A Proof of Arrow’s Impossibility Theorem

Navid Hassanpour
Department of Philosophy
Stanford University

October 03, 2008

Logical Methods in the Humanities
http://ai.stanford.edu/~epacuit/lmh/

*Arrow's Impossibility Theorem, Origins and Beyond*

Social Choice & Individual Values by Kenneth J. Arrow (1951)
Arrow’s Impossibility Theorem

Let $X$ be a finite set of Alternative preferences with \textit{at least three elements}.

Assume each individual has a transitive and complete preference over $X$ (ties are allowed).

Consider a society with $N$ individuals, each with a transitive preference over $X$.

A constitution is a function which associates with each $N$-tuple (or profile) of preferences a transitive preference called the social preference.
Arrow’s Theorem, Preliminaries

- Transitivity
- Unanimity: if every individual puts $\alpha$ strictly above $\beta$, society also puts $\alpha$ strictly above $\beta$
- Independence of Irrelevant Alternatives (IIA): If the social relative ranking of two alternatives $\alpha$ and $\beta$ only depends on their relative ranking by every individual
- Dictatorship by individual $n$ if for every pair $\alpha$ and $\beta$ society strictly prefers $\alpha$ to $\beta$ whenever $n$ strictly prefers $\alpha$ to $\beta$. 
Arrow's Theorem

Theorem (Arrow, 1951) Any constitution that respects transitivity, independence of irrelevant alternatives, and unanimity is a dictatorship.

Does not apply to

- Majority voting: transitivity
- Borda count: IIA
Arrow’s Theorem: Proof 1 (Geanakoplos, 2005)-I

- Assume $X = \{A, B, \ldots, C\}$
- Lemma: For any profile in which every individual puts alternative $B$ at the very top or the very bottom of his ranking, society must as well.
- Proof: If not then $B$ is in an intermediate position. For example $A \geq B$ and $B \geq C$. By IIA this holds if all the individuals move $C$ above $A$, because this can be done without changing any $A, B$ or $B, C$ relations; this follows from $B$ being at the extremes of each individual’s personal preferences. By unanimity $C > A$ while by transitivity $A \geq C$: contradiction. So $B$ should be either at the top or the bottom of the social preference.
Arrow’s Theorem: Proof 1 (Geanakoplos, 2005)-II

• There exists an individual $n^* = n(B)$ that by changing her vote at some profile she can move $B$ from the bottom of the social ranking to the top.

• Let’s assume each individual puts $B$ at the bottom of their ranking, by unanimity, the society does the same.

• Now beginning from the individual 1, we start to move $B$ from the bottom of each individual’s ranking to the the top. According to the above lemma, $B$’s social ranking will be either at top or the bottom. Assume $n^*$ is the the first individual whose moving $B$ causes the social ranking of $B$ to flip (by unanimity the change will happen the latest with $n^* = N$)
Arrow’s Theorem: Proof 1 (Geanakoplos, 2005)-III

- Name the profile of rankings just before the individual \( n^* \) moves \( B \) profile I, the one after her moving \( B \) to the top profile II (in profile I, \( B \) is at the bottom of social ranking, in profile II at the top)
- Now we argue that \( n^* \) is the dictator over any pair \( A, C \) not involving \( B \).
- To see this, let’s assume \( n^* \), moves \( A \) above \( B \) in profile II to have \( A >_{n^*} B >_{n^*} C \) and let other individuals \( n \neq n^* \) rearrange all their rankings of \( A \) and \( C \) while keeping \( B \)s in place. By IIA \( A > B \) because all \( A, B \) relations are like in profile I. The same way \( B > C \) because all \( B, C \) relations are like profile II. By transitivity society should have \( A > C \) agreeing with \( n^* \)’s preferences.
For any other pairs such as $A, B$, using the same argument as above there is a $n^*_C$ that is the dictator, but $n^*_B$ can dictate $A, B$ rankings in profiles I and II, therefore $n^*_C$ and $n^*_B$ should be the same.