Introduction to Neighborhood Semantics for Modal Logic

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Introduction

1. Motivation

2. Neighborhood Semantics for Modal Logic

3. Brief Survey of Results

4. Bisimulations

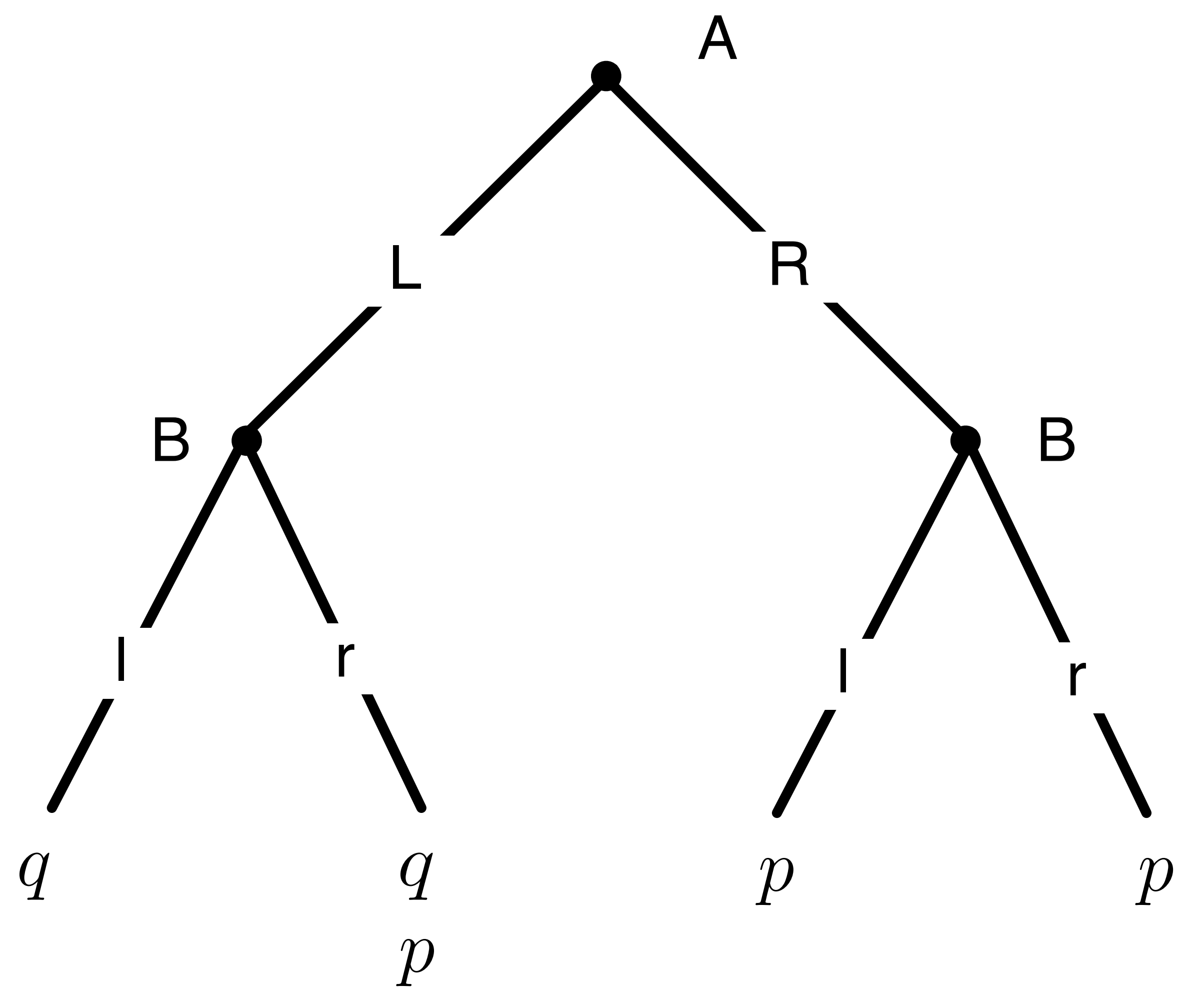
Game "Forcing" Operator

Let G be an extensive game and s a node in G.

We say an agent i can **force** a formula ϕ at s (written $s \models \Box_i \phi$) provided

1. i can move at s
2. there is a *strategy* for i such that for all strategies chosen by the other players, ϕ will become true.

Game "Forcing" Operator



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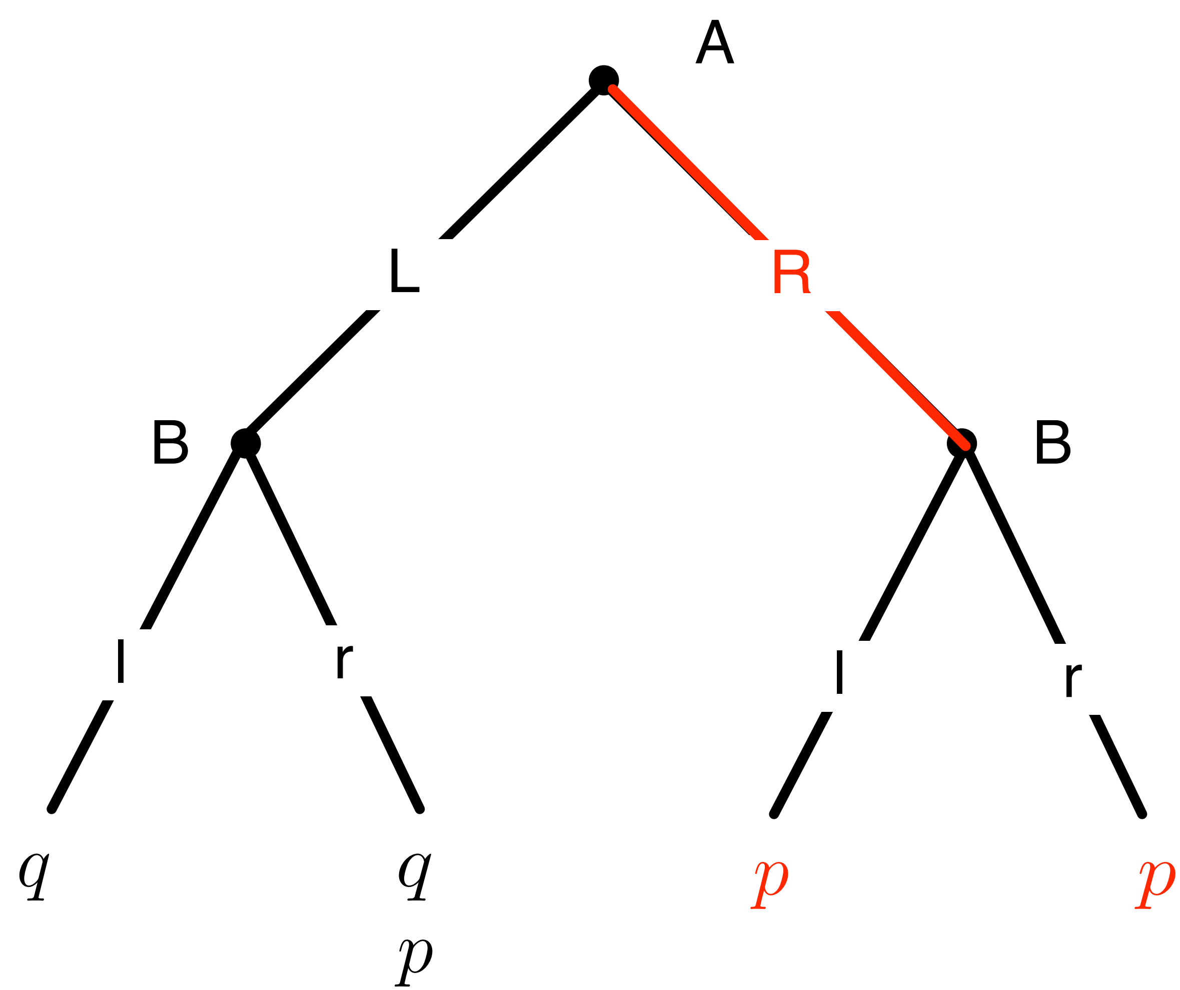

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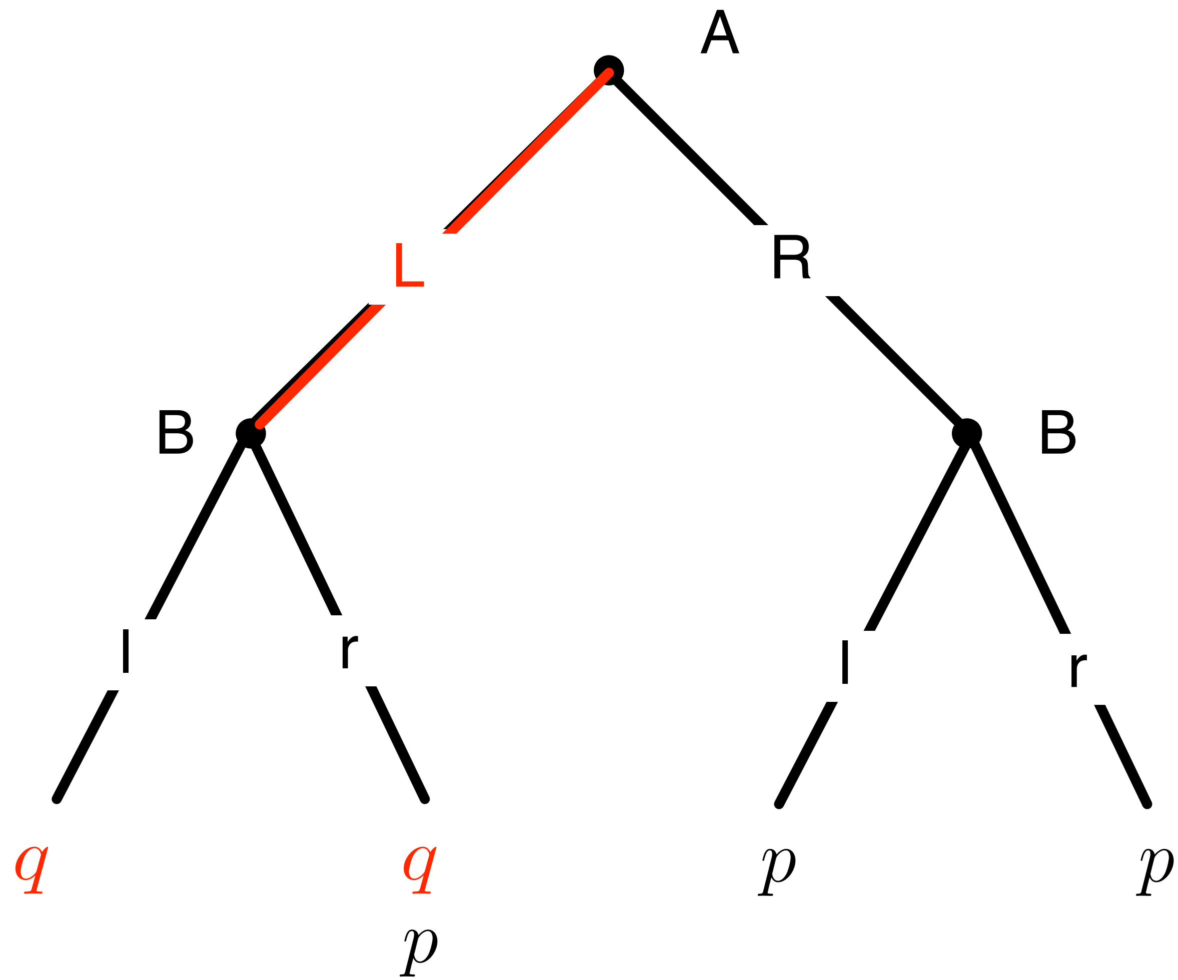

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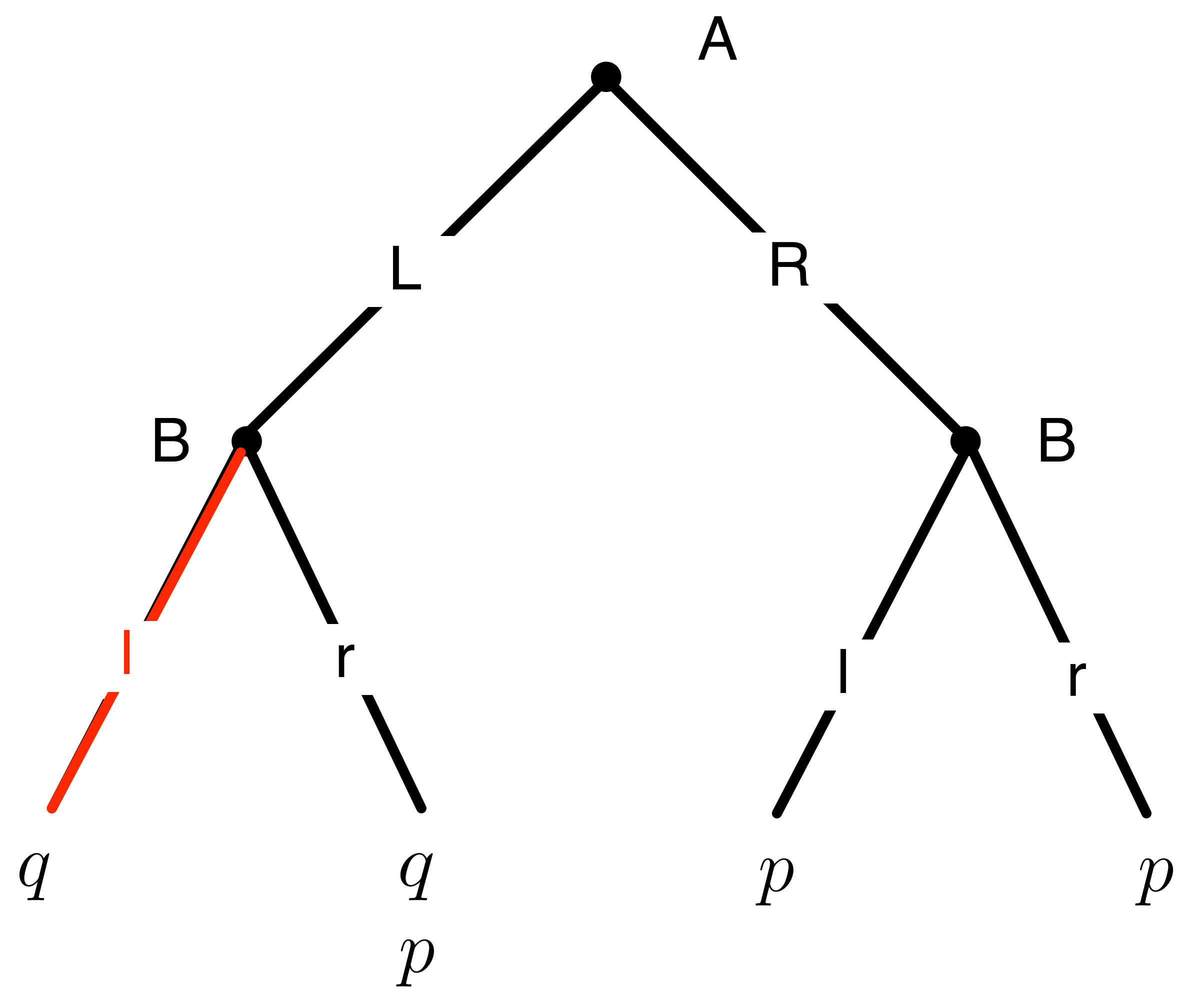

root $\not\models \Box_A(p \wedge q)$

Review: (Propositional) Modal Logic

Basic Modal Language: $\phi := p \mid \neg\phi \mid \phi \wedge \phi \mid \Box\phi$, where $p \in \Phi_0$.

Relational Semantics: A **Kripke model** is a tuple $\langle W, R, V \rangle$ where $R \subseteq W \times W$ and $V : \Phi_0 \to 2^W$.

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Let $R(w) = \{v \mid wRv\}$

Truth at a state: Let $\mathcal{M} = \langle W, R, V\rangle$ be a model with $w \in W$,

$$\mathcal{M}, w \models \Box\phi \text{ iff } R(w) \subseteq (\phi)^{\mathcal{M}}$$

where $(\phi)^{\mathcal{M}}$ is the **truth set** of ϕ.

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The set of **necessary** propositions: $\mathcal{N}_w = \{X \mid R(w) \subseteq X\}$

Fact: $\mathcal{N}_w$ is closed under intersections, supersets and contains the unit *and contains a minimal element:* $\forall w, \cap\mathcal{N}_w \in \mathcal{N}_w$

Other Examples

Fix $t \in [0, 1]$:

Intended Interpretation of $\Box\phi$: *ϕ is assigned (subjective) probability* $> t$

Fact: $\Box\phi \wedge \Box\psi \rightarrow \Box(\phi \wedge \psi)$ is **not valid** under this interpretation.

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Other examples: Concurrent Propositional Dynamic Logic, Parikh's Game Logic, or Pauly's Coalition Logic, Alternating-time Temporal Logic

Extensive literature on the "logical omniscience" problem, Deontic Logics

Neighborhood Semantics for Propositional Modal Logic

A **Neighborhood Frame** is a tuple $\langle W, N \rangle$ where $N : W \to 2^{2^W}$

A **Neighborhood Model** is a tuple $\langle W, N, V \rangle$ where $V : \Phi_0 \to 2^W$

Truth in a model is defined as follows

- $\mathcal{M}, w \models p$ iff $w \in V(p)$
- $\mathcal{M}, w \models \neg\phi$ iff $\mathcal{M}, w \not\models \phi$
- $\mathcal{M}, w \models \phi \wedge \psi$ iff $\mathcal{M}, w \models \phi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \Box\phi$ iff $\quad (\phi)^{\mathcal{M}} \in N(w)$

Some History

Neighborhood Models were first discussed in (Scott 1970, Montague 1970) — perhaps (McKinsey and Tarski 1944) should be cited? See (Segerberg 1971) and (Chellas 1980) for discussions of neighborhood semantics for propositional modal logics.

Non-normal Modal Logics

E $\Box\phi \leftrightarrow \neg\Diamond\neg\phi$

$$RE \quad \frac{\phi \leftrightarrow \psi}{\Box\phi \leftrightarrow \Box\psi}$$

M $\Box(\phi \wedge \psi) \rightarrow (\Box\phi \wedge \Box\psi)$

C $(\Box\phi \wedge \Box\psi) \rightarrow \Box(\phi \wedge \psi)$

N $\Box\top$

K $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$

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Fact: $\mathbf{K} = \mathbf{EMCN}$

Constraints on neighborhood frames

- **Monotonic or Supplemented:** If $X \cap Y \in N(w)$, then $X \in N(w)$ and $Y \in N(w)$
- **Closed under finite intersections:** If $X \in N(w)$ and $Y \in N(w)$, then $X \cap Y \in N(w)$
- **Contains the unit:** $W \in N(w)$
- **Augmented:** Supplemented plus for each $w \in W$, $\bigcap N(w) \in N(w)$

Definability Results

1. $\mathcal{F} \models \Box(\phi \wedge \psi) \rightarrow \Box\phi \wedge \Box\psi$ iff $\mathcal{F}$ is closed under supersets (monotonic frames).

2. $\mathcal{F} \models \Box\phi \wedge \Box\psi \rightarrow \Box(\phi \wedge \Box\psi)$ iff $\mathcal{F}$ is closed under finite intersections.

3. $\mathcal{F} \models \Box\top$ iff $\mathcal{F}$ contains the unit

4. $\mathcal{F} \models$ **EMCN** iff $\mathcal{F}$ is a filter

5. $\mathcal{F} \models \Box\phi \rightarrow \phi$ iff for each $w \in W$, $w \in \cap N(w)$

6. And so on...

Completeness Results

- **E** is sound and strongly complete with respect to the class of **all** neighborhood frames
- **EM** is sound and strongly complete with respect to the class of all **monotonic** neighborhood frames
- **EC** is sound and strongly complete with respect to the class of all neighborhood frames that are **closed under finite intersections**
- **EN** is sound and strongly complete with respect to the class of all neighborhood frames that **contain the unit**
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Questions

What is the precise connection between neighborhood semantics for modal logic and relational semantics for modal logic?

What is the expressive power of the basic modal language over *neighborhood frames*?

Some Results

- For each Kripke model $\langle W, R, V\rangle$, there is an pointwise equivalent *augmented* neighborhood model $\langle W, N, V\rangle$, and vice versa (see (Chellas, 1980) for more information).

- There are logics which are complete with respect to a class of neighborhood frames but not complete with respect to relational frames (Gabbay 1975, Gerson 1975, Gerson 1976).

- There are logics incomplete with respect to neighborhood frames (Martin Gerson, 1975; Litak, 200?).

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Kracht-Wolter Translation

Given a neighborhood model $\mathcal{M} = \langle W, \nu, V\rangle$, define a Kripke model $\mathcal{M}^\circ = \langle V, R_\ni, R_{\not\ni}, R_\nu, Pt, V\rangle$ as follows:

- $V = W \cup 2^W$
- $R_\ni = \{(v, w) \mid w \in W, v \in 2^W, v \in w\}$
- $R_{\not\ni} = \{(v, w) \mid w \in W, v \in 2^W, v \notin w\}$
- $R_\nu = \{(w, v) \mid w \in W, v \in 2^W, v \in \nu(w)\}$
- $Pt = W$

Kracht-Wolter Translation

Let $\mathcal{L}'$ be the language

$$\phi := p \mid \neg\phi \mid \phi \wedge \psi \mid [\ni]\phi \mid [\not\ni]\phi \mid [\nu]\phi \mid \mathsf{Pt}$$

where $p \in \mathsf{At}$ and Pt is a unary modal operator.

Define $ST : \mathcal{L}_{NML} \to \mathcal{L}'$ as follows

- $ST(p) = p$
- $ST(\neg\phi) = \neg ST(\phi)$
- $ST(\phi \wedge \psi) = ST(\phi) \wedge ST(\phi)$
- $ST(\Box\phi) = \langle\nu\rangle([\ni]ST(\phi) \wedge [\not\ni]\neg ST(\phi))$

Kracht-Wolter Translation

Theorem For each neighborhood model $\mathcal{M} = \langle W, \nu, V \rangle$ and each formuala $\phi \in \mathcal{L}_{NML}$, for any $w \in W$,

$$\mathcal{M}, w \models \phi \text{ iff } \mathcal{M}^\circ, w \models \mathsf{Pt} \rightarrow ST(\phi)$$

(The translation is simpler if monotonicity is assumed)

Expressive Power of Modal Logic (w.r.t. Relational Frames)

Van Benthem Characterization Theorem On the class of Kripke Structures, Modal Logic is the bisimulation invariant fragment of first-order logic.

Bisimulations for Neighborhood Structures

First Attempt: Let $\mathcal{M} = \langle W, N, V \rangle$ and $\mathcal{M}' = \langle W', N', V' \rangle$ be two neighborhood structures and $s \in W$ and $t \in W'$. A non-empty relation $Z \subseteq W \times W'$ is a bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ if

- (prop) If wZw' then w and w' satisfy the same formulas
- (back) If wZw' and $X \in N(w)$ then there is a $X' \subseteq W'$ such that $X' \in N'(w')$ and $\forall x' \in X' \; \exists x \in X$ such that xZx'
- (forth) If wZw' and $X' \in N'(w')$ then there is a $X \subseteq W$ such that $X \in N(w)$ and $\forall x \in X \; \exists x' \in X'$ such that xZx'

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Only works for monotonic modal logics

Bounded Morphism

Let $\mathcal{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, N_2, V_2 \rangle$ be two neighborhood models. A **bounded morphism** from $\mathcal{M}_1$ to $\mathcal{M}_2$ is a map $f : W_1 \to W_2$ such that for all $X \subseteq W_2$ and $w \in W_1$,

$$f^{-1}[X] \in N_1(w) \text{ iff } \; X \in N_2(f(w))$$

and for all p, $w \in V_1(p)$ iff $f(w) \in V_2(p)$

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Lemma Let $\mathcal{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, N_2, V_2 \rangle$ be two neighborhood models and $f : W_1 \to W_2$ a bounded morphism. Then for each modal formula $\phi \in \mathcal{L}_{NML}$ and state $w \in W_1$,

$$\mathcal{M}_1, w \models \phi \text{ iff } \mathcal{M}_2, f(w) \models \phi$$

Behavorial Equivalence

Two model-state pairs $\mathcal{M}_1, w_1$ and $\mathcal{M}_2, w_2$ are **behavorially equivalent** provided there is a neighborhood model $\mathcal{N} = \langle W, N, V \rangle$ such that there are boudned morphisms from f from $\mathcal{M}_1$ to $\mathcal{N}$ and g from $\mathcal{M}_2$ to $\mathcal{N}$ and $f(w_1) = g(w_2)$.

Two-Sorted First-Order Language for Neighborhood Structures

View $\mathcal{M}^\circ = \langle V, R_\ni, R_{\not\ni}, R_\nu, Pt, V\rangle$ as a **2-sorted** first-order structure.

Let $\mathcal{L}^2$ be a two-sorted first-order language (point variables and "set" variables)

Fact: First-order structures that are generated by neighborhood structures (i.e., of the form $\mathcal{M}^\circ$ for some neighborhood structure $\mathcal{M}$) can be axiomatized (in $\mathcal{L}^2$).

Standard Translation

We map formulas of the basic modal language to $\mathcal{L}^2$:

- $st_x(p) = Px$
- $st_x(\neg\phi) = \neg st_x(\phi)$
- $st_x(\phi \wedge \psi) = st_x(\phi) \wedge st_x(\psi)$.
- $st_x(\Box\phi) = \exists u(xR_\nu u \wedge (\forall z(uR_\ni z \rightarrow st_z(\phi)) \wedge \forall z'(uR_{\not\ni} z' \rightarrow \neg st_{z'}(\phi))$

$$ST_x(\phi) = Wx \rightarrow st_x(\phi)$$

Characterization Theorem for Classical Modal Logic

Theorem (Pauly) On the class of neighborhood models, monotonic modal logic is the monotonic bisimulation invariant fragment of $\mathcal{L}^2$

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Theorem (Pauly) On the class of neighborhood models, monotonic modal logic is the monotonic bisimulation invariant fragment of $\mathcal{L}^2$

Theorem On the class of neighbourhood models, modal logic is the behavioural equivalence-invariant fragment of $\mathcal{L}^2$.

Joint work with Helle Hvid Hansen and Clemens Kupke

Conclusions

- Model theory of modal logic with respect to neighborhood structures
- Monotonic Modal Logics have been studied (eg. Hansen, 2003)
- Large literature on the topological interpretation of modal logic

  Modal Language for Topology: Expressivity and Definability, B. ten Cate, D. Gabelaia and V. Sustretov, 2006.

Thank you.