Arrow’s Theorem
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Notation

- $X$ is a (finite or infinite) set of alternatives (or candidates).
- $N = \{1, \ldots, n\}$ is a set of voters
- Preferences: $\mathcal{P} = \{R \mid R \subseteq W \times W \text{ where } R \text{ is reflexive, transitive and connected}\}$.
- Given $R \in \mathcal{P}$, let $P$ be the strict preference generated by $R$: $xPy$ iff $xRy$ and not $yRx$ (we write $P_R$ if necessary).
- A profile is a tuple $(R_1, \ldots, R_n) \in \mathcal{P}^n$.
- Social Welfare Function: $F : \mathcal{D} \to \mathcal{P}$ where $\mathcal{D} \subseteq \mathcal{P}^n$ is the domain.

Axioms

- Universal Domain (UD): The domain of $F$ is $\mathcal{P}^n$ (i.e., $\mathcal{D} = \mathcal{P}^n$)
- Independence of Irrelevant Alternatives (IIA): $F$ satisfies IIA provide for all profiles $\vec{R}, \vec{R}' \in \mathcal{D}$, if $xR_iy$ iff $xR'_iy$ for all $i \in N$, then $xF(\vec{R})y$ iff $xF(\vec{R}')y$
- (weak) Pareto (P): For all profiles $\vec{R} \in \mathcal{D}$, if $xP_iy$ for all $i \in N$ then $xP_{F(\vec{R})}y$
- Agent $i$ is a dictator for $F$ provided for every preference profile and every pair $x, y \in X$, $xP_iy$ implies $xP_{F(\vec{R})}y$.

Arrows Impossibility Theorem: Suppose that $|X| \geq 3$ and $F$ satisfies UD, IIA and P. Then there is some $i \in N$ that is a dictator for $F$. 

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