Logics of Rational Agency

Lecture 1

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Plan for the Course

Part 1: Introduction, Motivation and Background

Part 2: Ingredients of a Logical Analysis of Rational Agency

▶ Logics of Informational Attitudes and Informative Actions
▶ Logics of Motivational Attitudes (Preferences)
▶ Time, Action and Agency

Part 3: Merging Logics of Rational Agency

▶ Two Models of Information Dynamics
▶ “Epistemizing” Logics of Action and Ability: knowing how to achieve $\varphi$ vs. knowing that you can achieve $\varphi$
▶ Entangling Knowledge/Beliefs and Preferences
▶ Preferences Change, Plans Change
▶ Logics of Rational Agency in Action: Logic and Games
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Course Website
ai.stanford.edu/~epacuit/lograt/nasslli2010.html

Reading Material
✓ Pointers to literature on the website

Concerning Modal Logic
✓ Course by Blackburn and Areces

✓ Modal Logic for Open Minds by Johan van Benthem (CSLI, 2010)
Part 1: Introduction, Motivation and Background
We are interested in reasoning about rational (and not-so rational) agents engaged in some form of social interaction.
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- Philosophy (social epistemology, philosophy of action)
- Game Theory
- Social Choice Theory
- AI (multiagent systems)
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What is a “rational agent”? What are we modeling?
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What is a “rational agent”? What are we modeling?

- has consistent preferences (complete, transitive)
- (acts as if she) maximizes expected utility
- reacts to observations
- revises beliefs when learning a surprising piece of information
- understands higher-order information
- plans for the future
- asks questions
- ????
We are interested in reasoning about rational (and not-so rational) agents engaged in some form of social interaction.

- playing a (card) game
- having a conversation
- executing a social procedure
- ....

Goal: incorporate/extend existing game-theoretic/social choice analyses
Introductory Remarks

We are interested in reasoning about rational (and not-so rational) agents engaged in some form of social interaction.

There is a jungle of logical frameworks!

- logics of informational attitudes (knowledge, beliefs, certainty)
- logics of action & agency
- temporal logics/dynamic logics
- logics of motivational attitudes (preferences, intentions)
- deontic logics

(Not to mention various game-theoretic/social choice models and logical languages for reasoning about them)
We are interested in reasoning about rational (and not-so rational) agents engaged in some form of social interaction.

- How can we compare different logical frameworks addressing similar aspects of rational agency and social interaction?
- How should we combine logical systems which address different aspects of social interaction towards the goal of a comprehensive (formal) theory of rational agency?
- How does a logical analysis contribute to the broader discussion of rational agency and social interaction within philosophy and the social sciences?
“We wish to find the mathematically complete principles which define ‘rational behavior’ for the participants.” (pg. 31)


“Game theory is a bag of analytical tools designed to help us understand the phenomena that we observe when decision-makers interact.” (pg. 1)

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Game Theory

“We wish to find the mathematically complete principles which define ‘rational behavior’ for the participants.” (pg. 31)


“Game theory is a bag of analytical tools designed to help us understand the phenomena that we observe when decision-makers interact.” (pg. 1)

1. a group of *self-interested* agents (players) involved in some interdependent decision problem, and

2. the players *recognize that they are engaged in a game situation*
What should Ann (Bob) do?
What does it mean for Ann to be *perfectly rational*?
Ann’s best choice depends on what she expects Bob to do, and this depends on what she thinks Bob expects her to do, and so on...

<table>
<thead>
<tr>
<th></th>
<th>Bob</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ann</td>
<td></td>
</tr>
<tr>
<td>U</td>
<td>1,2</td>
</tr>
<tr>
<td>D</td>
<td>0,0</td>
</tr>
</tbody>
</table>
Who is game theory about?
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1. **Classical view**: idealized world with *perfectly rational agents*

2. **Humanistic view**: real people in interactive situations

Who is game theory about?

1. **Classical view**: idealized world with *perfectly rational agents*

   - The game itself it taken to be a literal description of the strategic interaction

   “*We adhere to the classical point of view that the game under consideration fully describes the real situation — that any (pre) commitment possibilities, any repetitive aspect, any probabilities of error, or any possibility of jointly observing some random event, have already been modeled in the game tree.*” (pg. 1005)

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Who is game theory about?

1. **Classical view**: idealized world with *perfectly rational agents*
   - The game itself it taken to be a literal description of the strategic interaction
   - Any appropriate concept of equilibrium should be an *implication* of the information provided in the modeled interpreted through an assumption of perfect rationality.

2. **Humanistic view**: real people in interactive situations

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   - The game itself is taken to be a literal description of the strategic interaction
   - Any appropriate concept of equilibrium should be an *implication* of the information provided in the modeled interpretation through an assumption of perfect rationality.

2. **Humanistic view**: real people in interactive situations
   - The mathematical structures are *models* of interactive situations
   - The appropriate notion of equilibrium is part of the specification of the model

But, the game models are missing something...

*Formally, a game is described by its strategy sets and payoff functions.*
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Formally, a game is described by its strategy sets and payoff functions. But in real life, may other parameters are relevant; there is a lot more going on. Situations that substantively are vastly different may nevertheless correspond to precisely the same strategic game.
But, the game models are missing something...

Formally, a game is described by its strategy sets and payoff functions. But in real life, may other parameters are relevant; there is a lot more going on. Situations that substantively are vastly different may nevertheless correspond to precisely the same strategic game. For example, in a parliamentary democracy with three parties, the winning coalitions are the same whether the parties hold a third of the seats, or, say, 49%, 39%, and 12% respectively.
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What about a logical analysis?

Introductory Remarks

Eric Pacuit
What about a logical analysis?

- Which aspects of social situations should we focus on? Knowledge, Beliefs, Group Knowledge, Preferences, Desires, Ability, Actions, Intentions, Goals, Obligations, etc.
Introductory Remarks

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► Logics of rational agents in social situations. vs. Logics about rational agents in social situations.
Introductory Remarks

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- Which aspects of social situations should we focus on? Knowledge, Beliefs, Group Knowledge, Preferences, Desires, Ability, Actions, Intentions, Goals, Obligations, etc.

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- *Combining* systems is hard! (conceptually and technically)

- Logics *of* rational agents in social situations.
  vs.
  Logics *about* rational agents in social situations.

- Normative vs. Descriptive
Introductory Remarks

The point of view of this model is not normative; it is not meant to advise the players what to do. The players do whatever they do; their strategies are taken as given.
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Background

- Basic Modal Logic
- Weak Systems of Modal Logic
- Combining Logics
- Comparing Logics
Background

✓ Basic Modal Logic

⇒ Weak Systems of Modal Logic

⇒ Combining Logics

⇒ Comparing Logics
Weak Systems of Modal Logic
The Basic Modal Language: $\mathcal{L}$

$p \mid \neg \varphi \mid \varphi \land \psi \mid \Box \varphi \mid \Diamond \varphi$

where $p$ is an atomic proposition (At)
Kripke (Relational) Models

\[ \mathcal{M} = \langle W, R, V \rangle \]
Kripke (Relational) Models

\[ \mathcal{M} = \langle W, R, V \rangle \]

- \( W \neq \emptyset \)
- \( R \subseteq W \times W \)
- \( V : \text{At} \rightarrow 2^W \)
Truth in a Kripke Model

1. $M, w \models p$ iff $w \in V(p)$

2. $M, w \models \neg \varphi$ iff $M, w \not\models \varphi$

3. $M, w \models \varphi \land \psi$ iff $M, w \models \varphi$ and $M, w \models \psi$

4. $M, w \models \Box \varphi$ iff for each $v \in W$, if $wRv$ then $M, v \models \varphi$

5. $M, w \models \Diamond \varphi$ iff there is a $v \in W$ such that $wRv$ and $M, v \models \varphi$
Some Validities

(M) \( \Box (\varphi \land \psi) \rightarrow \Box \varphi \land \Box \psi \)

(C) \( \Box \varphi \land \Box \psi \rightarrow \Box (\varphi \land \psi) \)

(N) \( \Box \top \)

(K) \( \Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi) \)

(Dual) \( \Box \varphi \leftrightarrow \neg \Diamond \neg \varphi \)

(Nec) from \( \vdash \varphi \) infer \( \vdash \Box \varphi \)

(Re) from \( \vdash \varphi \leftrightarrow \psi \) infer \( \vdash \Box \varphi \leftrightarrow \Box \psi \)
Some Validities

(M) \[ \square (\varphi \land \psi) \rightarrow \square \varphi \land \square \psi \]

(C) \[ \square \varphi \land \square \psi \rightarrow \square (\varphi \land \psi) \]

(N) \[ \square \top \]

(K) \[ \square (\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi) \]

(Dual) \[ \square \varphi \leftrightarrow \neg \lozenge \neg \varphi \]

(Nec) from \[ \vdash \varphi \] infer \[ \vdash \square \varphi \]

(Re) from \[ \vdash \varphi \leftrightarrow \psi \] infer \[ \vdash \square \varphi \leftrightarrow \square \psi \]
In a topology, a \textit{neighborhood} of a point $x$ is any set $A$ containing $x$ such that you can “wiggle” $x$ without leaving $A$.

A \textit{neighborhood system} of a point $x$ is the collection of neighborhoods of $x$. 
Neighborhoods in Modal Logic

**Neighborhood Structure:** $\langle W, N, V \rangle$

- $W \neq \emptyset$
- $N : W \rightarrow \wp(\wp(W))$
- $V : \text{At} \rightarrow \wp(W)$
Some Notation

Given $\varphi \in \mathcal{L}$ and a model $\mathcal{M}$, the

- *proposition* expressed by $\varphi$
- *extension* of $\varphi$
- *truth set* of $\varphi$

is
Some Notation

Given \( \varphi \in \mathcal{L} \) and a model \( \mathcal{M} \), the

- **proposition** expressed by \( \varphi \)
- **extension** of \( \varphi \)
- **truth set** of \( \varphi \)

is

\[
(\varphi)^{\mathcal{M}} = \{ w \in W \mid \mathcal{M}, w \models \varphi \}
\]
\( \models \square \varphi \) if the truth set of \( \varphi \) is a neighborhood of \( w \)
\( w \models \square \varphi \) if the truth set of \( \varphi \) is a neighborhood of \( w \)

What does it mean to be a neighborhood?
$w \models \Box \varphi$ if the truth set of $\varphi$ is a neighborhood of $w$

neighborhood in some topology.

$w \models \Box \varphi$ if the truth set of $\varphi$ is a neighborhood of $w$

neighborhood in some topology.

contains all the immediate neighbors in some graph
\[ w \models \Box \varphi \] if the truth set of \( \varphi \) is a neighborhood of \( w \)

neighborhood in some topology.


contains all the immediate neighbors in some graph


an element of some distinguished collection of sets


To see the necessity of the more general approach, we could consider probability operators, conditional necessity, or, to invoke an especially perspicuous example of Dana Scott, the present progressive tense....Thus $N$ might receive the awkward reading ‘it is being the case that’, in the sense in which ‘it is being the case that Jones leaves’ is synonymous with ‘Jones is leaving’.

(Montague, pg. 73)

Segerberg’s Essay

This essay purports to deal with classical modal logic. The qualification “classical” has not yet been given an established meaning in connection with modal logic....Clearly one would like to reserve the label “classical” for a category of modal logics which—if possible—is large enough to contain all or most of the systems which for historical or theoretical reasons have come to be regarded as important, and which also posses a high degree of naturalness and homogeneity.

(pg. 1)
Example: Logics of High Probability

□φ means “φ is assigned ‘high’ probability”, where high means above some threshold $r \in [0, 1]$. 
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Claim: Mon is a valid rule of inference.
Example: Logics of High Probability

$\Box \varphi$ means “$\varphi$ is assigned ‘high’ probability”, where high means above some threshold $r \in [0, 1]$.

**Claim:** Mon is a valid rule of inference.

**Claim:** $C$ is not valid.
Example: Logics of High Probability

□φ means “φ is assigned ‘high’ probability”, where high means above some threshold \( r \in [0, 1] \).

Claim: Mon is a valid rule of inference.

Claim: C is not valid.


Example: Social Choice Theory

□α mean “the group accepts α.”
Example: Social Choice Theory

$\Box \alpha$ mean “the group accepts $\alpha$.”

*Note: the language is restricted so that $\Box \Box \alpha$ is not a wff.*
Example: Social Choice Theory

□α mean “the group accepts α.”

Consensus: α is accepted provided everyone accepts α.

(E) □α ↔ □β provided α ↔ β is a tautology

(M) □(α ∧ β) → (□α ∧ □β)

(C) (□α ∧ □β) → (□α ∧ □β)

(N) □⊤

(D) ¬□⊥
Example: Social Choice Theory

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**Theorem** The above axioms axiomatize consensus (provided \( n \geq 2^{|At|} \)).
Example: Social Choice Theory

$\square \alpha$ mean "the group accepts $\alpha$.

**Majority:** $\alpha$ is accepted if a *majority* of the agents accept $\alpha$. 
Example: Social Choice Theory

□α mean “the group accepts α.”

Majority: α is accepted if a majority of the agents accept α.

(E) □α ↔ □β provided α ↔ β is a tautology

(M) □(α ∧ β) → (□α ∧ □β)

(S) □α → ¬□¬α

(T) (≥[ϕ₁ ∧ ⋯ ∧ ≥[ϕₖ ∧ ≤[ψ₁ ∧ ⋯ ∧ ≤[ψₖ]) →
     ∧₁≤i≤ₖ([= [ϕᵢ ∧ [= [ψᵢ) where ∀v ∈ Vᵢ :
     |{i | v(ϕᵢ) = 1}| = |{i | v(ψᵢ) = 1}|}

Theorem The above axioms axiomatize majority.
Example: Social Choice Theory

\( \Box \alpha \) mean "the group accepts \( \alpha \)."


Many Other Examples

- **Epistemic Logic: the logical omniscience problem.**
  

- **Reasoning about coalitions**
  

- **Knowledge Representation**
  

- **Program logics: modeling concurrent programs**
  

- **More during this course....**
Let $W$ be a non-empty set of states.

Any map $N : W \rightarrow \mathcal{P}(\mathcal{P}(W))$ is called a neighborhood function.

**Definition**

A pair $\langle W, N \rangle$ is called a neighborhood frame if $W$ is a non-empty set and $N$ is a neighborhood function.
Some Terminology

Let $\mathcal{F} = \langle W, N \rangle$ be a neighborhood frame.

- $\mathcal{F}$ is **closed under intersections** if for any collections of sets $\{X_i\}_{i \in I}$ such that for each $i \in I$, $X_i \in N(w)$, then $\bigcap_{i \in I} X_i \in N(w)$.

- $\mathcal{F}$ is supplemented, or **closed under superset or monotonic** provided for each $X \subseteq W$, if $X \in N(w)$ and $X \subseteq Y \subseteq W$, then $Y \in N(w)$.

- $\mathcal{F}$ contains the unit provided $W \in N(w)$

- the set $\bigcap_{X \in \mathcal{F}} X$ the core of $\mathcal{F}$. $\mathcal{F}$ contains its core provided $\bigcap_{X \in \mathcal{F}} X \in \mathcal{F}$.

- $\mathcal{F}$ is augmented if $\mathcal{F}$ contains its core and is supplemented.
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From Kripke Frames to Neighborhood Frames

Let $R \subseteq W \times W$, define a map $R^\rightarrow : W \rightarrow \mathcal{P}W$:

for each $w \in W$, let $R^\rightarrow(w) = \{ v \mid wRv \}$
From Kripke Frames to Neighborhood Frames

Let \( R \subseteq W \times W \), define a map \( R^\rightarrow : W \rightarrow \mathcal{P}W \):

for each \( w \in W \), let \( R^\rightarrow(w) = \{ v \mid wRv \} \)

**Definition**

Given a relation \( R \) on a set \( W \) and a state \( w \in W \). A set \( X \subseteq W \) is **\( R \)-necessary at \( w \)** if \( R^\rightarrow(w) \subseteq X \).
Let $R \subseteq W \times W$, define a map $R^\rightarrow : W \to \mathcal{P}W$:

for each $w \in W$, let $R^\rightarrow (w) = \{v \mid wRv\}$

Let $\mathcal{N}_w^R$ be the set of sets that are $R$-necessary at $w$:

$$\mathcal{N}_w^R = \{X \mid R^\rightarrow (w) \subseteq X\}$$
From Kripke Frames to Neighborhood Frames

Let $R \subseteq W \times W$, define a map $R \rightarrow : W \rightarrow \wp(W)$:

for each $w \in W$, let $R \rightarrow (w) = \{ v \mid wRv \}$

Let $\mathcal{N}^R_w$ be the set of sets that are $R$-necessary at $w$:

$$\mathcal{N}^R_w = \{ X \mid R \rightarrow (w) \subseteq X \}$$

**Lemma**

Let $R$ be a relation on $W$. Then for each $w \in W$, $\mathcal{N}^R_w$ is augmented.
Properties of $R$ are reflected in $N^R_w$:

- If $R$ is reflexive, then for each $w \in W$, $w \in \cap N^R_w$

- If $R$ is transitive then for each $w \in W$, if $X \in N^R_w$, then 
  \[ \{ v \mid X \in N^R_v \} \in N^R_w. \]
Theorem

Let $\langle W, R \rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.

Let $\langle W, N \rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.

Proof.
For each $w \in W$, let $N(w) = N_{Rw}$.

Theorem
From Neighborhood Frames to Kripke Frames

Theorem

Let \( \langle W, R \rangle \) be a relational frame. Then there is an equivalent augmented neighborhood frame.

Let \( \langle W, N \rangle \) be an augmented neighborhood frame. Then there is an equivalent relational frame.

Proof.

For each \( w \in W \), let \( N(w) = N_w^R \).

for all \( X \subseteq W \), \( X \in N(w) \) iff \( X \in N_w^R \).
Theorem

✓ Let $\langle W, R \rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.

Let $\langle W, N \rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.

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For each $w \in W$, let $N(w) = N_R^w$. □
Theorem

Let \( \langle W, R \rangle \) be a relational frame. Then there is an equivalent augmented neighborhood frame.

✓ Let \( \langle W, N \rangle \) be an augmented neighborhood frame. Then there is an equivalent relational frame.

Proof.
For each \( w, v \in W \), \( wR_N v \) iff \( v \in \cap N(w) \).
Let $\mathcal{F} = \langle W, N \rangle$ be a neighborhood frame. A **neighborhood model** based on $\mathcal{F}$ is a tuple $\langle W, N, V \rangle$ where $V: \text{At} \to 2^W$ is a valuation function.
Truth in a Model

- $\mathcal{M}, w \models p$ iff $w \in V(p)$
- $\mathcal{M}, w \models \neg \varphi$ iff $\mathcal{M}, w \nvdash \varphi$
- $\mathcal{M}, w \models \varphi \land \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
Truth in a Model

- $\mathcal{M}, w \models p$ iff $w \in V(p)$

- $\mathcal{M}, w \models \neg \varphi$ iff $\mathcal{M}, w \not\models \varphi$

- $\mathcal{M}, w \models \varphi \land \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$

- $\mathcal{M}, w \models \Box \varphi$ iff $(\varphi)^m \in N(w)$

- $\mathcal{M}, w \models \Diamond \varphi$ iff $W - (\varphi)^m \notin N(w)$

where $(\varphi)^m = \{ w \mid \mathcal{M}, w \models \varphi \}$. 

Eric Pacuit
Detailed Example

Suppose \( \mathcal{W} = \{w, s, v\} \) is the set of states and define a neighborhood model \( \mathcal{M} = \langle \mathcal{W}, N, V \rangle \) as follows:

- \( N(w) = \{\{s\}, \{v\}, \{w, v\}\} \)
- \( N(s) = \{\{w, v\}, \{w\}, \{w, s\}\} \)
- \( N(v) = \{\{s, v\}, \{w\}, \emptyset\} \)

Further suppose that \( V(p) = \{w, s\} \) and \( V(q) = \{s, v\} \).
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Background: Weak Systems of Modal Logic

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- \( N(s) = \{\{w, v\}, \{w\}, \{w, s\}\} \)
- \( N(v) = \{\{s, v\}, \{w\}, \emptyset\} \)

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Detailed Example

\[ V(p) = \{w, s\} \text{ and } V(q) = \{s, v\} \]
**Detailed Example**

\[ V(p) = \{w, s\} \text{ and } V(q) = \{s, v\} \]

\[ \{s\} \quad \{v\} \quad \{w, v\} \quad \{w, s\} \quad \{w\} \quad \{s, v\} \quad \emptyset \]

\[ M, s \models \square p \]
Background: Weak Systems of Modal Logic

Detailed Example

\[ V(p) = \{ w, s \} \text{ and } V(q) = \{ s, v \} \]

\[ M, s \models \Box p \]
Detailed Example

\[ V(p) = \{w, s\} \text{ and } V(q) = \{s, v\} \]

\[
\begin{array}{ccccccc}
\{s\} & \{v\} & \{w, v\} & \{w, s\} & \{w\} & \{s, v\} & \emptyset \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
w & s & v & w & s & v & \\
\end{array}
\]

\[ M, s \models \Diamond p \]
Detailed Example

\[ V(p) = \{w, s\} \text{ and } V(q) = \{s, v\} \]

\[ M, s \models \Diamond p \]

\[ (\neg p)^M = \{v\} \]
Detailed Example

\[ V(p) = \{w, s\} \text{ and } V(q) = \{s, v\} \]

\[
\begin{array}{cccccc}
\{s\} & \{v\} & \{w, v\} & \{w, s\} & \{w\} & \{s, v\} & \emptyset \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
w & s & v & w & w & s & v
\end{array}
\]

\[
\mathcal{M}, w \models \Diamond \Box p? \\
\mathcal{M}, w \models \Box \Box p? \\
\mathcal{M}, v \models \Box \Diamond p? \\
\mathcal{M}, v \models \Diamond \Box p?
\]
Detailed Example

\[ V(p) = \{w, s\} \text{ and } V(q) = \{s, v\} \]

\[
\begin{align*}
\{s\} & \rightarrow w \\
\{v\} & \rightarrow w \\
\{w, v\} & \rightarrow s \\
\{w, s\} & \rightarrow v \\
\{w\} & \rightarrow \emptyset \\
\{s, v\} & \rightarrow \emptyset \\
\emptyset & \\
\end{align*}
\]

\[
\begin{align*}
M, w & \models \Diamond \Box p? \\
M, v & \models \Box \Diamond p \\
M, w & \models \Box \Box p? \\
M, v & \models \Diamond \Box p? \\
\end{align*}
\]
Detailed Example

\[ V(p) = \{w, s\} \text{ and } V(q) = \{s, v\} \]

\[
\begin{array}{cccccc}
\{s\} & \{v\} & \{w, v\} & \{w, s\} & \{w\} & \{s, v\} & \emptyset \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
w & s & s & w & s & v & \emptyset \\
\end{array}
\]

\[ M, w \models \Diamond \Box p? \quad \quad M, v \models \Box \Diamond p \]
\[ M, w \models \Box \Box p? \quad \quad M, v \models \Diamond \Box p? \]
**Detailed Example**

\[ V(p) = \{w, s\} \text{ and } V(q) = \{s, v\} \]

\[
\begin{array}{cccccc}
\{s\} & \{v\} & \{w, v\} & \{w, s\} & \{w\} & \{s, v\} & \emptyset \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
 w & s & s & w & s & s & \\
\end{array}
\]

\[
M, w \models \lozenge \Box p?
\]

\[
M, v \models \Box \lozenge p
\]

\[
M, w \models \Box \Box p?
\]

\[
M, v \models \lozenge \Box p?
\]
Detailed Example

\[ V(p) = \{w, s\} \text{ and } V(q) = \{s, v\} \]

\[ M, w \not\models \lozenge \square p \]
\[ M, v \models \square \lozenge p \]
\[ M, w \models \square \square p \]
\[ M, v \models \lozenge \square p \]
Detailed Example

\[ V(p) = \{w, s\} \text{ and } V(q) = \{s, v\} \]

\[
\begin{array}{ccccccc}
\{s\} & \{v\} & \{w, v\} & \{w, s\} & \{w\} & \{s, v\} & \emptyset \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
w & s & v & & & & \\
\end{array}
\]

\[
\begin{align*}
M, w & \not\models \lozenge \Box p \\
M, w & \models \Box \Box p \\
M, v & \models \Box \lozenge p \\
M, v & \models \lozenge \Box p \\
\end{align*}
\]
Detailed Example

\( V(p) = \{w, s\} \) and \( V(q) = \{s, v\} \)

\[
\begin{align*}
M, w & \not\models \lozenge \square p \\
M, w & \models \square \square p \\
M, v & \models \square \lozenge p \\
M, v & \models \lozenge \square p 
\end{align*}
\]
Background: Weak Systems of Modal Logic

Detailed Example

\( V(p) = \{w, s\} \) and \( V(q) = \{s, v\} \)

\[
\begin{array}{cccccc}
\{s\} & \{v\} & \{w, v\} & \{w, s\} & \{w\} & \{s, v\} & \emptyset \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
w & s & \text{red} & s & v & s & \\
\end{array}
\]

\[M, w \not\models \lozenge \Box p\]
\[M, w \models \Box \Box p\]
\[M, v \models \Box \lozenge p\]
\[M, v \models \lozenge \Box p\]
What can we say?

Definition
A modal formula \( \varphi \) defines a property \( P \) of neighborhood functions if any neighborhood frame \( \mathcal{F} \) has property \( P \) iff \( \mathcal{F} \) validates \( \varphi \).
What can we say?

**Lemma**

Let $\mathcal{F} = \langle W, N \rangle$ be a neighborhood frame. Then

$$\mathcal{F} \models \Box (\varphi \land \psi) \rightarrow \Box \varphi \land \Box \psi$$

iff $\mathcal{F}$ is closed under supersets.
What can we say?

**Lemma**

Let $\mathcal{F} = \langle W, N \rangle$ be a neighborhood frame. Then

$\mathcal{F} \models \Box (\varphi \land \psi) \rightarrow \Box \varphi \land \Box \psi$ iff $\mathcal{F}$ is closed under supersets.

**Lemma**

Let $\mathcal{F} = \langle W, N \rangle$ be a neighborhood frame. Then

$\mathcal{F} \models \Box \varphi \land \Box \psi \rightarrow \Box (\varphi \land \psi)$ iff $\mathcal{F}$ is closed under finite intersections.
What can we say?

Consider the formulas $\Diamond \top$ and $\Box \varphi \rightarrow \Diamond \varphi$. 
What can we say?

Consider the formulas $\diamond T$ and $\Box \varphi \rightarrow \diamond \varphi$.

On relational frames, these formulas both define the same property: seriality.
What can we say?

Consider the formulas $\Diamond \top$ and $\Box \varphi \rightarrow \Diamond \varphi$.

On relational frames, these formulas both define the same property: seriality.

On neighborhood frames:
- $\Diamond \top$ corresponds to the property $\emptyset \not\in N(w)$
What can we say?

Consider the formulas $\Diamond T$ and $\Box \varphi \rightarrow \Diamond \varphi$.

On relational frames, these formulas both define the same property: **seriality**.

On neighborhood frames:

- $\Diamond T$ corresponds to the property $\emptyset \notin N(w)$
- $\Box \varphi \rightarrow \Diamond \varphi$ is valid on $\mathcal{F}$ iff $\mathcal{F}$ is proper.
What can we say?

Lemma

Let $\mathcal{F} = \langle W, N \rangle$ be a neighborhood frame such that for each $w \in W$, $N(w) \neq \emptyset$.

1. $\mathcal{F} \models \Box \varphi \rightarrow \varphi$ iff for each $w \in W$, $w \in \cap N(w)$

2. $\mathcal{F} \models \Box \varphi \rightarrow \Box \Box \varphi$ iff for each $w \in W$, if $X \in N(w)$, then $\{v \mid X \in N(v)\} \in N(w)$
Find properties on frames that are defined by the following formulas:

1. $\Box \bot$
2. $\neg \Box \phi \rightarrow \Box \neg \Box \phi$
3. $\Diamond \phi \rightarrow \Box \phi$
4. $\Diamond \Box \phi \rightarrow \Box \Diamond \phi$
5. $\Box \Diamond \phi \rightarrow \Diamond \Box \phi$
Some Non-validities

1. $\Box(\varphi \land \psi) \rightarrow \Box\varphi \land \Box\psi$
2. $\Box\varphi \land \Box\psi \rightarrow \Box(\varphi \land \psi)$
3. $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
4. $\Box\top$
5. $\Box\varphi \rightarrow \varphi$
6. $\Box\varphi \rightarrow \Box\Box\varphi$
7. Many more...
Validities

(Dual) $\Box \varphi \leftrightarrow \neg \Diamond \neg \varphi$ is valid in all neighborhood models.

(Re) If $\varphi \leftrightarrow \psi$ is valid then $\Box \varphi \leftrightarrow \Box \psi$ is valid.
Other modal operators

- \( M, w \models \langle \rangle \varphi \) iff \( \exists X \in N(w) \) such that \( \exists v \in X, M, v \models \varphi \)
- \( M, w \models [\ ] \varphi \) iff \( \forall X \in N(w) \) such that \( \forall v \in X, M, v \models \varphi \)
- \( M, w \models \langle [\rangle \varphi \) iff \( \exists X \in N(w) \) such that \( \forall v \in X, M, v \models \varphi \)
- \( M, w \models [\langle \rangle \varphi \) iff \( \forall X \in N(w) \) such that \( \exists v \in X, M, v \models \varphi \)
Other modal operators

- $M, w \models \langle \rangle \varphi$ iff $\exists X \in N(w)$ such that $\exists v \in X, M, v \models \varphi$
- $M, w \models [\ ] \varphi$ iff $\forall X \in N(w)$ such that $\forall v \in X, M, v \models \varphi$

- $M, w \models \langle [ \ ] \rangle \varphi$ iff $\exists X \in N(w)$ such that $\forall v \in X, M, v \models \varphi$
- $M, w \models [\ ] \varphi$ iff $\forall X \in N(w)$ such that $\exists v \in X, M, v \models \varphi$
Other modal operators

- \( \mathcal{M}, w \models [\square] \varphi \iff \exists X \in N(w) \text{ such that } \forall v \in X, \mathcal{M}, v \models \varphi \)

- \( \mathcal{M}, w \models [\Diamond] \varphi \iff \forall X \in N(w) \text{ such that } \exists v \in X, \mathcal{M}, v \models \varphi \)
Other modal operators

- \( M, w \models \langle \square \varphi \rangle \iff \exists X \in N(w) \text{ such that } \forall v \in X, M, v \models \varphi \)
- \( M, w \models [ \Diamond \varphi ] \iff \forall X \in N(w) \text{ such that } \exists v \in X, M, v \models \varphi \)

Lemma

Let \( M = \langle W, N, V \rangle \) be a neighborhood model. The for each \( w \in W \),

1. if \( M, w \models \Box \varphi \) then \( M, w \models \langle \square \varphi \rangle \)
2. if \( M, w \models \Diamond \varphi \) then \( M, w \models \Diamond \varphi \)

However, the converses of the above statements are false.
Other modal operators

- $M, w \models \langle \ ] \varphi \iff \exists X \in N(w) \text{ such that } \forall v \in X, M, v \models \varphi$
- $M, w \models [ \ \rangle \varphi \iff \forall X \in N(w) \text{ such that } \exists v \in X, M, v \models \varphi$

Lemma

1. If $\varphi \to \psi$ is valid, then so is $\langle \ ] \varphi \to \langle \ ] \psi$.
2. $\langle \ ](\varphi \land \psi) \to (\langle \ ] \varphi \land \langle \ ] \psi)$ is valid

Investigate analogous results for the other modal operators defined above.
A multi-relational Kripke model is a triple $\mathcal{M} = \langle W, R, V \rangle$ where $R \subseteq \wp(W \times W)$. 
A **multi-relational** Kripke model is a triple $\mathcal{M} = \langle W, R, V \rangle$ where $R \subseteq \wp(W \times W)$.

$\mathcal{M}, w \models \Box \varphi$ iff $\exists R \in R$ such that $\forall v \in W$, if $wRv$ then $\mathcal{M}, v \models \varphi$. 
A multi-relational Kripke model is a triple $\mathcal{M} = \langle W, \mathcal{R}, V \rangle$ where $\mathcal{R} \subseteq \wp(W \times W)$.

$\mathcal{M}, w \models \Box \varphi$ iff $\exists R \in \mathcal{R}$ such that $\forall v \in W$, if $wRv$ then $\mathcal{M}, v \models \varphi$.

Are multi-relational semantics equivalent to neighborhood semantics?
A multi-relational Kripke model is a triple $\mathcal{M} = \langle W, R, V \rangle$ where $R \subseteq \wp(W \times W)$.

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Are multi-relational semantics equivalent to neighborhood semantics? Almost
Different Semantics

A multi-relational Kripke model is a triple $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$ where $\mathcal{R} \subseteq \wp(\mathcal{W} \times \mathcal{W})$.

$\mathcal{M}, w \models \Box \phi$ iff $\exists R \in \mathcal{R}$ such that $\forall v \in \mathcal{W}$, if $wRv$ then $\mathcal{M}, v \models \phi$.

A world is called queer if nothing is necessary and everything is possible.
Different Semantics

A **multi-relational** Kripke model is a triple $\mathcal{M} = \langle W, R, V \rangle$ where $R \subseteq \wp(W \times W)$.

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$w$ is queer iff $N(w) = \emptyset$.
A multi-relational Kripke model is a triple $M = \langle W, R, V \rangle$ where $R \subseteq \wp(W \times W)$.

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$w$ is queer iff $N(w) = \emptyset$

A multi-relational model with queer worlds is a quadruple $M = \langle W, Q, R, V \rangle$. 
A multi-relational Kripke model is a triple $\mathcal{M} = \langle W, R, V \rangle$ where $R \subseteq \wp(W \times W)$.

$\mathcal{M}, w \models \Box \varphi$ iff $\exists R \in R$ such that $\forall v \in W$, if $wRv$ then $\mathcal{M}, v \models \varphi$.

$w$ is queer iff $N(w) = \emptyset$

A multi-relational model with queer worlds is a quadruple $\mathcal{M} = \langle W, Q, R, V \rangle$.

$\mathcal{M}, w \models \Box \varphi$ iff $w \notin Q$ and $\exists R \in R$ such that $\forall v \in W$, if $wRv$ then $\mathcal{M}, v \models \varphi$. 
Different Semantics


**PC** Propositional Calculus

- **E** \(\Box \varphi \leftrightarrow \neg \Diamond \neg \varphi\)
- **M** \(\Box (\varphi \land \psi) \rightarrow (\Box \varphi \land \Box \psi)\)
- **C** \((\Box \varphi \land \Box \psi) \rightarrow \Box (\varphi \land \psi)\)
- **N** \(\Box \top\)
- **K** \(\Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)\)

**RE**

\[\begin{array}{c}
\varphi \leftrightarrow \psi \\
\hline \\
\Box \varphi \leftrightarrow \Box \psi
\end{array}\]

**Nec**

\[\begin{array}{c}
\varphi \\
\hline
\Box \varphi
\end{array}\]

**MP**

\[\begin{array}{c}
\varphi \\
\varphi \rightarrow \psi \\
\hline
\psi
\end{array}\]
**PC** Propositional Calculus

- **E** \( \Box \varphi \leftrightarrow \neg \Diamond \neg \varphi \)
- **M** \( \Box (\varphi \land \psi) \rightarrow (\Box \varphi \land \Box \psi) \)
- **C** \( (\Box \varphi \land \Box \psi) \rightarrow \Box (\varphi \land \psi) \)
- **N** \( \Box \top \)
- **K** \( \Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi) \)

**RE**  

\[
\frac{\varphi \leftrightarrow \psi}{\Box \varphi \leftrightarrow \Box \psi}
\]

**Nec**  

\[\frac{\varphi}{\Box \varphi}\]

**MP**  

\[
\frac{\varphi \quad \varphi \rightarrow \psi}{\psi}
\]

A modal logic \( L \) is **classical** if it contains all instances of \( E \) and is closed under \( RE \).
A modal logic $L$ is classical if it contains all instances of $E$ and is closed under $RE$.

$E$ is the smallest classical modal logic.
Background: Weak Systems of Modal Logic

**PC** Propositional Calculus

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<table>
<thead>
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<td>$E$</td>
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<td>$Nec$</td>
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<td></td>
<td>$\Box \varphi$</td>
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<tr>
<td>$MP$</td>
<td>$\varphi \varphi \rightarrow \psi$</td>
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<tr>
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<td>$\psi$</td>
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</tbody>
</table>

**E** is the smallest **classical** modal logic.

In **E**, **M** is equivalent to

(\textit{Mon})

\[
\begin{array}{c}
\varphi \rightarrow \psi \\
\Box \varphi \rightarrow \Box \psi
\end{array}
\]
Background: Weak Systems of Modal Logic

**PC** Propositional Calculus

E [\(\square \varphi \leftrightarrow \neg \diamondsuit \neg \varphi\)]

**Mon**

\[
\begin{align*}
&\varphi \rightarrow \psi \\
&\square \varphi \rightarrow \square \psi
\end{align*}
\]

C [\((\square \varphi \land \square \psi) \rightarrow \square (\varphi \land \psi)\)]

N [\(\square \top\)]

K [\(\square (\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi)\)]

**RE**

\[
\begin{align*}
&\varphi \leftrightarrow \psi \\
&\square \varphi \leftrightarrow \square \psi
\end{align*}
\]

**Nec**

\[
\begin{align*}
&\varphi \\
&\square \varphi
\end{align*}
\]

**MP**

\[
\begin{align*}
&\varphi \varphi \rightarrow \psi \\
&\psi
\end{align*}
\]

\(E\) is the smallest classical modal logic.

**EM** is the logic \(E + Mon\).
6. Propositional Calculus

**E** \( \square \varphi \leftrightarrow \neg \Diamond \neg \varphi \)

**Mon**
\[
\frac{\varphi \rightarrow \psi}{\square \varphi \rightarrow \square \psi}
\]

**C**
\[
(\square \varphi \land \square \psi) \rightarrow \square(\varphi \land \psi)
\]

**N** \( \square \top \)

**K** \( \square(\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi) \)

**RE**
\[
\frac{\varphi \leftrightarrow \psi}{\square \varphi \leftrightarrow \square \psi}
\]

**Nec**
\[
\frac{\varphi}{\square \varphi}
\]

**MP**
\[
\frac{\varphi \quad \varphi \rightarrow \psi}{\psi}
\]

**E** is the smallest **classical** modal logic.

**EM** is the logic **E** + **Mon**

**EC** is the logic **E** + **C**
**PC** Propositional Calculus

\[ E \quad \square \varphi \leftrightarrow \neg \Diamond \neg \varphi \]

**Mon**

\[ \frac{\varphi \rightarrow \psi}{\square \varphi \rightarrow \square \psi} \]

\[ C \quad (\square \varphi \land \square \psi) \rightarrow \square (\varphi \land \psi) \]

**N**

\[ \square \top \]

**K**

\[ \square (\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi) \]

**RE**

\[ \frac{\varphi \leftrightarrow \psi}{\square \varphi \leftrightarrow \square \psi} \]

**Nec**

\[ \frac{\varphi}{\square \varphi} \]

**MP**

\[ \frac{\varphi \quad \varphi \rightarrow \psi}{\psi} \]

**E** is the smallest classical modal logic.

**EM** is the logic **E** + **Mon**

**EC** is the logic **E** + **C**

**EMC** is the smallest regular modal logic
**PC** Propositional Calculus

\[ E \quad \square \varphi \leftrightarrow \neg \Diamond \neg \varphi \]

**Mon**

\[ \frac{\varphi \rightarrow \psi}{\square \varphi \rightarrow \square \psi} \]

**C**

\[ (\square \varphi \land \square \psi) \rightarrow \square (\varphi \land \psi) \]

**N**

\[ \square \top \]

**K**

\[ \square (\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi) \]

**RE**

\[ \frac{\varphi \leftrightarrow \psi}{\square \varphi \leftrightarrow \square \psi} \]

**Nec**

\[ \frac{\varphi}{\square \varphi} \]

**MP**

\[ \frac{\varphi \quad \varphi \rightarrow \psi}{\psi} \]

**E** is the smallest classical modal logic.

**EM** is the logic **E** + **Mon**

**EC** is the logic **E** + **C**

**EMC** is the smallest regular modal logic

A logic is normal if it contains all instances of **E**, **C** and is closed under **Mon** and **Nec**
**PC** Propositional Calculus

**E** \( \square \varphi \leftrightarrow \neg \Diamond \neg \varphi \)

**Mon**

\[ \varphi \rightarrow \psi \]

\[ \square \varphi \rightarrow \square \psi \]

**C** \( (\square \varphi \land \square \psi) \rightarrow \square (\varphi \land \psi) \)

**N** \( \square \top \)

**K** \( \square (\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi) \)

**RE** \( \varphi \leftrightarrow \psi \)

\[ \square \varphi \leftrightarrow \square \psi \]

**Nec** \( \varphi \)

\[ \square \varphi \]

**MP** \( \varphi \quad \varphi \rightarrow \psi \)

\[ \psi \]

**E** is the smallest **classical** modal logic.

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PC  Propositional Calculus

E  \( \Box \varphi \leftrightarrow \neg \Diamond \neg \varphi \)

Mon  \[ \frac{\varphi \rightarrow \psi}{\Box \varphi \rightarrow \Box \psi} \]

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N  \( \Box \top \)

K  \[ \Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi) \]

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E is the smallest classical modal logic.

EM is the logic \( E + \text{Mon} \)

EC is the logic \( E + C \)

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K = EMCN

<table>
<thead>
<tr>
<th><strong>PC</strong></th>
<th>Propositional Calculus</th>
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<tbody>
<tr>
<td><strong>E</strong></td>
<td>$\Box \varphi \leftrightarrow \neg \lozenge \neg \varphi$</td>
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<tr>
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</tr>
<tr>
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<tr>
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**E** is the smallest **classical** modal logic.

**EM** is the logic **E + Mon**

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**EMC** is the smallest **regular** modal logic

$$K = PC(+E) + K + \text{Nec} + \text{MP}$$
Useful Fact

Theorem (Uniform Substitution)

The following rule can be derived in $E$

$$
\frac{\psi \leftrightarrow \psi'}{
\varphi \leftrightarrow \varphi[\psi/\psi']}
$$
Some Facts (1)

Theorem

*The logic $E$ is sound and strongly complete with respect to the class of all neighborhood frames.*
Some Facts (1)

**Theorem**

The logic $\mathbf{E}$ is sound and strongly complete with respect to the class of all neighborhood frames.

**Lemma**

If $C \in \mathbf{L}$, then $\langle M_{\mathbf{L}}, N_{\mathbf{L}}^{\text{min}} \rangle$ is closed under finite intersections.

**Theorem**

The logic $\mathbf{EC}$ is sound and strongly complete with respect to the class of neighborhood frames that are closed under intersections.
Some Facts (2)

**Fact:** The canonical model for EM is not closed under supersets.
Some Facts (2)

**Fact**: The canonical model for $EM$ is not closed under supersets.

**Lemma**

Suppose that $M = sup(M_{EM}^{min})$. Then $M$ is canonical for $EM$.

**Theorem**

The logic $EM$ is sound and strongly complete with respect to the class of supplemented frames.
Some Facts (2)

Theorem

*The logic $\mathbf{K}$ is sound and strongly complete with respect to the class of filters.*

Theorem

*The logic $\mathbf{K}$ is sound and strongly complete with respect to the class of augmented frames.*
Incompleteness?

Are all modal logics complete with respect to some class of neighborhood frames?
Incompleteness?

Are all modal logics complete with respect to some class of neighborhood frames? No
Incompleteness


Presents two logics $L$ and $L'$ that are incomplete with respect to neighborhood semantics.
Incompleteness


Presents two logics $L$ and $L'$ that are **incomplete with respect to neighborhood semantics**.

(there are formulas $\varphi$ and $\varphi'$ that are valid in the class of frames for $L$ and $L'$ respectively, but $\varphi$ and $\varphi'$ are not deducible in the respective logics).
Incompleteness


Presents two logics $L$ and $L'$ that are *incomplete with respect to neighborhood semantics.*

$L$ is between $T$ and $S4$

$L'$ is above $S4$ (adapts Fine’s incomplete logic)
Comparing Relational and Neighborhood Semantics
Comparing Relational and Neighborhood Semantics

Fact: If a (normal) modal logic is complete with respect to some class of relational frames then it is complete with respect to some class of neighborhood frames.

What about the converse?

Are there normal modal logics that are incomplete with respect to relational semantics, but complete with respect to neighborhood semantics?
Comparing Relational and Neighborhood Semantics

**Fact:** If a (normal) modal logic is complete with respect to some class of relational frames then it is complete with respect to some class of neighborhood frames.

What about the converse?

Are there normal modal logics that are incomplete with respect to relational semantics, but complete with respect to neighborhood semantics? Yes!
Comparing Relational and Neighborhood Semantics

There is

▶ an extension of $K$

D. Gabbay. *A normal logic that is complete for neighborhood frames but not for Kripke frames.* Theoria (1975).
Comparing Relational and Neighborhood Semantics

There is

- an extension of $K$

D. Gabbay. *A normal logic that is complete for neighborhood frames but not for Kripke frames*. Theoria (1975).

- An extension of $T$

Comparing Relational and Neighborhood Semantics

There is
- an extension of \( K \)

D. Gabbay. *A normal logic that is complete for neighborhood frames but not for Kripke frames.* Theoria (1975).

- An extension of \( T \)


- An extension of \( S4 \)

 ✓ Basic Modal Logic

 ✓ Weak Systems of Modal Logic

 ⇒ Combining Logics

 ⇒ Comparing Logics

Transfer Results

Given a family $L$ of modal logics and a combination method $C$, do certain properties of the component logics $L \in L$ transfer to their combination $C(L)$?
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Typical Assumptions:

1. $C$ is only defined on finite families $L$ of modal logics
2. $C(L)$ is a (multi-)modal logic
3. $C(L)$ is an extension of each component logic $L \in L$
Transfer Results

Given a family $\mathbf{L}$ of modal logics and a combination method $C$, do certain properties of the component logics $\mathbf{L} \in \mathbf{L}$ transfer to their combination $C(\mathbf{L})$?

Typical Assumptions:

1. $C$ is only defined on finite families $\mathbf{L}$ of modal logics
2. $C(\mathbf{L})$ is a (multi-)modal logic
3. $C(\mathbf{L})$ is an extension of each component logic $\mathbf{L} \in \mathbf{L}$

Does (recursive) axiomatizability, decidability, complexity transfer?
Fusions

Let $L_1$ and $L_2$ be two normal (multi-)modal logics.
Background: Combining Logics

Fusions

Let $L_1$ and $L_2$ be two normal (multi-)modal logics with disjoint sets of modal operators (say $\{\Box_1, \ldots, \Box_n\}$ and $\{\Box_{n+1}, \ldots, \Box_{n+m}\}$).

The fusion of $L_1$ and $L_2$, denoted $L_1 \oplus L_2$, is the smallest normal modal logic in the joint language containing both $L_1$ and $L_2$. If $C_1 = \{\langle W, R_1, \ldots, R_n \rangle\}$ and $C_2 = \{\langle W, S_1, \ldots, S_m \rangle\}$ are classes of Kripke frames, let $C_1 \oplus C_2 = \{\langle W, R_1, \ldots, R_n, S_1, \ldots, S_m \rangle\}$. Theorem: $L_1 \oplus L_2$ is a conservative extension of $L_1$ and $L_2$. If $L_1$ and $L_2$ are characterized by $C_1$ and $C_2$ respectively, then $L_1 \oplus L_2$ is characterized by $C_1 \oplus C_2$. Eric Pacuit
Background: Combining Logics

Fusions

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**Theorem.** \( L_1 \oplus L_2 \) is a conservative extension of (consistent) \( L_1 \) and \( L_2 \).
Fusions

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**Theorems.** Many other properties transfer (eg., decidability). See the Kurucz paper for details.
Suppose $F_1 = \langle W_1, R_1^1, \ldots, R_1^n \rangle$ and $F_2 = \langle W_2, R_2^1, \ldots, R_2^m \rangle$ be Kripke frames. Define the \textbf{product}: 

$$F_1 \times F_2 = \langle W_1 \times W_2, R_1^h, \ldots, R_n^h, R_1^v, \ldots, R_m^v \rangle$$

where

$$(w_1, w_2) R_i^h (v_1, v_2) \text{ iff } w_1 R_i^1 v_1 \text{ and } w_2 = v_2$$

$$(w_1, w_2) R_i^v (v_1, v_2) \text{ iff } w_2 R_i^2 v_2 \text{ and } w_1 = v_1$$

Let $L_1, L_2$ be (Kripke complete) logics, the \textbf{product} is

$$L_1 \times L_2 = \text{Log(}\{F_1 \times F_2 \mid F_1 \in \text{Fr}(L_1) \text{ and } F_2 \in \text{Fr}(L_2)\})$$
Let $L_1$ and $L_2$ be Kripke complete (uni-)modal logics. Let $[L_1, L_2]$ be the smallest modal logic containing $L_1, L_2$ and the axioms:
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$\begin{align*}
&\text{(comm)} \quad \lozenge_1 \lozenge_2 p \leftrightarrow \lozenge_2 \lozenge_1 p
\end{align*}$
Let $L_1$ and $L_2$ be Kripke complete (uni-)modal logics. Let $[L_1, L_2]$ be the smallest modal logic containing $L_1, L_2$ and the axioms:

(comm) $\Diamond_1 \Diamond_2 p \leftrightarrow \Diamond_2 \Diamond_1 p$

(cr) $\Diamond_1 \Box_2 p \rightarrow \Box_2 \Diamond_1 p$
Products: Some Interesting Facts
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- \([L_1, \ldots, L_n] \subseteq L_1 \times \cdots \times L_n\)
Products: Some Interesting Facts

- $[L_1, \ldots, L_n] \subseteq L_1 \times \cdots \times L_n$
- very few transfer results: eg., $[K4, K4] = K4 \times K4$, but is \textit{undecidable}.
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- $[S4, \text{Log}(\langle \omega, < \rangle)]$ is $\Pi^1_1$-hard (another example coming later...)

Eric Pacuit
Background: Combining Logics

Products: Some Interesting Facts

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The general situation is very interesting, but beyond the scope of this course (see the Kurucz paper).
[L_1, \ldots, L_n] \subseteq L_1 \times \cdots \times L_n

▶ very few transfer results: eg., [K4, K4] = K4 \times K4, but is undecidable.

▶ [S4, Log(⟨ω, <⟩)] is \(\Pi_1^1\)-hard (another example coming later...)

The general situation is very interesting, but beyond the scope of this course (see the Kurucz paper).

We will see many examples of combining/merging modal logics during the course...
Background

✓ Basic Modal Logic

✓ Weak Systems of Modal Logic

✓ Combining Logics

⇒ Comparing Logics
What is the precise relationship between Neighborhood Models and Relational Models?
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We can *simulate* any non-normal modal logic with a bi-modal normal modal logic.
The key idea is to replace neighborhood models with a two-sorted Kripke model.
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Given a neighborhood model $\mathcal{M} = \langle W, N, V \rangle$, define a Kripke model $\mathcal{M}^\circ = \langle W', R_N, R_\neg, R_N, Pt, V \rangle$ as follows:

- $W' = W \cup \mathcal{P}(W)$
The key idea is to replace neighborhood models with a two-sorted Kripke model.

Given a neighborhood model $\mathcal{M} = \langle W, N, V \rangle$, define a Kripke model $\mathcal{M}^\circ = \langle W', R_N, R_\varnothing, R_N, Pt, V \rangle$ as follows:

- $W' = W \cup \wp(W)$
- $R_\varnothing = \{ (u, w) \mid w \in W, u \in \wp(W), w \in u \}$
The key idea is to replace neighborhood models with a two-sorted Kripke model.

Given a neighborhood model $\mathcal{M} = \langle W, N, V \rangle$, define a Kripke model $\mathcal{M}^\circ = \langle W', R_N, R_\exists, R_\forall, Pt, V \rangle$ as follows:

- $W' = W \cup \wp(W)$
- $R_\exists = \{ (u, w) \mid w \in W, u \in \wp(W), w \in u \}$
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- $R_N = \{(w, u) \mid w \in W, u \in \wp(W), u \in N(w)\}$
- $Pt = W$
The key idea is to replace neighborhood models with a two-sorted Kripke model.

Given a neighborhood model $\mathcal{M} = \langle W, N, V \rangle$, define a Kripke model $\mathcal{M}^\circ = \langle W', R_N, R_\not\in, R_N, Pt, V \rangle$ as follows:

- $W' = W \cup \wp(W)$
- $R_\exists = \{(u, w) | w \in W, u \in \wp(W), w \in u\}$
- $R_\not\in = \{(u, w) | w \in W, u \in \wp(W), w \not\in u\}$
- $R_N = \{(w, u) | w \in W, u \in \wp(W), u \in N(w)\}$
- $Pt = W$

Let $\mathcal{L}'$ be the language

$$\varphi := p | \neg \varphi | \varphi \land \psi | [\exists] \varphi | [\not\in] \varphi | [N] \varphi | Pt$$

where $p \in \text{At}$ and $Pt$ is a unary modal operator.
Define $ST : \mathcal{L} \rightarrow \mathcal{L}'$ as follows.
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$ST(p) = p$
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- $ST(\varphi \land \psi) = ST(\varphi) \land ST(\varphi)$
- $ST(\Box \varphi) = \langle N \rangle ([\exists] ST(\varphi) \land [\forall] \neg ST(\varphi))$
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- $ST(\varphi \land \psi) = ST(\varphi) \land ST(\varphi)$
- $ST(\Box \varphi) = \langle N \rangle (\exists] ST(\varphi) \land [\not\exists] \neg ST(\varphi))$

**Lemma**

*For each neighborhood model $\mathcal{M} = \langle W, N, V \rangle$ and each formula $\varphi \in \mathcal{L}$, for any $w \in W$,*

$$\mathcal{M}, w \models \varphi \text{ iff } \mathcal{M}^\circ, w \models ST(\varphi)$$
Lemma

On Monotonic Models

\[ \langle N \rangle ([\exists] ST(\varphi) \land [\emptyset] \neg ST(\varphi)) \] is equivalent to

\[ \langle N \rangle ([\exists] ST(\varphi)) \]
What can we infer from the fact that bi-modal normal modal logic can simulate non-normal modal logics?
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Can we read off a notion of bisimulation?
What can we infer from the fact that bi-modal normal modal logic can simulate non-normal modal logics?

Can we read off a notion of bisimulation? **Not clear.**

What can we infer from the fact that bi-modal normal modal logic can simulate non-normal modal logics?

Can we read off a notion of bisimulation? Not clear.


- Decidability of the satisfiability problem
- Canonicity
- Salqhvist Theorem


Background

✓ Basic Modal Logic

✓ Weak Systems of Modal Logic

✓ Combining Logics

✓ Comparing Logics
Ingredients of a Logical Analysis of Rational Agency

What are the basic building blocks?

- The nature of time (continuous or discrete/branching or linear)
- How (primitive) events or actions are represented
- How causal relationships are represented
- What constitutes a state of affairs

Single agent vs. many agents.

What are the primitive operators?

- Informational attitudes
- Motivational attitudes
- Normative attitudes

Static vs. dynamic
Ingredients of a Logical Analysis of Rational Agency

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  - Informational attitudes
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- Static vs. dynamic
Ingredients of a Logical Analysis of Rational Agency

⇒ informational attitudes (eg., knowledge, belief, certainty)
⇒ time, actions and ability
⇒ motivational attitudes (eg., preferences)
⇒ group notions (eg., common knowledge and coalitional ability)
⇒ normative attitudes (eg., obligations)
End of Part 1.