Decidability of the PAL Substitution Core

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Introduction

The substitution core of a logic is the set of formulas all of whose substitution instances are valid [van Benthem, 2006].
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- Typically the substitution core of a logic coincides with its set of validities, in which case the logic is substitution-closed.
- However, many dynamic logics axiomatized using reduction axioms are not substitution-closed.
- A classic example is Public Announcement Logic (PAL).
Not in the Core...

Example
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\([p \land \neg \Box p](p \land \neg \Box p)\) is not valid—this is the well-known problem of “unsuccessful” formulas.

Since \([\varphi]\varphi\) is not valid for arbitrary \(\varphi\), \([p]p\) is not “schematically valid.”

It is not in the substitution core.
Not in the Core...

Example

Recall the PAL reduction axioms:

1. \( \langle \varphi \rangle p \leftrightarrow (\varphi \land p) \)
2. \( \langle \varphi \rangle (\psi \land \chi) \leftrightarrow (\langle \varphi \rangle \psi \land \langle \varphi \rangle \chi) \)
3. \( \langle \varphi \rangle \neg \psi \leftrightarrow (\varphi \land \neg \langle \varphi \rangle \psi) \)
4. \( \langle \varphi \rangle \Diamond \psi \leftrightarrow (\varphi \land \Diamond \langle \varphi \rangle \psi) \)
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4. \( \langle \varphi \rangle \Box \psi \leftrightarrow (\varphi \land \Box \langle \varphi \rangle \psi) \)

The formula \( \langle q \rangle p \leftrightarrow (q \land p) \) is valid. However, its substitution instance \( \langle p \rangle Kp \leftrightarrow (p \land Kp) \) is not valid, so it is not schematically valid.

It is not in the substitution core.
In the Core…

What is in the substitution core?

1. \( \langle \varphi \rangle (\psi \land \chi) \leftrightarrow (\langle \varphi \rangle \psi \land \langle \varphi \rangle \chi) \)

2. \( \langle \varphi \rangle \neg \psi \leftrightarrow (\varphi \land \neg \langle \varphi \rangle \psi) \)

3. \( \langle \varphi \rangle 3 \psi \leftrightarrow (\varphi \land 3 \langle \varphi \rangle \psi) \)

Plus:

\( \langle \varphi \rangle \langle \psi \rangle \chi \leftrightarrow \langle \langle \varphi \rangle \psi \rangle \chi \)
In the Core...

What is in the substitution core?

All the reduction axioms except the first:

2. $\langle \varphi \rangle(\psi \land \chi) \leftrightarrow (\langle \varphi \rangle \psi \land \langle \varphi \rangle \chi)$

3. $\langle \varphi \rangle \neg \psi \leftrightarrow (\varphi \land \neg \langle \varphi \rangle \psi)$

4. $\langle \varphi \rangle \lozenge \psi \leftrightarrow (\varphi \land \lozenge \langle \varphi \rangle \psi)$
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We have identified other principles of the substitution core.
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Call a formula **purely epistemic** if every propositional variable and occurrence of $\langle \varphi \rangle$ appears within the scope of a $\Diamond$. Note that if $\varphi$ is purely epistemic, so is any substitution instance of $\varphi$. 
In the Core...

Proposition

Formulas 1-3 are schematically valid. In 2 and 3, $\chi$ is purely epistemic.

1. $\Box \varphi \rightarrow (\psi \rightarrow \langle \varphi \rangle \psi)$

2. $\langle (\chi \lor \varphi_1) \land \varphi_2 \rangle \psi \leftrightarrow (\chi \land \langle \varphi_2 \rangle \psi) \lor (\neg \chi \land \langle \varphi_1 \land \varphi_2 \rangle \psi)$

3. $\langle (\chi \land \varphi_1) \lor \varphi_2 \rangle \psi \leftrightarrow (\chi \land \langle \varphi_1 \lor \varphi_2 \rangle \psi) \lor (\neg \chi \land \langle \varphi_2 \rangle \psi)$
In the Core...

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3. $\langle (\chi \land \phi_1) \lor \phi_2 \rangle \psi \leftrightarrow (\chi \land \langle \phi_1 \lor \phi_2 \rangle \psi) \lor (\neg \chi \land \langle \phi_2 \rangle \psi)$

Other schematic validities can be derived from these and the previous.
Question 1 from “Open Problems in Logical Dynamics”:

Is the substitution core of PAL is **decidable**?

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Open problems in logical dynamics.

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Is the substitution core of PAL is **decidable**?


Open problems in logical dynamics.


We have proven the following:

**Theorem**

*The substitution core of single-agent PAL is decidable.*
Thank you!

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