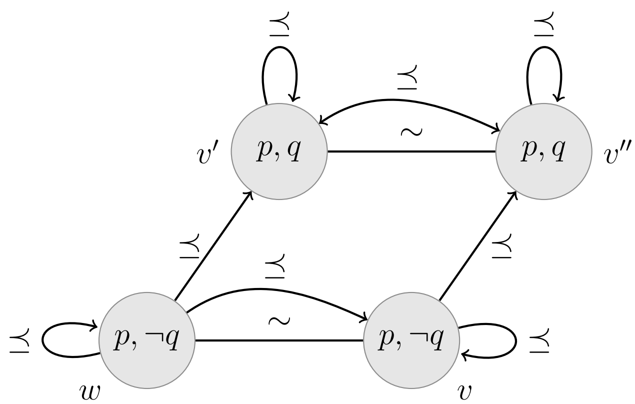
   Claim 1 *$K(\psi \to \langle\preceq \cap \sim\rangle\varphi) \to K(\psi \to \langle\preceq\rangle\varphi)$ is valid*

   Proof. Suppose that $\mathcal{M}, w \models K(\psi \to \langle\preceq \cap \sim\rangle\varphi)$. Suppose that there is a v such that $w \sim v$. We must show $\mathcal{M}, v \models \psi \to \langle\preceq\rangle\varphi$. Suppose that $\mathcal{M}, v \models \psi$. Since, $\mathcal{M}, w \models K(\psi \to \langle\preceq \cap \sim\rangle\varphi)$ and $w \sim v$, we have $\mathcal{M}, v \models \psi \to \langle\preceq \cap \sim\rangle\varphi$. This implies $\mathcal{M}, v \models \langle\preceq \cap \sim\rangle\varphi$. Hence, there is a v' such that $v(\preceq \cap \sim)v'$ and $\mathcal{M}, v' \models \varphi$. Since, $(\preceq \cap \sim) \subseteq \preceq$, we have $v \preceq v'$. Hence, $\mathcal{M}, v \models \langle\preceq\rangle\varphi$, as desired. Hence, $\mathcal{M}, w \models K(\psi \to \langle\preceq\rangle\varphi)$. QED

Claim 2 *$A(\psi \to \langle\preceq\rangle\varphi)$ and $K(\psi \to \langle\sim\cap\preceq\rangle\varphi$ are independent*

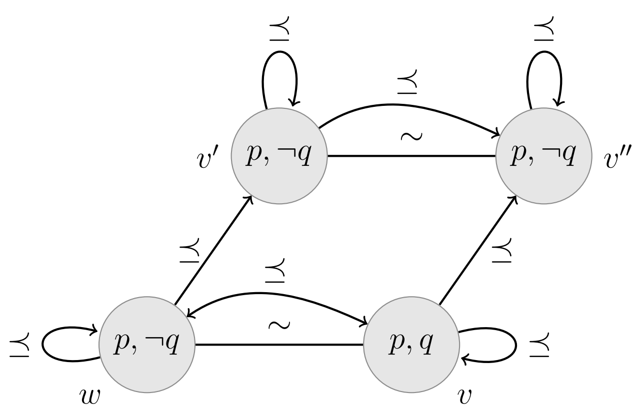
Proof. We can construct two models: one where $A(p \to \langle\preceq\rangle q)$ is true but $K(p \to \langle\preceq \cap \sim\rangle q)$ is false, and vice versa. The models are drawn below (with the undirected lines denoting the information relation $\sim$ and the arrows denoting the preference relation where an arrow from w to v means $w \preceq v$).

Example 1



Then we have $\mathcal{M}, w \models A(p \to \langle\preceq\rangle q)$ but $\mathcal{M}, w \not\models K(p \to \langle\sim\cap\preceq\rangle q)$

Example 2



Then we have $\mathcal{M}, w \not\models A(p \to \langle\preceq\rangle q)$ but $\mathcal{M}, w \models K(p \to \langle\sim\cap\preceq\rangle q)$

QED