1 Arrow’s Theorem

Notation

- $X$ is a (finite or infinite) set of alternatives (or candidates).
- $N = \{1, \ldots, n\}$ is a set of voters.
- Preferences: $\mathcal{P} = \{R \mid R \subseteq W \times W \text{ where } R \text{ is reflexive, transitive and connected}\}$
- Given $R \in \mathcal{P}$, let $P$ be the strict preference generated by $R$: $xPy$ iff $xRy$ and not $yRx$ (we write $P_R$ if necessary).
- A profile is a tuple $(R_1, \ldots, R_n) \in \mathcal{P}^n$.
- Social Welfare Function: $F : D \rightarrow \mathcal{P}$ where $D \subseteq \mathcal{P}^n$ is the domain.

Axioms

- Universal Domain (UD): The domain of $F$ is $\mathcal{P}^n$ (i.e., $D = \mathcal{P}^n$).
- Independence of Irrelevant Alternatives (IIA): $F$ satisfies IIA provide for all profiles $\vec{R}, \vec{R}' \in D$, if $xR_i y$ iff $xR'_i y$ for all $i \in N$, then $xF(\vec{R}) y$ iff $xF(\vec{R}') y$.
- (weak) Pareto (P): For all profiles $\vec{R} \in D$, if $xP_i y$ for all $i \in N$ then $xP_{F(\vec{R})} y$. 


• Agent $i$ is a **dictator** for $F$ provided for every preference profile and every pair $x, y \in X$, $xP_i y$ implies $xF(\vec{R}) y$.

**Arrows (Im)possibility theorem**: Suppose that $|X| \geq 3$ and $F$ satisfies UD, IIA and P. Then there is some $i \in N$ that is a dictator for $F$.

**Key Lemmas**

First, some key definitions. To simplify the notation, for a $\vec{R} \in D$, we write $S$ for the social ordering given by $F$, i.e., $F(\vec{R}) = Q$.

For a set of voters $S \subseteq \{1, \ldots, n\}$, we say

- $S$ is **decisive for** $x$ **over** $y$ if for some preference profile $\vec{R}$ we have $xP_i x$ for all $i \in S$, $yP_i x$ for all $i \notin S$ and $xP_Q y$.

- $S$ is **strictly decisive for** $x$ **over** $y$ if for every preference profile $\vec{R}$ satisfying $xP_i y$ for all $i \in S$, we have $xP_Q y$

- $S$ is **decisive** if it is strictly decisive for every pair of distinct alternatives.

**Lemma 1** Suppose that for some $x$ and $y$, $S$ is decisive for $x$ over $y$, then $S$ is decisive.

**Lemma 2** If $S$ and $T$ are decisive then so is $S \cap T$

**Lemma 3** If $S$ is not decisive, then $S^C = N - S$ is decisive.

**Arrow’s Theorem**: There is a singleton decisive set.


## 2 Sen’s Theorem

**Notation**

- *Linear Preferences*: $\mathcal{L} = \{> | < \subseteq X \times X$ is a linear order$\}$

- *Social choice function*: $C : \mathcal{L}^n \rightarrow X$
Axioms

- **(weak) Unanimity**: $C$ satisfies weak unanimity provided if for every preference profile $\succ \in \mathcal{L}^n$, if there is a pair of alternatives $x$ and $y$ such that $x >_i y$ for all $i \in N$, then $C(\succ) \neq y$.

- **Liberalism**: $C$ satisfies liberalism provided if for every individual $i$, there exists two distinct alternatives $x, y \in X$ such that $i$ is two-way decisive on $x$ and $y$: If $x >_i y$, then $C(\succ) \neq y$; and if $y >_i x$, then $C(\succ) \neq x$.

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**Sen’s Impossibility of the Paretian Liberal**: No social choice function satisfies both liberalism and the weakly unanimity conditions.


3 Muller-Satterthwaite Theorem

Axioms

- **Monotonicity**: $C$ is monotonic provided if for every preference profile $\succ \in \mathcal{L}^n$ such that $C(\succ) = x$, if $\succ'$ is another profile such that $x >'_i y$ whenever $x >_i y$ for every agent $i$ and alternative $y$, then $C(\succ') = x$.

- **Dictator**: A voter $i$ is a dictator in a social choice function $C$ if $C$ always selects $i$'s top choice: for every preference profile $\succ$, $C(\succ) = a$ iff for all $y \in X$ different from $x$, $x >_i y$.

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**Muller-Satterthwaite Theorem**: If $|X| \geq 3$, then any social choice function that is weakly unanimous and monotonic is also dictatorial.

Proof

- A set of voters $S$ is winning if, for any profile $\succ \in \mathcal{L}^n$ in which every $i \in S$ ranks some alternative $x$ on top of her preference, $C(\succ) = x$. 

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• A set of voters $S$ is **blocking** if there exists some profile $> \in \mathcal{L}^n$ such that $C(>) = x$ and $x$ is ranked on the top by all $i \in S$ and ranked at the bottom for all $i \in N - S$.

• **Theorem** (Tang and Sandholm). If $|X| \geq 3$, then for any social choice function satisfying weak unanimity and strong monotonicity, a coalition is winning iff it is blocking.
