Dynamic Epistemic Logic II:
Logics of Information Change

June 8, 2012

Abstract

Dynamic epistemic logic, broadly conceived, is the study of logics of *information change*. Inference, communication and observation are typical examples of informative events which have been subjected to a logical analysis. This article will introduce and critically examine a number of different logical systems that have been used to reason about the knowledge and beliefs of a group of agents during the course of a social interaction or rational inquiry. The goal here is not to be comprehensive, but rather to discuss the key conceptual and technical issues that drive much of the research in this area.

1 Modeling Informative Events

The logical frameworks introduced in the first paper all describe the (rational) agents' knowledge and belief at a fixed moment in time. This is only the beginning of a general logical analysis of rational inquiry and social interaction. A comprehensive logical framework must also describe how a rational agent's knowledge and belief *change* over time. The general point is that *how* the agent(s) come to know or believe that some proposition p is true is as important (or, perhaps, more important) than the fact that the agent(s) knows or believes that p is the case (cf. the discussion in van Benthem, 2009, Section 2.5). In this article, I will introduce various *dynamic* extensions of the static logics of knowledge and belief introduced earlier. This is a well-developed research area attempting to balance sophisticated logical analysis with philosophical insight — see van Ditmarsch et al. (2007) and van Benthem (2010) for textbook presentations of this rapidly developing area.

A concrete example of the type of situation that motivates much of the work on dynamic epistemic logics is found in the following clever cartoon:

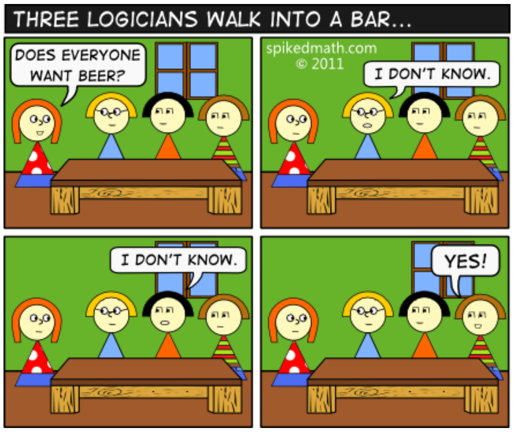


Figure 1: Three logicians walk into a bar...

Initially, none of the logicians have enough information to truthfully answer the bartender either 'yes, we all want a beer' or 'no, not all of us want a beer'. After repeated announcements of this fact (something which is commonly known among the logicians), the third logician can deduce that everyone must want a beer. This cartoon highlights the important interplay between inference, observation and communication for a general logic of information flow. A formal analysis of this cartoon using ideas presented in this paper can be found in Appendix A.

Suppose that Ann knows (in the sense discussed in Section 2 of the first paper) that both φ and $\varphi \rightarrow \psi$ are true, i.e., $K_a\varphi \wedge K_a(\varphi \rightarrow \psi)$ is true at a state w. Then, at state w, it must be the case that Ann also knows that ψ is true (i.e., $K_a\psi$ is must be true at state w). Whether Ann actually realizes that ψ (or for that matter either φ or $\varphi \rightarrow \psi$) is true or how she arrived at this knowledge is not specified in a standard epistemic model. It is only assumed that Ann *found out that* ψ is true in some way or the other. There are many ways that Ann could have come to this conclusion: For example, she may have deduced ψ from φ and $\varphi \rightarrow \psi$, directly observed ψ, or been told that ψ is true from a trusted source. In the static logics of knowledge and belief introduced in the first paper, this epistemic activity is not explicitly represented. The logical frameworks introduced in this article rectify this situation by including machinery to explicitly describe these epistemic actions.

Rational agents engage in a wide range of cognitive activities as they interact with their environment and each other. In this paper, I focus on the logical properties of one such activity: *finding out that* φ *is true*. The key idea is to formalize this epistemic action as an operation that transforms an epistemic (-plausibility/-probability) model. The general picture to keep in mind is:

$$\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\beta_i\}_{i\in\mathcal{A}}, V\rangle$$

$$\Big\Vert$$

$$\textbf{Find out that } \varphi$$

$$\Big\Downarrow$$

$$\mathcal{M}' = \langle W', \{\sim_i'\}_{i\in\mathcal{A}}, \{\beta_i'\}_{i\in\mathcal{A}}, V|_{W'}\rangle$$

where the β_i are either plausibility orderings ($\preceq_i$) or probability measures (π_i) and $V|_{W'}$ is the restriction of V to the new set of states W'. As noted above there are many ways in which a group of agents can find out that φ is true. Correspondingly, there are various ways to transform an epistemic (-plausibility/-probability) model. The model transformations that I introduce below focus on two key features of the change induced in the relevant agents' informational attitudes when a group of agents collectively find out that φ is true:

1. The *type* of change triggered by the learning event. Agents may differ in precisely how they incorporate new information into their epistemic states. These differences are based, in part, on the agents' perception of the *source* of the information. For example, an agent may consider a particular source of information *infallible* (not allowing for the possibility of error) or merely *trustworthy* (accepting the information as reliable, though allowing for the possibility of a mistake).

2. The agents' *observational* powers. There are two issues here. First, agents may perceive the same event differently, and this can be described in terms of what agents do or do not observe. Examples range from *public announcements*, where everyone witnesses the same event, to private communications between two or more agents, with no other agents aware that an event took place. Second, it is often (implicitly) assumed that not only do a group of agents find out that φ is true, but they also find out the fact that everyone in the group just realized that φ is true. Thus, it is important to track both changes in what the agents know and believe about the proposition that is learned and also what the agent know and believe about each other.

This last point is related to a more general methodological issue that is important to highlight at this stage. Many of the recent developments in this area have been driven by analyzing *concrete* examples. These range from toy examples, such as the infamous muddy children puzzle or the three logicians cartoon discussed in the introduction, to philosophical quandaries, such as Fitch's Paradox, to everyday examples of social interaction. Different logical systems are then judged, in part, on how well they conform to the analyst's intuitions about the relevant set of examples. But this raises an important methodological issue: Implicit assumptions about what the actors know and believe about the situation being modeled often guide the analyst's intuitions. In many cases, it is crucial to make these underlying assumptions explicit (cf. the discussion in Stalnaker, 2009, Section 4). The main point is that informal analyses of the logical systems introduced below often rely on assumptions about the agents' "meta-information", such as how "trusted" or "reliable" the sources of the information are. This is particularly important when analyzing how an agent's beliefs change over an extended period of time. For example, rather than taking a stream of contradictory incoming evidence (i.e., the agent receives the information that p, then the information that q, then the information that $\neg p$, then the information that $\neg q$) at face value (and performing the suggested belief revisions), a rational agent may consider the stream itself as evidence that the source is not reliable[1]. There is much more to say about logical frameworks that incorporate notions of trust and reliability, but, in this paper, these issues do not play a central role.

2 Finding Out That φ is True

At a fixed moment in time, the agents are in some *epistemic state* (which is described by an epistemic -plausibility/-probability model). The question addressed in this section is: How does (the model of) this epistemic state change after a group of agents find out that φ is true? As is well known from the belief revision literature, there are many ways to incorporate new information into a plausibility ordering (Rott, 2006). I do not have the space to survey this entire literature here (see van Benthem, 2010; Baltag and Smets, 2009, for modern introductions). The different ways of transforming an epistemic (-plausibility/-probability) model can be categorized in terms of the opinions that the agent(s) have about the *source(s)* of information. For example, an agent will have an opinion about how trustworthy or reliable the source is, whether the source is trying to deceive her, and so on.

The simplest type of informational change treats the source of the information as *infallible*. The effect of finding out that φ is true from an infallible source should be clear: *Remove* all states that do not satisfy φ. In the epistemic logic literature this operation is called a *public announcement* (Plaza, 1989; Gerbrandy, 1999). However, calling this an "announcement" is misleading since audience

[1]Cf. the very interesting discussion of *higher-order evidence* in the (formal) epistemology literature (Christensen, 2010)

members typically do not assume that speakers are infallible. A better terminology might be "public observation", though one can still have skeptical worries about the fallibility of perception. I will stick with the first terminology to be consistent with the literature. In any case, the epistemic event that is being modeled is one where φ is made publicly available and not only do all the agents *take it for granted* that φ is true, but they all take it for granted that all the agents take it for granted that φ is true, and so on *ad infinitum*. The formal definition is:

Definition 2.1 (Public Announcement) Suppose that $\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, V\rangle$ is an epistemic model and φ is a formula (in the language $\mathcal{L}_K$). After all the agents find out that φ is true (i.e., φ is **publicly announced**), the resulting model is $\mathcal{M}^{!\varphi} = \langle W^{!\varphi}, \{\sim_i^{!\varphi}\}, V^{!\varphi}\rangle$ where $W^{!\varphi} = \{w \in W \mid \mathcal{M}, w \models \varphi\}$, $\sim_i^{!\varphi} = \sim_i \cap W^{!\varphi} \times W^{!\varphi}$ for all $i \in \mathcal{A}$, and for all $p \in \mathsf{At}$, $V^{!\varphi}(p) = V(p) \cap W^{!\varphi}$. $\triangleleft$

The same definition applies *mutatis mutandi* to epistemic-plausibility and epistemic-probability[2] models. The models $\mathcal{M}$ and $\mathcal{M}^{\varphi}$ describe two different moments in time, with $\mathcal{M}$ describing the current or initial information state of the agents and $\mathcal{M}^{\varphi}$ the information state *after* the all the agents find out that φ is true. This temporal dimension can also be represented in the logical language with modalities of the form $[!\varphi]\psi$. The intended interpretation of $[!\varphi]\psi$ is "ψ is true after all the agents find out that φ is true", and truth is defined as

- $\mathcal{M}, w \models [!\varphi]\psi$ iff if $\mathcal{M}, w \models \varphi$ then $\mathcal{M}^{!\varphi}, w \models \psi$.

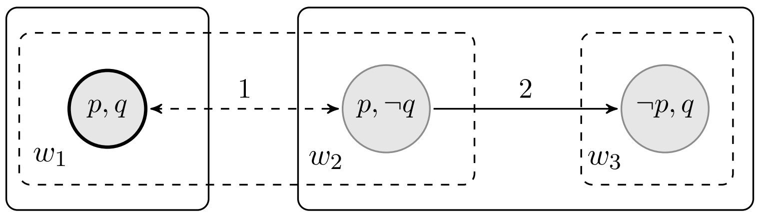
Consider the formula $\neg K_i\psi \wedge [!\varphi]K_i\psi$: This says that "the agent (currently) does not know ψ, but after the finding out that φ is true, the agent knows that ψ". So, languages with these announcement modalities can describe what is true both before and after the announcement. A fundamental insight is that there is a strong logical relationship between what is true before and after an announcement in the form of so-called *recursion axioms* (for the language $\mathcal{L}_{KB}$ over the class of all epistemic-plausibility models):

$$
\begin{array}{lcl}
[!\varphi]p & \leftrightarrow & \varphi \rightarrow p, \quad \text{where } p \in \mathsf{At} \\
[!\varphi]\neg\psi & \leftrightarrow & \varphi \rightarrow \neg[!\varphi]\psi \\
[!\varphi](\psi \wedge \chi) & \leftrightarrow & [!\varphi]\psi \wedge [!\varphi]\chi \\
[!\varphi]K_i\psi & \leftrightarrow & \varphi \rightarrow K_i(\varphi \rightarrow [!\varphi]\psi) \\
[!\varphi]B_i\psi & \leftrightarrow & B_i^{\varphi}[!\varphi]\psi \\
[!\varphi]B_i^{\psi}\chi & \leftrightarrow & \varphi \rightarrow B_i^{\varphi\wedge[!\varphi]\psi}[!\varphi]\chi
\end{array}
$$

[2] Of course, the probability measures need to be renormalized.

These recursion axioms[3] provide an insightful syntactic analysis of announcements that complements the semantic analysis: The recursion axioms describe the effect of an announcement in terms of what is true before the announcement.

It is important to clarify the relationship between conditional belief $B^{\varphi}\psi$ and beliefs after a public announcement $[!\varphi]B\psi$. *Prima facie*, the two statements seem to express the same thing; and, in fact, they are equivalent provided that ψ is a *true ground formula* (i.e., does not contain any modal operators). However, the formulas are not equivalent in general as the following example illustrates:



In this model, the solid lines represent agent 2's hard and soft information (the box is 2's hard information $\sim_2$ and the arrow represent 2's soft information $\preceq_2$) while the dashed lines represent 1's hard and soft information. (Reflexive arrows are not drawn to keep down the clutter in the picture.) Note that at state w_1, agent 2 *knows* p and q (eg., $w_1 \models K_2(p \wedge q)$), and agent 1 believes p but not q ($w_1 \models B_1 p \wedge \neg B_1 q$). Now, although agent 1 does not *know* that agent 2 knows q, agent 1 does believe that agent 2 believes q ($w_1 \models B_1 B_2 q$). Furthermore, agent 1 maintains this belief *conditional on* p: $w_1 \models B_1^p B_2 q$. However, after the agents found out that p is true, state w_3 is removed, and so we have $w_1 \models [!p]\neg B_1 B_2 q$. Thus a belief in ψ conditional on φ is *not* the same as a belief in ψ *after* the public announcement of φ. This point is worth reiterating: the reader is invited to check that $B_i^p(p \wedge \neg K_i p)$ is satisfiable but $[!p]B_i(p \wedge \neg K_i p)$ is not satisfiable.[4]

2.1 Finding Out From a Fallible Source

A public announcement is only one type of informative action. For the other transformations discussed in this paper, while the agents do *trust* the source of φ, they do not treat the source as infallible. Perhaps the most ubiquitous policy

[3]There are also recursion axioms for the other notions of belief (robust, strong and probabilistic belief) discussed above, but I do not discuss them here (see van Benthem, 2010, for a discussion).

[4]The situation is nicely summarized as follows: "$B^{\psi}\varphi$ says that if the agent would learn φ, then she would come to believe that ψ was the case (before the learning)...$[!\varphi]B\psi$ says that after learning φ, the agent would come to believe that ψ is the case (in the worlds after the learning)." (Baltag and Smets, 2008b, pg. 2). So, the conditional beliefs *encode* how the agent beliefs will change in the presence of new information.

is *conservative upgrade* ($\uparrow\varphi$), which lets the agent only tentatively accept the incoming information φ by making the best φ-worlds the new minimal set and keeping the old plausibility ordering the same on all other worlds. A second, stronger, operation is *radical upgrade* ($\Uparrow\varphi$) which moves *all* φ worlds before all the $\neg\varphi$ worlds and otherwise keeps the plausibility ordering the same. Before giving the formal definition we need some notation: Given an epistemic-doxastic model $\mathcal{M}$, let $[\![\varphi]\!]_i^w = \{x \mid \mathcal{M}, x \models \varphi\} \cap [w]_i$ denote the set of all φ-worlds that i considers possible and $best_i(\varphi, w) = Min_{\preceq_i}([w]_i \cap \{x \mid \mathcal{M}, x \models \varphi\})$ the best φ-worlds at state w according to agent i.

Definition 2.2 (Conservative and Radical Upgrade) Given an epistemic-plausibility model $\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\preceq_i\}_{i\in\mathcal{A}}, V\rangle$ and a formula $\varphi \in \mathcal{L}_{KB}$, the conservative/radical upgrade of $\mathcal{M}$ with φ is the model $\mathcal{M}^{*\varphi} = \langle W^{*\varphi}, \{\sim_i^{*\varphi}\}_{i\in\mathcal{A}}, \{\preceq_i^{*\varphi}\}_{i\in\mathcal{A}}, V^{*\varphi}\rangle$ with $W^{*\varphi} = W$, for each i, $\sim_i^{*\varphi} = \sim_i$, $V^{*\varphi} = V$ where $* = \uparrow, \Uparrow$. The relations $\preceq_i^{\uparrow\varphi}$ and $\preceq_i^{\Uparrow\varphi}$ are the smallest relations satisfying:

Conservative Upgrade

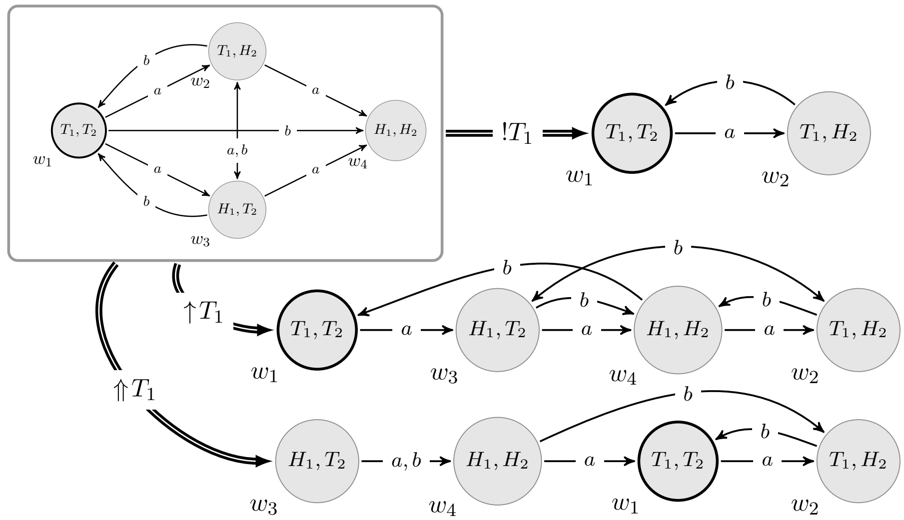
1. If $x \in best_i(\varphi, w)$ and $y \in [w]_i$, then $x \preceq_i^{\uparrow\varphi} y$, and
2. for all $x, y \in [w]_i - best_i(\varphi, w)$, $x \preceq_i^{\uparrow\varphi} y$ iff $x \preceq_i y$.

Radical Upgrade

1. for all $x \in [\![\varphi]\!]_i^w$ and $y \in [\![\neg\varphi]\!]_i^w$, set $x \prec_i^{\Uparrow\varphi} y$,
2. for all $x, y \in [\![\varphi]\!]_i^w$, set $x \preceq_i^{\Uparrow\varphi} y$ iff $x \preceq_i y$, and
3. for all $x, y \in [\![\neg\varphi]\!]_i^w$, set $x \preceq_i^{\Uparrow\varphi} y$ iff $x \preceq_i y$. $\lhd$

As the reader is invited to check, a conservative upgrade is a special case of a radical upgrade: the conservative upgrade of φ at w is the radical upgrade of $best_i(\varphi, w)$.[5] An example will help clarify the differences between the three transformations introduced above. Recall the running example of an epistemic-plausibility from the first paper. In this model, both Ann and Bob believe that the coin in the first drawer is laying heads up, but their conditional beliefs are different. Now, suppose that Ann and Bob find out that φ is true. I have introduced three different ways to incorporate this into an epistemic-plausibility model, and they are illustrated below:

[5] Of course, $best_i(\varphi, w)$ may not be expressible in the formal language.



Naturally, after learning that T_1 is true, both Ann and Bob believe that the coin in the first drawer is laying tails up. There are two features of these operations that I want to point out. First, note that Ann and Bob have different beliefs about the status of the coin in the second drawer: In each of the updated models, Ann believes that H_2 is true while Bob believes that T_2 is true. This makes sense given the underlying assumption that Bob believes the coins in the two drawers are correlated in some way, while Ann is under the impression that the position of the coins are independent. Second, while the agents' beliefs are the same in each of the updated models, their conditional beliefs are different. For example, Ann believes that $T_1 \wedge H_2$ is true in all three models. However, under the supposition that the coin in the second drawer is laying tails up (i.e., T_2 is true), Ann believes that H_1 is true in the second model (after the conservative upgrade with T_1) and T_1 is true in the third model (after the radical upgrade with T_1).

A logical analysis of these operations include formulas of the from $[\uparrow\varphi]\psi$ intended to mean "after everyone conservatively upgrades with φ, ψ is true" and $[\Uparrow\varphi]\psi$ intended to mean "after everyone radically upgrades with φ, ψ is true" . The definition of truth for these formula is as expected:

- $\mathcal{M}, w \models [\uparrow\varphi]\psi$ iff $\mathcal{M}^{\uparrow\varphi}, w \models \psi$
- $\mathcal{M}, w \models [\Uparrow\varphi]\psi$ iff $\mathcal{M}^{\Uparrow\varphi}, w \models \psi$

Note that unlike with public announcements, there is no precondition for these operations. Recursion axioms are also available for these dynamic operators: The dynamic modalities commute with the Boolean connectives and the knowledge modality, so I only give the key axioms involving conditional belief (note that I leave out the i subscripts to make the formulas easier to read):

$$[\uparrow\varphi]B^{\psi}\chi \quad\leftrightarrow\quad (B^{\varphi}\neg[\uparrow\varphi]\psi \wedge B^{[\uparrow\varphi]\psi}[\uparrow\varphi]\chi) \vee (\neg B^{\varphi}\neg[\uparrow\varphi]\psi \wedge B^{\varphi\wedge[\uparrow\varphi]\psi}[\uparrow\varphi]\chi)$$
$$[\Uparrow\varphi]B^{\psi}\chi \quad\leftrightarrow\quad (L(\varphi \wedge [\Uparrow\varphi]\psi) \wedge B^{\varphi\wedge[\Uparrow\varphi]\psi}[\Uparrow\varphi]\chi) \vee (\neg L(\varphi \wedge [\Uparrow\varphi]\psi) \wedge B^{[\Uparrow\varphi]\psi}[\Uparrow\varphi]\chi)$$

I conclude this section with a few comments on the recursion axiom methodology. The recursion axioms *reduce* the complexity of a given formula in the sense that that going from left to right either the number of dynamic modalities is reduced or the complexity of the formulas within the scope of dynamic modalities is reduced. This gives a method for proving completeness of logics with dynamic modalities. For example, public announcement logic consists of the standard axiomatization of epistemic logic plus the recursion axioms given above. It is easy to see that this is a complete axiom systems: Given a formula containing an announcement operator, one can completely eliminate the announcement by repeatedly applying the recursion axioms. In this way one produces a formula of epistemic logic. That is, given any formula φ of public announcement logic, repeated applications of the recursion axioms gives a *provably* equivalent formula φ' in the language of epistemic logic. In particular, this shows that announcement modalities do not add any expressive power to the basic epistemic language (similarly for the language $\mathcal{L}_{KB}$ and the conservative/radical upgrade operators). However, Lutz (2006) has shown that in the case of public announcement logic at least, the language is more *succinct* than the language of epistemic logic (there is a formula scheme in the epistemic language with the public announcement operator such that every equivalent formula scheme in the language of epistemic logic is exponentially longer). This suggests that languages with public announcement operators describe the epistemic action at an appropriate level of abstraction.

When a logical language becomes strictly more expressive by adding dynamic operators, recursion axioms are not available. Adding public announcement operators to epistemic logic with common knowledge is such a case. It was shown by Baltag et al. (1998) than the language of epistemic logic with common knowledge and public announcements is more expressive than epistemic logic with common knowledge. Therefore, a reduction axiom for formulas of the form $[!\varphi]C_G\psi$ does not exist. Nonetheless, a reduction axiom-style analysis is still possible (van Benthem et al., 2006). The key idea is to introduce a *conditional common knowledge* operator:

- $\mathcal{M}, w \models C^{\varphi}\psi$ iff ψ is true in all worlds reachable (via $\bigcup_{i\in G} \sim_i$) by a finite path starting at w *going through states satisfying* φ.[6]

[6]More formally, $\mathcal{M}, w \models C^{\varphi}\psi$ iff for all $v_0, v_1, \ldots, v_n \in W$ with $w = v_0 \sim_{i_1} v_1 \sim_{i_2} v_2 \sim_{i_3} \cdots \sim_{i_n} v_n$, where $i_1, i_2, \ldots, i_n \in G$, if $v_j \in [\![\varphi]\!]_{\mathcal{M}}$ for $j = 0, \ldots, n$, then $\mathcal{M}, v_n \models \psi$.

The are recursion axioms in this more expressive language:

$$\begin{array}{lcl} [!\psi]C\varphi & \leftrightarrow & (\psi \to C^{\psi}[!\psi]\varphi) \\ [!\psi]C^{\alpha}\varphi & \leftrightarrow & (\psi \to C^{\psi\wedge[!\psi]\alpha}[!\psi]\varphi) \end{array}$$

2.2 Protocol Information

The above recursion axioms also illustrate the mixture of factual and *procedural* truth that drives conversations or processes of observation. To be more explicit, consider the formula $\langle!\varphi\rangle\top$ (with $\langle!\varphi\rangle\psi = \neg[!\varphi]\neg\psi$ the dual of $[!\varphi]$) which means "φ is *announceable*". It is not hard to see that $\langle!\varphi\rangle\top \leftrightarrow \varphi$ is derivable using standard modal reasoning and the above recursion axioms. The left-to-right direction represents a semantic fact about public announcements (only true facts can be announced), but the right-to-left direction represents specific *procedural information*: every true formula is available for announcement. But this is only one of many different protocols and different assumptions about the protocol is reflected in a logical analysis. Consider the following variations of the knowledge recursion axiom (cf. van Benthem et al., 2009, Section 4):

1. $\langle!\varphi\rangle K_i\psi \leftrightarrow (\varphi \wedge K_i\langle!\varphi\rangle\psi)$
2. $\langle!\varphi\rangle K_i\psi \leftrightarrow (\langle!\varphi\rangle\top \wedge K_i(\varphi \to \langle!\varphi\rangle\psi))$
3. $\langle!\varphi\rangle K_i\psi \leftrightarrow (\langle!\varphi\rangle\top \wedge K_i(\langle!\varphi\rangle\top \to \langle!\varphi\rangle\psi))$

Each of these axioms represent a different assumption about the underlying protocol and how that affects the agents' knowledge. The first is the above recursion axiom (in the dual form) and assumes a specific protocol (which is common knowledge) where all true formulas are always available for announcement. The second (weaker) axiom is valid when there is a fixed protocol that is common knowledge. Finally, the third adds a requirement that the agents must know which formulas are currently available for announcement. Of course, the above three formulas are all *equivalent* given our definition of truth in an epistemic model (Definition 2.2 from the first paper) and public announcement (Definition 2.1). In order to see a difference, the *protocol information* must be explicitly represented in the model (cf. van Benthem et al., 2009).

3 Uncertain Observations

The previous section focused on different ways an agent's epistemic state (as described by an epistemic -plausibility/-probability model) can change after finding out that φ is true. There are two basic assumptions built into the formal definition of this epistemic event. The same information is conveyed to *all* the participants, and it is completely transparent *what* is being observed or communicated. In this section I extend the logical analysis of the previous section by weakening these two assumptions.

In order to model situations where the agent is *misinformed* or *uncertain* about what she is observing or what is communicated, there must be a way to describe this uncertainty. In this section, I will introduce three ways to describe this uncertainty. The first is to model such a complex epistemic event as a relational structure (Baltag et al., 1998).

Definition 3.1 (Event Model) An **event model** is a tuple $\langle E, \{S_i\}_{i\in\mathcal{A}}, \mathsf{pre}\rangle$, where E is a nonempty finite[7] set of **primitive events**, for each $i \in \mathcal{A}$, $S_i \subseteq E \times E$ and $\mathsf{pre} : E \to \mathcal{L}_K$ is the **precondition function**. $\triangleleft$

The only difference with an epistemic model (Definition 2.1 from part 1) is that the precondition function assigns a single formula to each primitive event (whereas a valuation function in effect assigns a set of atomic propositions to each state). The intuition is that $\mathsf{pre}(e)$ describes what must be true in order for event e to happen. Alternatively, $\mathsf{pre}(e)$ describes the *content* of what is observed given that event e took place. Given two primitive events e and f, the intuitive meaning of eS_if is "the occurrence of event e appears to agent i as event f". I.e., if event e takes place, then agent i *thinks* it is event f". For example, if $\mathsf{pre}(e) = \varphi$, $\mathsf{pre}(f) = \psi$ and eS_if, then the event e will appear to agent i to be an public announcement of ψ (rather than φ which is the correct information conveyed by the event e). So, an event model describes what the agents perceive about the epistemic event that is taking place. The resulting change in an epistemic model is given by the so-called product update rule:

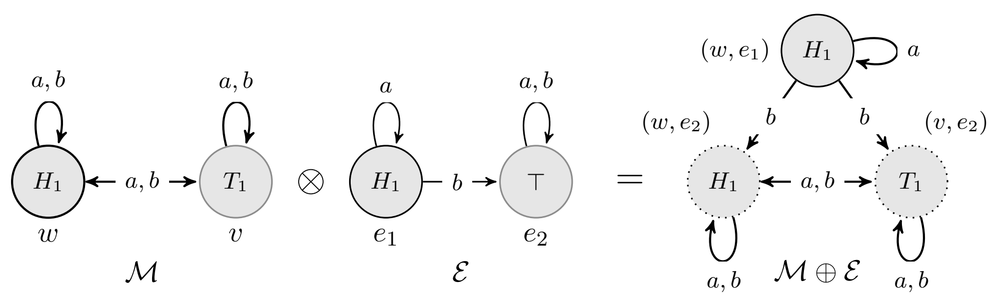
Definition 3.2 (Product Update) The **product update** $\mathcal{M} \otimes \mathcal{E}$ of an epistemic model $\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, V\rangle$ and event model $\mathcal{E} = \langle E, \{S_i\}_{i\in\mathcal{A}}, \mathsf{pre}\rangle$ is the epistemic model $\langle W', \{\sim'_i\}_{i\in\mathcal{A}}, V'\rangle$ with:

1. $W' = \{(w, e) \mid w \in W, e \in E \text{ and } \mathcal{M}, w \models \mathsf{pre}(e)\}$;
2. For each $i \in \mathcal{A}$, $(w, e) \sim'_i (w', e')$ iff $w \sim_i w'$ in $\mathcal{M}$ and eS_ie' in $\mathcal{E}$; and
3. for all $p \in \mathsf{At}$, $(s, e) \in V'(p)$ iff $s \in V(p)$ $\triangleleft$

I illustrate this operation with the following example. Consider an initial model where neither Ann nor Bob knows whether the coin in the first drawer is laying

[7]Finiteness of the set of primitive events is imposed in order to simplify the logical analysis. In general, an event model can have an infinite number of primitive events.

heads up. Suppose that Ann looks at the coin in the drawer while Bob is not paying attention (eg., Bob does not observe Ann looking at the coin). This epistemic event has different effects on Ann and Bob's information: Ann finds out that H_1 is true, while Bob acquires a false belief that neither agent knows the position of the coin. The event model $\mathcal{E}$ below describes this epistemic event. Using product update (Definition 3.2), a new model ($\mathcal{M}\otimes\mathcal{E}$) is created describing Ann and Bob's epistemic states after the event takes place. Of course, this will not be an epistemic model since Bob has acquired a false believe that Ann does not know whether H_1 is true. For this reason, I use labeled directed arrows to indicate the epistemic possibilities for each agent at the different states (for consistency, I also do this in the picture of the initial model $\mathcal{M}$).



Notice that the public announcement event (Definition 2.1) is a special case of Definition 3.1. Given a formula $\varphi \in \mathcal{L}_K$, the public announcement of φ is the event model $\mathcal{E}_\varphi = \langle \{e\}, \{S_i\}\mathsf{pre}\rangle$ where eS_ie for all $i \in \mathcal{A}$ and $\mathsf{pre}(e) = \varphi$. As the reader is invited to verify, the product update of an epistemic model $\mathcal{M}$ with a public announcement event $\mathcal{E}_\varphi$ ($\mathcal{M} \otimes \mathcal{E}_\varphi$) is (isomorphic to) the model $\mathcal{M}^\varphi$ of Definition 2.1.

The logical analysis of the product update operation is similar to what is found in the previous section. Extend the epistemic language $\mathcal{L}_K$ with modal operators $[\mathcal{E}, e]\varphi$ where $\mathcal{E}$ is an event model and e a primitive event in $\mathcal{E}$. The intended interpretation is "after the epistemic event described by $\mathcal{E}$ (with e the actual event) takes place, ψ is true". The definition of truth is:

- $\mathcal{M}, w \models [\mathcal{E}, e]\varphi$ iff if $\mathcal{M}, w \models \mathsf{pre}(e)$, then $\mathcal{M} \otimes \mathcal{E}, (w, e) \models \varphi$.

The key recursion axiom is similar to the one for public announcements:

$$[\mathcal{E}, e]K_i\varphi \quad \leftrightarrow \quad (\mathsf{pre}(e) \rightarrow \bigwedge_{eS_if} K_i(\mathsf{pre}(e) \rightarrow [\mathcal{E}, f]\varphi))$$

The analysis of language with common knowledge operators is not as straightforward requiring a more expressive epistemic language (see van Benthem et al., 2006, for details).

So, event models and product update generalize the public announcement operation to situations where agents may not *know* (in the sense discussed in Section 2 of part 1) precisely which event they are observing. Similarly, the following doxastic action model and product operation generalize the weaker operations of radical and conservative upgrade. The basic idea is to extend an event model with a plausibility ordering over the set of primitive actions. This plausibility ordering encodes changes in beliefs as beliefs about actions. Formally, a **doxastic action model** is a tuple $\langle E, \{\preceq_i\}_{i\in\mathcal{A}}, \{\sim_i\}, \mathsf{pre}\rangle$, where E is a nonempty finite[8] set of **primitive events**, the relations $\preceq_i$ and $\sim_i$ satisfy the same conditions as in an epistemic-plausibility model (Definition **??**), and $\mathsf{pre} : E \to \mathcal{L}_{KB}$ is the **precondition function**. The appropriate notion of "product update" between an epistemic-plausibility model and a doxastic-action model is:

Definition 3.3 (Update) The **update** $\mathcal{M} \otimes \mathcal{E}$ of an epistemic-plausibility model $\mathcal{M} = \langle W, \{\preceq_i\}_{i\in\mathcal{A}}, \{\sim_i\}_{i\in\mathcal{A}}, V\rangle$ and a doxastic action model $\mathcal{E} = \langle E, \{\preceq_i\}_{i\in\mathcal{A}}, \{\sim_i\}_{i\in\mathcal{A}}, \mathsf{pre}\rangle$ is the epistemic model $\langle W', \{\preceq_i'\}_{i\in\mathcal{A}}, \{\sim_i'\}_{i\in\mathcal{A}}, V'\rangle$ defined as follows: W', $\sim_i'$ and V' are defined as in Definition 3.2, and for each $i \in \mathcal{A}$, $\preceq_i'$ is a

Anti-lexicographic ordering: $(w, e) \preceq_i' (w', e')$ iff either $e \prec_i e'$ in $\mathcal{E}$ or ($e \preceq_i e'$ and $e' \preceq_i e$ in $\mathcal{E}$ and $w \preceq_i w'$ in $\mathcal{M}$) ◁

There is much more to say about the logical analysis of this operation which I do not go into here — see (Baltag and Smets, 2006) and (van Benthem, 2010, Chapter 7) for details. Instead, I briefly describe a probabilistic variant of an event model and product update.

Van Benthem et al. (2009) identify three types of probabilistic information that influence how an agent changes her (graded) beliefs in response to a learning event: .

1. *Prior probabilities* over the states states representing agents' initial beliefs.
2. *Occurrence probabilities for events* representing the agents' views on what sort of process is producing the new information.
3. *Observational probabilities* reflecting the agents' uncertainty as to which event is currently being observed.

An epistemic-probability model describes the agents' prior probabilities while a **probabilistic action model** describes the agents' occurrence and observational probabilities.

[8]Again, finiteness is only imposed to simplify the logical analysis.

Definition 3.4 (Probabilistic Action Model) A **probabilistic action model** is a tuple $\mathcal{E} = \langle E, \{\sim_i\}_{i\in\mathcal{A}}, \Phi, \mathsf{pre}, \{P_i\}_{i\in\mathcal{A}}\rangle$ where E is a non-empty finite set of primitive events, for each $i \in \mathcal{A}$, $\sim_i$ is an equivalence relation on E, and

- Φ is a set of pairwise inconsistent sentences (of $\mathcal{L}_{KB}^{prob}$) called **preconditions**
- pre assigns to each preconditions $\varphi \in \Phi$ a probability distribution over E (denoted $\mathsf{pre}(\varphi, e)$)
- For each i, P_i is a probability measure on E (that is weakly regular in the sense that for all $e \in E$, $P_i([e]_i) > 0$, where $[e]_i$ is the equivalence class of $\sim_i$ at state e). ◁

The probability measure P_i is analogous to the prior π_i in an epistemic-probability model. If e is the actual event, then $P_i(f \mid [e]_i)$ is the probability that agent i observed event f. The preconditions Φ and probability measure pre describes the underlying protocol producing the new information (see van Benthem et al. (2009) for a discussion and examples or such protocols). So, $\mathsf{pre}(\cdot, e)$ is the probability that e occurs in different possible situations.

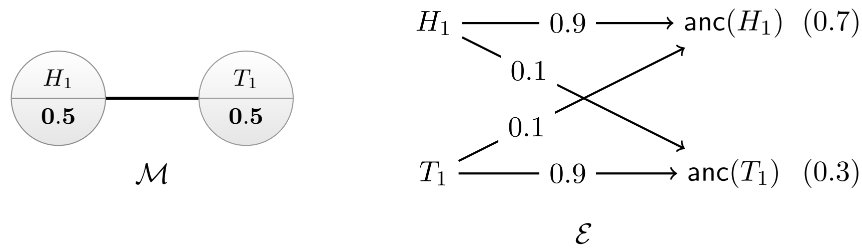
Definition 3.5 (Probabilistic Update Rule) Let $\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\pi_i\}_{i\in\mathcal{A}}, V\rangle$ be an epistemic-probability model and $\mathcal{E} = \langle E, \{\sim_i\}_{i\in\mathcal{A}}, \Phi, \mathsf{pre}, \{P_i\}_{i\in\mathcal{A}}\rangle$ a probabilistic action model. The probabilistic update model $\mathcal{M} \otimes \mathcal{E} = \langle W', \{\sim'_i\}_{i\in\mathcal{A}}, \{\pi_i\}_{i\in\mathcal{A}}, V'\rangle$ is defined as follows:

- $W' = \{(w, e) \mid w \in W, e \in E \text{ and } \mathsf{pre}(s, e) > 0\}$ ($pre(s, e) = pre(\varphi, e)$ where $\varphi \in \Phi$ is the element of Φ true at s (if none exists, set $pre(s, e) = 0$).
- $(s, e) \sim'_i (s', e')$ iff $s \sim_i s'$ and $e \sim_i e'$
- $\pi'_i((w, e)) :=$

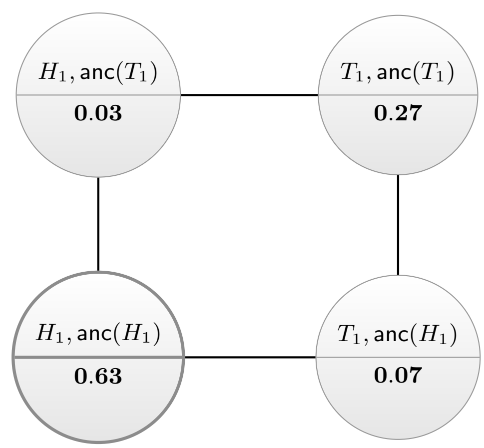
$$\frac{\pi_i(w \mid [w]_i) \cdot \mathsf{pre}(w, e) \cdot P_i(e \mid [e]_i)}{\sum_{w' \in W, e' \in E} \pi_i(w' \mid [w]_i) \cdot \mathsf{pre}(w', e') \cdot P_i(e' \mid [e]_i)}$$

  (set to 0 if the denominator is 0) ◁

Suppose that initially Ann does not know the position of the coin in the first drawer (furthermore, suppose that she assigns equal probability to H_1 and T_1). Consider the event where Bob looks at the coin and announces what he sees. Now, Bob is the type of person that typically tells the truth; however Bob mumbled, and so Ann is not sure what exactly she heard. Giving precise probabilities to these events (suppose that Bob tells the truth 90% of the time and the probability Ann assigns to Bob saying "H_1" is 0.7), this event can be describe by the following probabilistic action model:



Combining the epistemic probability model $\mathcal{M}$ with the epistemic action model $\mathcal{E}$ results in the epistemic-probability model $\mathcal{M} \otimes \mathcal{E}$ pictured below:



There is much more to say about how a rational agent's graded beliefs (should) change in light of new evidence. A complete overview of this literature is beyond the scope of this paper. Consult (van Benthem et al., 2009) for a logical analysis of probabilistic product update, and see (Baltag and Smets, 2008a; Bovens and Ferreira, 2010; Bovens, 2010; Halpern and Grünwald, 2003; Halpern and Tuttle, 1993; Shafer, 1985) for discussions related to issues raised in this paper.

4 The Dynamic Turn in Logic

The general approach described in this paper has been dubbed the "dynamic turn" in logic. The central idea is that informational acts, such as observation and communication, have their own valid laws that can be brought out in an appropriate logic. This research program has been developed in great detail and has been applied to a number of different logical frameworks beyond the epistemic and doxastic logics surveyed in this paper. I briefly mentions a few of these below (I do not have the space to go into details, but see the references and van Benthem (2010) for a more extensive discussion).

- *Preference*: Modal preference logics study an agent's preferences over complete states of affairs (eg., given agent i's overall evaluations of the possible words, i concludes that w is at least as good as v) and over partial descriptions in the forms of propositions (eg., agent i judges that φ is at least as good as ψ).[9] These preferences are intimately connected to the agents' information attitudes, and so the epistemic actions studied in this paper may also change an agent's evaluative attitudes. But dynamic preference logics also study acts of "pure preference change", such as commands or other imperatives. See Liu (2011) for an extensive overview of dynamic preference logics.

- *Inference*: Dynamic inference logics treat acts of logical inference (eg., the agent concludes ψ from φ and $\varphi \to \psi$) as first-class citizens providing a more fine-grained analysis of how an agent's information changes during a rational inquiry. These logics also shed some light on the vexing problem of logical omniscience. A logical analysis of the dynamics of logical reasoning in the style presented in this paper can be found in (van Benthem, 2008; Velázques-Quesada, 2009).

- *Awareness*: There are (at least) two general reasons why an agent might not know that a certain fact φ is true. One reason is that the agent may not have enough information to conclude that φ is true (or has enough information, but has not performed the required inference). A second reason is that the agent may not even be *aware* of the proposition in question. Representing (un)awareness requires going beyond the state-based models discussed in this paper (Dekel et al., 1998; Halpern and Rego, 2009). Nonetheless, dynamic logics of awareness in the style presented in this paper have been developed (see van Benthem and Velázques-Quesada, 2010; van Ditmarsch and French, 2011, for a discussion).

The dynamic turn in logic is not only concerned with dynamic extensions of (modal) logical frameworks and their applications. The general methodology outlined in this paper provides new insights and perspectives on a number of philosophical issues. In the interest of space, I mention just one: Fitch's paradox.

Fitch's Paradox Fitch's paradox[10] shows that the *verification thesis* ("everything that is true can be known") trivializes the notion of knowledge. More for-

[9]See van Benthem et al. (2009) and references therein for an overview of modal preference logics.

[10]See Brogaard and Salerno (2011) for references and a general discussion of this paradox.

mally, the verification thesis corresponds to the validity of the scheme $\varphi \to \Diamond K\varphi$ (if φ is true then it is *possible* that the agent knows that φ is true"). Substituting $\neg\psi \wedge K\psi$ for φ, one can derive $\psi \to K\psi$ using a minimal modal logic for K and $\Diamond$.[11] So, the verification thesis implies (in a minimal modal logic) that if a formula is true then it must be known. There is an extensive literature devoted to this paradox, but here I limit my discussion to the analysis of this paradox in dynamic epistemic logic (following van Benthem, 2004).

From the perspective of dynamic epistemic logic, Fitch's argument does not uncover a paradox that must be solved; rather, it suggests a new and interesting line of research. The point is to acknowledge the paradoxical behavior of the "Moore sentences" (formulas of the form $\psi \wedge \neg K\psi$) and use (variants of) public announcement logic to *explain* the paradox by studying the "dynamic underpinnings" of the verification thesis. This means that rather than focusing on modifying the underlying logic of knowledge and "possibility", the focus is on the verification thesis itself. Focusing on the verification thesis raises two important issues. The first is that the verification thesis, as stated, holds for *any* formula (in the language of epistemic logic). One natural response is to restrict the thesis to only those formulas φ where $K\varphi$ is consistent. [12] The second issue involves the formalization of "coming to know that...". The standard formalization represents this as a bare possibility operators $\Diamond$. However, "coming to know" is an epistemic activity which I have argued is most naturally modeled as the public announcement operation of Section 2. This suggests the following semantic definition of the '$\Diamond$' operator:

$$\mathcal{M}, w \models \Diamond\varphi \text{ iff there is a } \psi \in \mathcal{L}_K \text{ such that } \mathcal{M}, w \models \psi \text{ and } \mathcal{M}^{!\psi}, w \models \varphi$$

The above discussion raises a number of interesting technical and conceptual questions. First, what is the logic of the above *arbitrary announcement operator*? This is answered by Balbiani et al. (2008) for the epistemic language with public announcement and arbitrary public announcement operators. The answer is much more complicated if the language also contains a common knowledge operator (Miller and Moss, 2005). Returning to the verification thesis, using the language of public announcement logic, there are three variants of this thesis (recall that $\langle !\psi\rangle\varphi$ means "after the public announcement of ψ, φ is true"):

[11] The derivation proceeds as follows: $(\psi \wedge \neg K\psi) \to \Diamond K(\psi \wedge \neg K\psi)$ is an instance of the verification thesis. Using $K(\alpha \wedge \beta) \to (K\alpha \wedge K\beta)$ and the inference rule "from $\alpha \to \beta$, we can derive $\Diamond\alpha \to \Diamond\beta$", we have $\Diamond K(\psi \wedge \neg K\psi) \to \Diamond(K\psi \wedge K\neg K\psi)$. Using $K\alpha \to \alpha$, the inference rule for $\Diamond$, and propositional reasoning, we have $\Diamond(K\psi \wedge K\neg K\psi) \to \Diamond(K\psi \wedge \neg K\psi)$. Under the basic assumption that $\Diamond\bot \to \bot$ and putting everything together we have $(\psi \wedge \neg K\psi) \to \bot$, which implies (using propositional reasoning) that $\psi \to K\psi$.

[12] See (Tennant, 2002) for an argument in favor of this proposal. Note that this restriction rules out Fitch's argument since $K(\psi \wedge \neg K\psi)$ is not consistent in any epistemic logic satisfying negative introspection (cf. van Benthem, 2004, Sections 2 and 3).

1. *Learnability*: If φ is true, then there exists a formula ψ such that $\langle !\psi \rangle K\varphi$ is true.

2. *Uniform Learnability*: There is a formula ψ such that, if φ is true, then $\langle !\psi \rangle K\varphi$ is true.

3. *Self-Fulfillment*: If φ is true, then $\langle !\varphi \rangle K\varphi$ is true.

These all have group versions where the individual knowledge modality K is replaced by a group knowledge modality (eg., common knowledge or distributed knowledge). The question is: Can we characterize the formulas that are learnable/uniformly learnable and self-fulfilling? A partial answer is provided by Holiday and Icard (2010) (cf. also van Ditmarsch et al., 2011).

These are not the only applications of dynamic logics of knowledge and belief surveyed in this paper. The dynamic perspective has provided new insights on a number of different topics, including the surprise examination paradox (Gerbrandy, 2007), Willamson's *margin of error* paradox (Egré and Bonnay, 2009), foundational issues in game theory (van Benthem et al., 2011) and quantum logic (Baltag and Smets, 2012), with many more yet to come.

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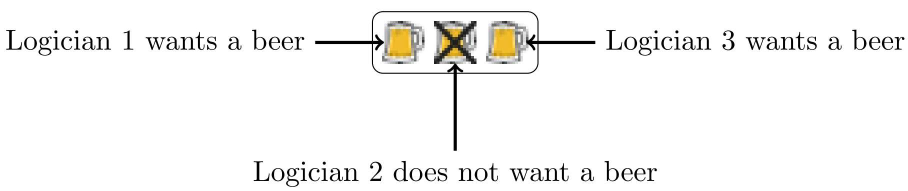
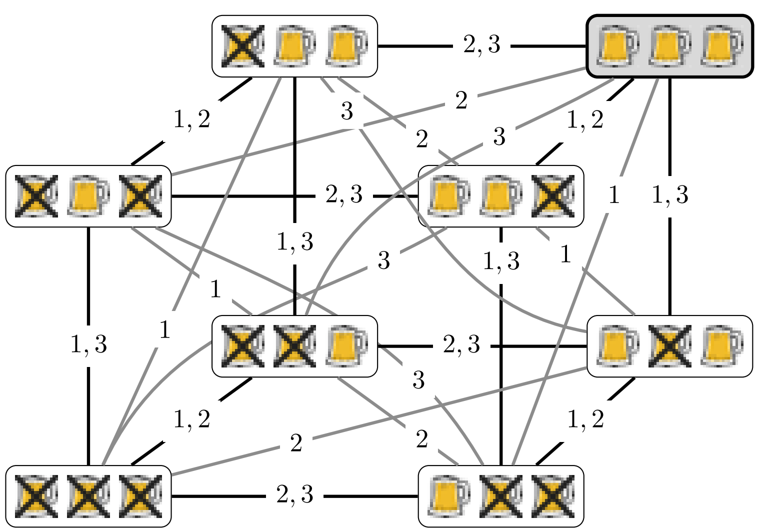
A *Three Logicians Walk Into a Bar...*

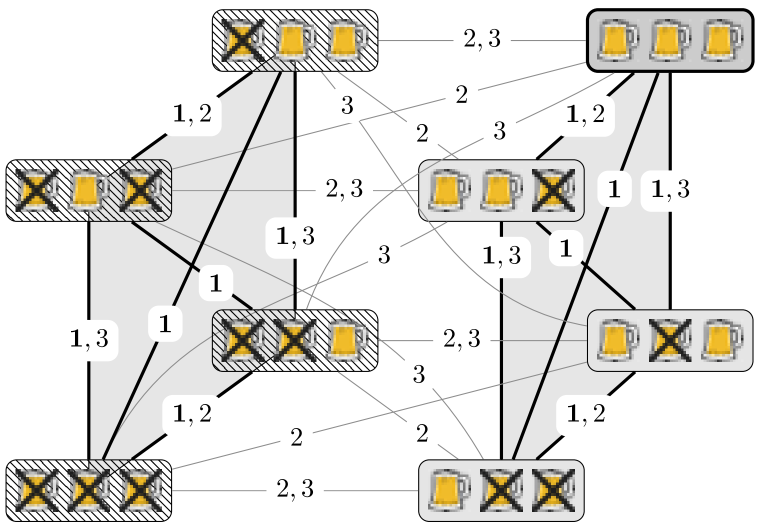
In this appendix, I will present a formal rendering of the "three logicians cartoon" in terms of sequences of public announcements. After the bartender asks the question "does everyone want a beer?", the information of the three logicians is represented below. Each logician knows whether or not she wants a beer, but has no information about the whether the other logicians want a beer. This fact is common knowledge among all the logicians. This initial situation can be represented by the following figure. There are eight states describing the different configurations of who wants a beer. I use the convention that a picture of a beer in the ith position indicates that logician i wants a beer, and a picture of a beer with an 'X' through in the ith position indicates that logician i does not want a beer. For example, the state where logicians 1 and 3 want a beer and logician 2 does not want a beer is:



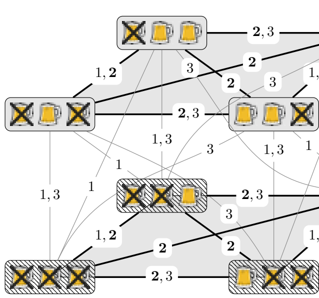
Then, the information of all the logicians after the question of the bartender is represented in the following figure (I do not draw any reflexive edges to keep down the clutter):



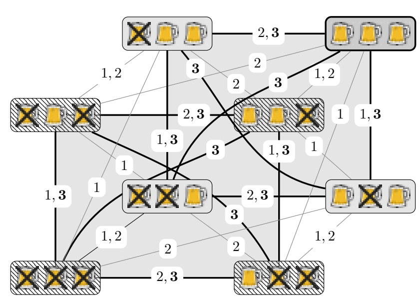
The first person to speak is the logician 1. Let B be an atomic proposition[13] that is true at states where everyone does want a beer (so, in this example, B is true at the state in the top left corner and false everywhere else). Since we assume the logicians are honest and not attempting to deceive the bartender, in order to truthfully answer "I don't know", logician 1 must not know whether everyone wants a beer. More formally, the precondition for the event 'logician 1 utters "I don't know" in response the the bartender's question' is $\neg(K_1 B \vee K_1 \neg B)$. To help visualize the effect of this event on the model, I highlighted logician 1's information cells below. The states with the diagonal lines are the ones where $\neg(K_1 B \vee K_1 \neg B)$ is false:



For completeness, I also highlight logician 2 and 3's information:



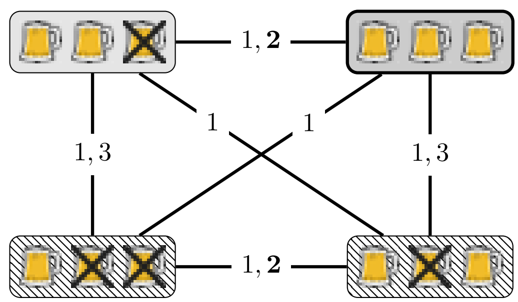
Logician 2's information cells



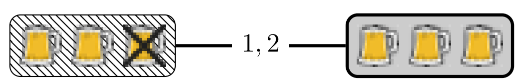
Logician 3's information cells

[13] Of course, we could set things up differently and assume there are three atomic propositions Y_i meaning logician i wants a beer. Then B would be defined as $Y_1 \wedge Y_2 \wedge Y_3$.

(The diagonal lines indicate that the logicians do not know whether everyone wants a beer.) After logician 1 publicly announces “I don’t know” in response to the bartender’s question, all states from the initial model where the precondition of that event is false are removed (i.e., the states marked by the diagonal lines). The effect of the public announcement is pictured below (note that the states with diagonal lines are the states where logician 2 does not know whether all the logicians want a beer):



Now, it is common knowledge that logician 1 wants a beer, but the second logician still does not know whether everyone wants a beer (he does not have enough information to rule out the possibility that logician 3 does not want a beer). So, logician 2 also answers “I don’t know” which reduces the model even further:



At this point, logician 3 knows that all three logicians wants a beer; and so, she answers “yes”. This reduces the model to a single state where it is common knowledge among the logicians that everyone wants a beer: