PHIL 308S: Voting Theory and Fair Division

Lecture 13

Eric Pacuit

Department of Philosophy
University of Maryland, College Park
ai.stanford.edu/~epacuit
epacuit@umd.edu

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What does it mean to vote strategically?

- Voting as a game vs. voting as an act of communication

“...in many situations, differences of opinion arise from differences in values, not erroneous judgments. In this case it seems better to adopt the view that group choice is an exercise in finding a compromise between conflicting opinions.” (Young, p. 60)

Dodgson Method

1. The score for each candidate $A$ is the fewest number of swaps needed to make $A$ the Condorcet winner.
2. The candidate with the lowest score is declared the winner.
Dodgson Method

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2. The candidate with the lowest score is declared the winner.

Distance-Base Rationalization

- For some profiles, there is a clear winner (e.g., Condorcet winner, unanimous top choice, unanimous rankings, majority winner)
- If the profile $P$ is not a consensus profile, then find the closest consensus profile, according to some notion of distance
Kemeny Metric

Suppose that \( R \) and \( R' \) are two rankings

\[
d(R, R') = \text{number of pairs of alternatives on which they differ}
\]

Examples:

\[
d(a > b > c > d, d > a > b > c) = 3
\]

\[
d(a > b > c > d, c > d > a > b) = 4
\]
Kemeny Metric

**Kemeny metric**: Suppose that $R$ and $R'$ are two rankings

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Examples:
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d(a > b > c > d, d > a > b > c) = 3
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d(a > b > c > d, c > d > a > b) = 4
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Let $P = (P_1, \ldots, P_n)$ be a sequence of linear orders on $X$. If $x \in X$, let $U(x) = \{P \in P | x = \text{top}(P_i) \text{ for all } i\}$, then it is unanimous that $x$ should be the winner. $x$ is a relative unanimous winner provided the distance between $P$ and $U(x)$ is no larger than the distance between $P$ and $U(y)$ for all other alternatives $y$. 

Reaching Consensus
Reaching Consensus

Let \( P = (P_1, \ldots, P_n) \) be a sequence of linear orders on \( X \).

\[ x \in X, \text{ let } U(x) = \{ P \in \mathcal{P}^n \mid x = \text{top}(P_i) \text{ for all } i \} \]
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Reaching Consensus

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\( P \in U(x) \), then it is unanimous that \( x \) should be the winner.

\( x \) is a **relative unanimous winner** provided the *distance* between \( P \) and \( U(x) \) is no larger than the distance between \( P \) and \( U(y) \) for all other alternatives \( y \).
Distance
Distance

\[ d(P, Q) = \sum_{i=1}^{n} d(P_i, Q_i) \]
Distance

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\[ d(P, Y) = \min_{Q \in Y} d(P, Q) \]
Distance

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\[ U^*(x) = \{ P \in \mathcal{P}^n \mid d(P, U(x)) \leq d(P, U(y)) \text{ for all } x \in X \} \]
**Fact.** An alternative $x$ has the highest Borda score iff it is a relative unanimous winner.
\[ \delta_2(P_i, Q_i) = \begin{cases} 
0 & \text{if } \text{top}(P_i) = \text{top}(Q_i) \\
1 & \text{otherwise} 
\end{cases} \]

**Fact** An alternative is the plurality winner iff it is closest to the unanimous profile using the \( \delta_2 \) measure.
Distance-Based Judgement Aggregation


Given \((A_1, \ldots, A_n)\), select the set consistent and complete \(A\) that minimizes the total distance from the individual judgement sets: find \(A\) such that \(\sum_{i \in N} d(A, A_i)\) is minimized.

Hamming Metric:
\[ d(A', A) = \text{the number of propositions for which } A \text{ and } A' \text{ disagree} \]

\[ d_{\text{H}}(\{p, q, p \land q\}, \{p, \neg q, \neg (p \land q)\}) = 2 \]
Given \((A_1, \ldots, A_n)\), select the set consistent and complete \(A\) that minimizes the total distance from the individual judgement sets: find \(A\) such that \(\sum_{i \in \mathbb{N}} d(A, A_i)\) is minimized.

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\]

Duddy and Piggins: shouldn’t
\[
d(\{p, q, p \land q\}, \{p, \neg q, \neg (p \land q)\}) = 1?
\]
Duddy and Piggins Measure

Judgement set $C$ is between judgement sets $A$ and $B$ if $A$, $B$ and $C$ are distinct and, on each proposition $C$ agrees with $A$ or with $B$ (or both). ($C$ is a compromise between $A$ and $B$)
Duddy and Piggins Measure

Judgement set $C$ is between judgement sets $A$ and $B$ if $A$, $B$ and $C$ are distinct and, on each proposition $C$ agrees with $A$ or with $B$ (or both). ($C$ is a compromise between $A$ and $B$)

Draw a graph where the nodes are possible judgement sets and there is an edge between $A$ and $B$ provided there is no judgement set between them.

The distance between $A$ and $B$ is the length of the shortest path from $A$ to $B$. 
\neg p, q, \neg (p \land q)

p, q, (p \land q)

\neg p, \neg q, \neg (p \land q)

p, \neg q, \neg (p \land q)
Axioms

Axiom 1 \( d(A, B) = 0 \) iff \( A = B \)
Axiom 2 \( d(A, B) = d(B, A) \)
Axiom 3 \( d(A, B) \leq d(A, C) + d(C, B) \)
Axioms

**Axiom 1** \( d(A, B) = 0 \) iff \( A = B \)

**Axiom 2** \( d(A, B) = d(B, A) \)

**Axiom 3** \( d(A, B) \leq d(A, C) + d(C, B) \)

For all \( A, B, C \), \( C \) is between \( A \) and \( B \) provided \( A \neq B \neq C \) and \((A \cap B) \subset C\).
Axioms

**Axiom 1** \( d(A, B) = 0 \) iff \( A = B \)

**Axiom 2** \( d(A, B) = d(B, A) \)

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For all \( A, B, C \), \( C \) is between \( A \) and \( B \) provided \( A \neq B \neq C \) and \( (A \cap B) \subset C \).

**Axiom 4** If there is a judgement set between \( A \) and \( B \) then there exists \( C \) different from \( A \) and \( B \) such that \( d(A, B) = d(A, C) + d(C, B) \)
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**Axiom 4** If there is a judgement set between \( A \) and \( B \) then there exists \( C \) different from \( A \) and \( B \) such that
\[
d(A, B) = d(A, C) + d(C, B)
\]

**Axiom 5** If there is no judgement set between \( A \) and \( B \) with \( A \neq B \) then \( d(A, B) = 1 \)
Theorem (Duddy & Piggins) The previously defined metric is the unique metric satisfying Axioms 1 - 5.
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Differing on \( \{a, b \land c\} \) may be considered more consequential than differing on \( \{a, a \land b\} \).
Differing on \( \{ a, b \land c \} \) may be considered more consequential than differing on \( \{ a, a \land b \} \).

Let \( \mathcal{F} \) be the set of all judgement sets and \( \mathcal{F}^\circ \) the set of all consistent judgement sets.

\[
d : \mathcal{F} \times \mathcal{F} \to \mathbb{R}
\]

**Axiom 1** \( d(A, B) = 0 \) iff \( A = B \)

**Axiom 2** \( d(A, B) = d(B, A) \)

**Axiom 3** \( d(A, B) \leq d(A, C) + d(C, B) \)

\[
d(J, J') = \sum_{i \leq n} d(J_i, J'_i)
\]
For a profile $P$, $M(P) \in \mathcal{F}$ the judgement set resulting from majority rule. $P$ is majority consistent provided $M(P) \in \mathcal{F}^\circ$.

Fix a metric $d$ and a profile $J \in \mathcal{F}^\circ$. 

For a profile $P$, $M(P) \in \mathcal{F}$ the judgement set resulting from majority rule. $P$ is majority consistent provided $M(P) \in \mathcal{F}^\circ$

Fix a metric $d$ and a profile $J \in \mathcal{F}^\circ$

- $\text{Full}_d(J)$ is the collection of $M(J') \in \mathcal{F}^\circ$ such that $J'$ minimizes $d(J, J')$ over all majority consistent profiles $J'$ in $\mathcal{F}^\circ$
For a profile \( P \), \( M(P) \in \mathcal{F} \) the judgement set resulting from majority rule. \( P \) is majority consistent provided \( M(P) \in \mathcal{F}^\circ \)

Fix a metric \( d \) and a profile \( J \in \mathcal{F}^\circ \)

- \( \text{Full}_d(J) \) is the collection of \( M(J') \in \mathcal{F}^\circ \) such that \( J' \) minimizes \( d(J, J') \) over all majority consistent profiles \( J' \) in \( \mathcal{F}^\circ \)

- \( \text{Output}_d(J) \) is the collection of \( M(J') \in \mathcal{F}^\circ \) such that \( J' \) minimizes \( d(J, J') \) over all majority consistent profiles \( J' \) in \( \mathcal{F} \) (allowing inconsistencies)
For a profile $P$, $M(P) \in \mathcal{F}$ the judgement set resulting from majority rule. $P$ is majority consistent provided $M(P) \in \mathcal{F}^\circ$

Fix a metric $d$ and a profile $J \in \mathcal{F}^\circ$

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- $\text{Endpoint}_d(J)$ is the collection of $K \in \mathcal{F}^\circ$ that minimize $d(J, J')$ over all majority consistent profiles $J'$
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Fix a metric $d$ and a profile $J \in \mathcal{F}^\circ$

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- $\text{Output}_d(J)$ is the collection of $M(J') \in \mathcal{F}^\circ$ such that $J'$ minimizes $d(J, J')$ over all majority consistent profiles $J'$ in $\mathcal{F}$ (allowing inconsistencies)

- $\text{Endpoint}_d(J)$ is the collection of $K \in \mathcal{F}^\circ$ that minimize $d(J, J')$ over all majority consistent profiles $J'$

- $\text{Prototype}_d(J)$ is the collection of $K \in \mathcal{F}^\circ$ that minimize $\sum_{i \leq n} d(J_i, K)$ over all $K \in \mathcal{F}^\circ$
For $J, K$ let $\text{Ham}(J, K)$ denote the Hamming distance (the number of items on which $J$ and $K$ disagree)

$$d(J, K) = \begin{cases} 
0.9 & \text{if } J \text{ and } K \text{ disagree only on } a \land b \\
\sqrt{\text{Ham}(p, q)} & \text{otherwise}
\end{cases}$$
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$\text{Full}(J) = \text{TTT}(\text{d}(\text{FTF}, \text{FFF}) = 1)$

$\text{Output} \text{d}(J) = \text{TTT}(\text{d}(\text{TFF}, \text{TFT}) = 0)$

$\text{Endpoint} \text{d}(J) = \text{TTT}(\text{d}(\text{TTF}, \text{TTT}) = 0)$

$\text{Prototype} \text{d}(J) = \{\text{TTT}, \text{TFF}\}$
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$\text{Full}_d(J) = TFF \ (d(FTF, FFF) = 1)$
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  2 & T & T & T & T & T & T & T & T \\
  3 & T & F & F & T & F & F & T & F & T \\
  4 & T & F & F & T & F & F & T & F & F \\
  5 & F & T & F & F & F & F & F & T & F & T \\
  \hline
  M & T & T & F & T & F & F & T & T & T & T \\
\end{array}
\]

- \( \text{Full}_d(J) = TFF \) \( d(FTF, FFF) = 1 \)
- \( \text{Output}_d(J) = TTT \) \( d(TFF, TFT) = 0.9 \)
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4 & T & F & F \\
5 & F & T & F \\
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M & T & T & T \\
\end{array}
\]

- \( \text{Full}_d(J) = TFF \ (d(FTF, FFF) = 1) \)
- \( \text{Output}_d(J) = TTT \ (d(TFF, TFT) = 0.9) \)
- \( \text{Endpoint}_d(J) = TTT \ (d(TTF, TTT) = 0.9) \)
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- $\text{Full}_d(J) = TFF$ ($d(FTF, FFF) = 1$)
- $\text{Output}_d(J) = TTT$ ($d(TFF, TFT) = 0.9$)
- $\text{Endpoint}_d(J) = TTT$ ($d(TTF, TTT) = 0.9$)
- $\text{Prototype}_d(J) = \{TTT, TFF\}$ ($\sum_i d(J_i, TTT) = 3\sqrt{2}$, $\sum_i d(J_i, TFF) = 3\sqrt{2}$, $\sum_i d(J_i, FTF) = 4\sqrt{2}$, $\sum_i d(J_i, FFF) = 2\sqrt{3} + 3$)