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Assumes the existence of an absolute scale over which the utilities of individuals are measured and compared.
Let $U(A)$ be the set of real-valued “utility functions” defined over $A$, an *aggregation function* is

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Assumes the existence of an absolute scale over which the utilities of individuals are measured and compared.

At the other extreme, rule out any kind of cardinal information and interpersonal comparability, partitions $U(A)^n$ into cells which are *ordinally equivalent*:

$$F : W(A)^n \rightarrow \wp(A) - \{\emptyset\}$$

where $W(A)$ is the set of *weak orderings* on $A$. 
Approval Voting

\[ F : \mathcal{W}(A \cup \{0\})^n \to \wp(A) - \{\emptyset\} \]

where \(0\) separates the “good” and “bad” elements.
Approval Voting

\[ F : W(A \cup \{0\})^n \to \wp(A) - \{\emptyset\} \]

where 0 separates the “good” and “bad” elements.

“approval is not a strategic action but has an intrinsic meaning: It refers to the alternatives which are qualified as good.”
Combining Approval and Preference

Under Approval Voting (AV), voters are asked which candidates the voter approves.
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What about asking for *both* pieces of information?

Assumptions

Assume each voter has a (linear) preference over the candidates.

Each voter is asked to rank the candidates from most preferred to least preferred (ties are not allowed).

Voters are then asked to specify which candidates are acceptable.

**Consistency Assumption** Given two candidates $a$ and $b$, if $a$ is approved and $b$ is disapproved then $a$ is ranked higher than $b$.

For example, we denote this approval ranking for a set $\{a, b, c, d\}$ of candidates as follows

$$a \ d \ | \ c \ b$$
Preference Approval Voting (PAV)

1. If no candidate, or exactly one candidate, receives a majority of approval votes, the PAV winner is the AV winner.
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2. If two or more candidates receive a majority of approval votes, then
   2.1 If one of these candidates is preferred by a majority to every other majority approved candidate, then he or she is the PAV winner.
   
   2.2 If there is not one majority-preferred candidate because of a cycle among the majority-approved candidates, then the AV winner among them is the PAV winner.
Rule 1

I. 1 voter: \( a b \mid c \)

II. 1 voter: \( b \mid a c \)

III. 1 voter: \( c \mid a b \)
PAV vs. Condorcet

Rule 1

I. 1 voter: \( a \ b \mid c \)

II. 1 voter: \( b \mid a \ c \)

III. 1 voter: \( c \mid a \ b \)

\( b \) is the AV winner.
PAV vs. Condorcet

Rule 1

I. 1 voter: $a$ $b$ | $c$
II. 1 voter: $b$ | $a$ $c$
III. 1 voter: $c$ | $a$ $b$

$b$ is the AV winner.
$b$ is also the PAV winner.
PAV vs. Condorcet

Rule 1

I. 1 voter: \( a \ b \ | \ c \)
II. 1 voter: \( b \ | \ a \ c \)
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\( b \) is the AV winner.
\( b \) is also the PAV winner.
\( a \) is the Condorcet winner.
PAV vs. Condorcet

Rule 2(a)

I. 1 voter: \( a \ b \ c \ | \ d \)

II. 1 voter: \( b \ c \ | \ a \ d \)

III. 1 voter: \( d \ | \ a \ c \ b \)
PAV vs. Condorcet

Rule 2(a)

I. 1 voter: \( a \ b \ c \ | \ d \)
II. 1 voter: \( b \ c \ | \ a \ d \)
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\( b \) is the PAV winner.
PAV vs. Condorcet

Rule 2(a)

I. 1 voter: \( a \ b \ c \ | \ d \)

II. 1 voter: \( b \ c \ | \ a \ d \)

III. 1 voter: \( d \ | \ a \ c \ b \)

\( b \) is the PAV winner.
\( a \) is the Condorcet winner.
PAV vs. Condorcet

Rule 2(b)

I. 1 voter: \( d \ a \ b \ c \ | \ e \)
II. 1 voter: \( d \ b \ c \ a \ | \ e \)
III. 1 voter: \( e \ | \ d \ c \ a \ b \)
IV. 1 voter: \( a \ b \ c \ | \ d \ e \)
V. 1 voter: \( c \ | \ b \ a \ d \ e \)
PAV vs. Condorcet

Rule 2(b)

I. 1 voter: $d \ a \ b \ c \ | \ e$
II. 1 voter: $d \ b \ c \ a \ | \ e$
III. 1 voter: $e \ | \ d \ c \ a \ b$
IV. 1 voter: $a \ b \ c \ | \ d \ e$
V. 1 voter: $c \ | \ b \ a \ d \ e$

$a$ (3 votes), $b$ (3 votes), and $c$ (4 votes) are all majority approved.
PAV vs. Condorcet

Rule 2(b)

I. 1 voter: \( d \ a \ b \ c \mid e \)

II. 1 voter: \( d \ b \ c \ a \mid e \)

III. 1 voter: \( e \mid d \ c \ a \ b \)

IV. 1 voter: \( a \ b \ c \mid d \ e \)

V. 1 voter: \( c \mid b \ a \ d \ e \)

\( a \) (3 votes), \( b \) (3 votes), and \( c \) (4 votes) are all majority approved. \( c \) is the PAV winner.
PAV vs. Condorcet

Rule 2(b)

I. 1 voter: \(\; d \; a \; b \; c \; | \; e \)

II. 1 voter: \(\; d \; b \; c \; a \; | \; e \)

III. 1 voter: \(\; e \; | \; d \; c \; a \; b \)

IV. 1 voter: \(\; a \; b \; c \; | \; d \; e \)

V. 1 voter: \(\; c \; | \; b \; a \; d \; e \)

\(a\) (3 votes), \(b\) (3 votes), and \(c\) (4 votes) are all majority approved. \(c\) is the PAV winner. \(d\) is the Condorcet winner.
Example

I. 3 voters: $a \ b \ c \ | \ d$

II. 3 voters: $d \ a \ c \ | \ b$

III. 2 voters: $b \ d \ c \ | \ a$
Example

I. 3 voters: \(a\ b\ c\ |\ d\)

II. 3 voters: \(d\ a\ c\ |\ b\)

III. 2 voters: \(b\ d\ c\ |\ a\)

\(c\) is approved by all 8 voters.
Example

I. 3 voters: $a \ b \ c \ | \ d$
II. 3 voters: $d \ a \ c \ | \ b$
III. 2 voters: $b \ d \ c \ | \ a$

c is approved by all 8 voters.
There is a top cycle $a > b > d > a$ which are all preferred by majorities to $c$ (the AV winner).
a is the PAV winner
Example

I. 3 voters: $a b c | d$

II. 3 voters: $d a c | b$

III. 2 voters: $b d c | a$

$a$ is the PAV winner.
$c$ is the AV winner.
$d$ is the STV winner.
Example

I. 2 voters:  \[ a \ c \ b | d \]

II. 2 voters:  \[ a \ c \ d | b \]

III. 3 voters:  \[ b \ c \ d | a \]
Example

I. 2 voters: \(a\ c\ b\ \mid\ d\)
II. 2 voters: \(a\ c\ d\ \mid\ b\)
III. 3 voters: \(b\ c\ d\ \mid\ a\)

c is approved by all 7 voters.
a is the least approved candidate.
a is the PAV winner.
\(BC(a) = 12\)
\(BC(c) = 14\)
Fallback Voting (FV)

1. Voters indicate all candidates of whom they approve, who may range from no candidate to all candidate. Voters rank only those candidates whom they approve.

2. The highest-ranked candidate of all voters is considered. If a majority agree on the highest-ranked candidate, this candidate is the FV winner (level 1).
Fallback Voting (FV)

1. If there is no level 1 winner, the next-highest ranked candidate of all voters in considered. If a majority of voters agree on one candidate as either their highest or their next-highest ranked candidate, this candidate is the FV winner (level 2). If more than one receive majority approval, then the candidate with the largest majority is the FV winner.

2. If no level 2 winner, the voters descend — one level at a time — to lower ranks of approved candidates stopping when one or more candidates receives majority approval. If more than one receives majority approval then the candidate with the largest majority is the FV winner. If the descent reaches the bottom and no candidate has won, then the candidate with the most approval is the FV winner.
Example

I. 4 voters:  $a \ b \ c \ | \ d$

II. 3 voters:  $b \ c \ | \ a \ d$

III. 2 voters:  $d \ a \ c \ | \ b$
Example

I. 4 voters: \( a b c \mid d \)

II. 3 voters: \( b c \mid a d \)

III. 2 voters: \( d a c \mid b \)

\( b \) is the FV winner.
\( c \) is the AV winner.
\( a \) is the PAV winner.
Neither PAV nor FV may elect the Condorcet winner.

Both PAV and FV are monotonic (approval-monotonic and rank-monotonic)

Truth-telling strategies of voters under PAV and FV may not be in equilibrium
\[ q_i(x; R) = G \text{ iff } xP_i0 \]
\[ q_i(x; R) = G \text{ iff } xP_i0 \]

\[ n^G(x; R) = |\{i \in N \mid q_i(x; R) = G\}| \]
\[ q_i(x; R) = G \text{ iff } xP_i0 \]

\[ n^G(x; R) = |\{ i \in N \mid q_i(x; R) = G \}| \]

\[ \gamma(R) = \{ x \in A \mid n^G(x; R) \geq n/2 \} \]
\[ q_i(x; R) = G \text{ iff } xP_i0 \]

\[ n^G(x; R) = |\{i \in N \mid q_i(x; R) = G\}| \]

\[ \gamma(R) = \{x \in A \mid n^G(x; R) \geq n/2\} \]

\[ f : W(A \cup \{0\}^n \rightarrow \varnothing(A) - \emptyset \text{ satisfies majoritarian approval} \text{ iff we have } f(R) \subseteq \gamma(R) \text{ for every } R \in W(A \cup \{0\}^n \text{ where } \gamma(R) \neq \emptyset \]
\[ f : W(A \cup \{0\})^n \rightarrow \mathcal{P}(A) - \emptyset \] satisfies approval independence iff we have \[ f(R) = f(R') \] for every \( R, R' \in W(A \cup \{0\})^n \) where \( xR_iy \) iff \( xR'_iy \) for all \( x, y \in A \) and \( i \in N \).
\[ f : W(A \cup \{0\})^n \rightarrow 2^A - \emptyset \text{ satisfies approval independence iff we have } f(R) = f(R') \text{ for every } R, R' \in W(A \cup \{0\})^n \text{ where } xR_i y \text{ iff } xR'_i y \text{ for all } x, y \in A \text{ and } i \in N. \]

**Fact.** Majoritarian approval and approval independence are logically incompatible.
Two Questions

1. How to refine the set of socially good alternatives, when this set contains more than one alternative?
2. Which alternative to choose when none of them is socially good?
Majoritarian Compromise

1. The highest-ranked candidate of all voters is considered. If a majority of voters agree on one highest-ranked candidate, this candidate is the MC winner.

2. If there is no level 1 winner, the next-highest ranked candidate of all voters is considered. If a majority of voters agree on one candidate as either their highest or their next-highest ranked candidate, this candidate is the MC winner. If more than one candidate receives a majority support, then the candidate with highest support is the MC winner. The procedure stops.

3. If there is no level 2 winner, the voters descend one level at a time to lower and lower ranks, stopping when, for the first time, one or more candidates receive a majority support. If more than one candidate receives a majority support, then the candidate with the highest majority support is the MC winner.
Majoritarian Approval Compromise

3 voters $a \mid b\ c\ d$
2 voters $b\ a\ c \mid d$
2 voters $c \mid a\ b\ d$
2 voters $d\ b\ c \mid a$
Majoritarian Approval Compromise

3 voters \( a \mid b \ c \ d \)
2 voters \( b \ a \ c \mid d \)
2 voters \( c \mid a \ b \ d \)
2 voters \( d \ b \ c \mid a \)

\( a \) is the MAC winner.
Approval Voting with Runoff

Given $R \in W(A \cup \{0\})^n$, let $\rho(R) = \{x, y\}$ be the pair of alternatives which receive the highest approval. AVR picks the pairwise majority winner among the runoff winners.
$R'$ is a lifting of $x$ with respect to $R$ provided for all $i \in N$, 
$xR_iy$ implies $xR_i'y$ for all $y \in A$

$XP_iy$ implies $XP_i'y$ for all $y \in A$

$XP_i0$ implies $XP_i'0$

and $yP_iz$ iff $yP_i'z$ for all $y, z \in A - \{x\} \cup \{0\}$. 
$R'$ is a lifting of $x$ with respect to $R$ provided for all $i \in N$,
$xR_iy$ implies $xR'_iy$ for all $y \in A$
$xP_iy$ implies $xP'_iy$ for all $y \in A$
$xP_i0$ implies $xP'_i0$
and $yP_iz$ iff $yP'_iz$ for all $y, z \in A - \{x\} \cup \{0\}$.

$f$ is monotonic iff $x \in f(R)$ implies $x \in f(R')$ for all $R'$ that are
liftings of $x$ with respect to $R$. 
$R'$ is a lifting of $x$ with respect to $R$ provided for all $i \in N$,

$XR_iy$ implies $XR'_iy$ for all $y \in A$

$XP_iy$ implies $XP'_iy$ for all $y \in A$

$XP_i0$ implies $XP'_i0$

and $yP_iz$ iff $yP'_iz$ for all $y, z \in A - \{x\} \cup \{0\}$.

$f$ is monotonic iff $x \in f(R)$ implies $x \in f(R')$ for all $R'$ that are

liftings of $x$ with respect to $R$.

**Theorem.** Approval Voting, Majoritarian Approval Compromise, Preference- Approval Voting and Approval Voting with a runoff are all monotonic.
\[ B = A \cup \{x^*\}. \]

We say that \( R \in W(A \cup \{0\})^n \) and \( R' \in W(B \cup \{0\})^n \) agree provided for all \( x, y \in A \), \( xR_i y \) iff \( xR'_i y \) and \( xR_i 0 \) iff \( xR'_i 0 \).
\[ B = A \cup \{ x^* \}. \]

We say that \( R \in W(A \cup \{ 0 \})^n \) and \( R' \in W(B \cup \{ 0 \})^n \) agree provided for all \( x, y \in A \), \( xR_i y \) iff \( xR'_i y \) and \( xR_i 0 \) iff \( xR'_i 0 \).

We call \( x^* \) a spoiler if its presence can change the social choice without \( x^* \) being chosen.

\[ x^* \not\in f(R') \neq f(R) \] at some \( R \in W(A \cup \{ 0 \}) \) that agrees with \( R' \in W(B \cup \{ 0 \}) \)
\[ B = A \cup \{ x^* \}. \]

We say that \( R \in W(A \cup \{0\})^n \) and \( R' \in W(B \cup \{0\})^n \) agree provided for all \( x, y \in A \), \( xR_i y \) iff \( xR'_i y \) and \( xR_i 0 \) iff \( xR'_i 0 \).

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\[ x^* \notin f(R') \neq f(R) \] at some \( R \in W(A \cup \{0\}) \) that agrees with \( R' \in W(B \cup \{0\}) \).

\( f \) is independent if it does not admit a spoiler.
\[ B = A \cup \{ x^* \}. \]

We say that \( R \in W(A \cup \{ 0 \})^n \) and \( R' \in W(B \cup \{ 0 \})^n \) agree provided for all \( x, y \in A \), \( xR_i y \) iff \( xR'_i y \) and \( xR_i 0 \) iff \( xR'_i 0 \).

We call \( x^* \) a spoiler if its presence can change the social choice without \( x^* \) being chosen.

\[ x^* \notin f(R') \neq f(R) \at \text{at some } R \in W(A \cup \{ 0 \}) \text{ that agrees with } R' \in W(B \cup \{ 0 \}) \]

\( f \) is independent if it does not admit a spoiler.

**Theorem.** Majoritarian Approval Compromise, Preference-Approval Voting and Approval Voting with a runoff fail independence.