Heuristic (Informed) Search
(Where we try to choose smartly)

R&N: Chap. 4, Sect. 4.1-3

Best-First Search
- It exploits state description to estimate how “good” each search node is
- An evaluation function $f$ maps each node $N$ of the search tree to a real number $f(N) \geq 0$
  [Traditionally, $f(N)$ is an estimated cost; so, the smaller $f(N)$, the more promising $N$]
- Best-first search sorts the FRINGE in increasing $f$
  [Arbitrary order is assumed among nodes with equal $f$]

How to construct $f$?
- Typically, $f(N)$ estimates:
  - either the cost of a solution path through $N$
    Then $f(N) = g(N) + h(N)$, where
    - $g(N)$ is the cost of the path from the initial node to $N$
    - $h(N)$ is an estimate of the cost of a path from $N$ to a goal node
  - or the cost of a path from $N$ to a goal node
    Then $f(N) = h(N)$ → Greedy best-search
- But there are no limitations on $f$. Any function of your choice is acceptable.
  But will it help the search algorithm?

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  Then $f(N) = h(N)$ → Greedy best-search
- But there are no limitations on $f$. Any function of your choice is acceptable.
  But will it help the search algorithm?
The heuristic function $h(N) \geq 0$ estimates the cost to go from $\text{STATE}(N)$ to a goal state. Its value is independent of the current search tree; it depends only on $\text{STATE}(N)$ and the goal test $\text{GOAL}$.

- **Example:**

  $\begin{bmatrix}
  5 & 8 & 1 & 2 & 3 \\
  4 & 2 & 1 & 4 & 5 & 6 \\
  7 & 3 & 6 & 7 & 8
  \end{bmatrix}$

  $h_1(N) = \text{number of misplaced numbered tiles} = 6$

  [Why is it an estimate of the distance to the goal?]

### Heuristic Function

- **Example:**

  $\begin{bmatrix}
  5 & 8 & 1 & 2 & 3 \\
  4 & 2 & 1 & 4 & 5 & 6 \\
  7 & 3 & 6 & 7 & 8
  \end{bmatrix}$

  $h_1(N) = \text{number of misplaced numbered tiles} = 6$

  $h_2(N) = \sum \text{distances of numbered tiles to their goals}$

  $h_3(N) = \sum \text{permutation inversions}$

- **Other Examples**

  $\begin{bmatrix}
  5 & 8 & 1 & 2 & 3 \\
  4 & 2 & 1 & 4 & 5 & 6 \\
  7 & 3 & 6 & 7 & 8
  \end{bmatrix}$

  - $h_1(N) = \text{number of misplaced numbered tiles} = 6$
  - $h_2(N) = \text{sum of the (Manhattan) distance of every numbered tile to its goal position}$
    \[= 2 + 3 + 0 + 1 + 3 + 0 + 3 + 1 = 13\]
  - $h_3(N) = \text{sum of permutation inversions}$
    \[= n_5 + n_8 + n_4 + n_2 + n_1 + n_7 + n_3 + n_6 = 16\]

### 8-Puzzle

$f(N) = h(N) = \text{number of misplaced numbered tiles}$

- **Diagram:**

  The white tile is the empty tile.

### 8-Puzzle

$f(N) = g(N) + h(N)$

- **Diagram:**

  with $h(N) = \text{number of misplaced numbered tiles}$

### Robot Navigation

$h_1(N) = \sqrt{(x_N - x_g)^2 + (y_N - y_g)^2}$ (L₂ or Euclidean distance)

$h_2(N) = |x_N - x_g| + |y_N - y_g|$ (L₁ or Manhattan distance)
Best-First \(\Rightarrow\) Efficiency

Local-minimum problem

\[ f(N) = h(N) = \text{straight distance to the goal} \]

Can we prove anything?

- If the state space is infinite, in general the search is not complete
- If the state space is finite and we do not discard nodes that revisit states, in general the search is not complete
- If the state space is finite and we discard nodes that revisit states, the search is complete, but in general is not optimal

Admissible Heuristic

- Let \( h^*(N) \) be the cost of the optimal path from \( N \) to a goal node
- The heuristic function \( h(N) \) is admissible if:
  \[ 0 \leq h(N) \leq h^*(N) \]
- An admissible heuristic function is always optimistic!

Admissible Heuristic

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- An admissible heuristic function is always optimistic!

\[ G \text{ is a goal node} \Rightarrow h(G) = 0 \]

8-Puzzle Heuristics

\[ h_1(N) = \text{number of misplaced tiles} = 6 \]

\[ h_2(N) = \text{sum of the (Manhattan) distances of every tile to its goal position} = 2 \times 3 + 0 \times 1 + 3 \times 0 + 3 \times 1 = 13 \]

\[ h_3(N) = \text{sum of permutation inversions} \]

\[ h_4(N) = \text{sum of Manhattan distances of tiles} \]

\[ h_5(N) = \text{number of tiles not in original position} \]

\[ h_6(N) = \text{number of tiles not in final position} \]

\[ h_7(N) = \text{number of tiles not in the same column} \]

\[ h_8(N) = \text{number of tiles not in the same row} \]

\[ h_9(N) = \text{number of tiles not in the same row or column} \]

\[ h_{10}(N) = \text{number of tiles not in the same position} \]

\[ h_{11}(N) = \text{number of tiles not in the same row or column} \]

\[ h_{12}(N) = \text{number of tiles not in the same position} \]

\[ h_{13}(N) = \text{number of tiles not in the same row} \]

\[ h_{14}(N) = \text{number of tiles not in the same column} \]

\[ h_{15}(N) = \text{number of tiles not in the same row or column} \]

\[ h_{16}(N) = \text{number of tiles not in the same position} \]

\[ h_{17}(N) = \text{number of tiles not in the same row or column} \]

\[ h_{18}(N) = \text{number of tiles not in the same position} \]

\[ h_{19}(N) = \text{number of tiles not in the same row or column} \]

\[ h_{20}(N) = \text{number of tiles not in the same position} \]
8-Puzzle Heuristics

- $h_1(N)$ = number of misplaced tiles = 6 is admissible
- $h_2(N)$ = sum of the (Manhattan) distances of every tile to its goal position
  
  $= 2 + 3 + 0 + 1 + 3 + 0 + 3 + 1 = 13$ is admissible
- $h_3(N)$ = sum of permutation inversions
  
  $= 4 + 6 + 3 + 1 + 0 + 2 + 0 + 0 = 16$ is not admissible

Goal state

STATE(N)

1 2 3
4 2 1
5 6

8-Puzzle Heuristics

- $h_1(N)$ = number of misplaced tiles = 6 is admissible
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  $= 2 + 3 + 0 + 1 + 3 + 0 + 3 + 1 = 13$ is admissible
- $h_3(N)$ = sum of permutation inversions
  
  $= 4 + 6 + 3 + 1 + 0 + 2 + 0 + 0 = 16$ is not admissible

Goal state

STATE(N)

1 2 3
4 2 1
5 6

Robot Navigation Heuristics

Cost of one horizontal/vertical step = 1
Cost of one diagonal step = $\sqrt{2}$

$h_1(N) = \sqrt{(x_N-x_g)^2 + (y_N-y_g)^2}$ is admissible

Goal state

STATE(N)

1 2 3
4 2 1
5 6

Robot Navigation Heuristics

Cost of one horizontal/vertical step = 1
Cost of one diagonal step = $\sqrt{2}$

$h_2(N) = |x_N-x_g| + |y_N-y_g|$ is not admissible

Goal state

STATE(N)

1 2 3
4 2 1
5 6

How to create an admissible $h$?

- An admissible heuristic can usually be seen as the cost of an optimal solution to a relaxed problem (one obtained by removing constraints)
- In robot navigation:
  - The Manhattan distance corresponds to removing the obstacles
  - The Euclidean distance corresponds to removing both the obstacles and the constraint that the robot moves on a grid
- More on this topic later
A* Search
(most popular algorithm in AI)

1) \( f(N) = g(N) + h(N) \), where:
   - \( g(N) \) = cost of best path found so far to \( N \)
   - \( h(N) \) = admissible heuristic function
2) for all arcs: \( c(N,N') \geq \varepsilon > 0 \)
3) SEARCH#2 algorithm is used

\( \Rightarrow \) Best-first search is then called A* search

Result #1

A* is complete and optimal
[This result holds if nodes revisiting states are not discarded]

Proof (1/2)
1) If a solution exists, A* terminates and returns a solution
   - For each node \( N \) on the fringe, \( f(N) = g(N) + h(N) \geq g(N) \geq d(N) \times \varepsilon \),
     where \( d(N) \) is the depth of \( N \) in the tree

Proof (2/2)
2) Whenever A* chooses to expand a goal node, the path to this node is optimal
   - \( C^* \) = cost of the optimal solution path
   - \( G' \): non-optimal goal node in the fringe
     \( f(G') = g(G') + h(G') = g(G') > C^* \)
   - A node \( K \) in the fringe lies on an optimal path:
     \( f(K) = g(K) + h(K) \leq C^* \)
   - So, \( G' \) will not be selected for expansion
**Time Limit Issue**

- When a problem has no solution, A* runs for ever if the state space is infinite. In other cases, it may take a huge amount of time to terminate.
- So, in practice, A* is given a time limit. If it has not found a solution within this limit, it stops. Then there is no way to know if the problem has no solution, or if more time was needed to find it.
- When AI systems are “small” and solving a single search problem at a time, this is not too much of a concern.
- When AI systems become larger, they solve many search problems concurrently, some with no solution. What should be the time limit for each of them? More on this in the lecture on Motion Planning ...

**8-Puzzle**

\[ f(N) = g(N) + h(N) \]

with \( h(N) \) = number of misplaced tiles

**Robot Navigation**

\[ f(N) = g(N) + h(N) \]

with \( h(N) \) = Manhattan distance to goal (A*)

\[
\begin{bmatrix}
8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\end{bmatrix}
\]
**Best-First Search**

- An evaluation function \( f \) maps each node \( N \) of the search tree to a real number \( f(N) \geq 0 \)
- Best-first search sorts the FRINGE in increasing \( f \)

**A* Search**

1) \( f(N) = g(N) + h(N) \), where:
   - \( g(N) \): cost of best path found so far to \( N \)
   - \( h(N) \): admissible heuristic function
2) for all arcs: \( c(N,N') \geq \varepsilon > 0 \)
3) SEARCH#2 algorithm is used

⇒ Best-first search is then called A* search

**Result #1**

A* is **complete and optimal**

[This result holds if nodes revisiting states are not discarded]

**What to do with revisited states?**

- If we discard this new node, then the search algorithm expands the goal node next and returns a non-optimal solution
- Instead, if we do not discard nodes revisiting states, the search terminates with an optimal solution

The heuristic \( h \) is clearly admissible
But ... 

If we do not discard nodes revisiting states, the size of the search tree can be exponential in the number of visited states.

It is not harmful to discard a node revisiting a state if the cost of the new path to this state is ≥ cost of the previous path [so, in particular, one can discard a node if it revisits a state already visited by one of its ancestors].

A* remains optimal, but states can still be revisited multiple times [the size of the search tree can still be exponential in the number of visited states].

Fortunately, for a large family of admissible heuristics – consistent heuristics – there is a much more efficient way to handle revisited states.

**Consistent Heuristic**

An admissible heuristic h is **consistent** (or **monotone**) if for each node N and each child N' of N:

\[ h(N) \leq c(N,N') + h(N') \]

(triangle inequality)

Intuition: a consistent heuristic becomes more precise as we get deeper in the search tree.

**Consistency Violation**

If h tells that N is 100 units from the goal, then moving from N along an arc costing 10 units should not lead to a node N' that h estimates to be 10 units away from the goal.

**Consistent Heuristic** (alternative definition)

A heuristic h is **consistent** (or **monotone**) if

1) for each node N and each child N' of N:

\[ h(N) \leq c(N,N') + h(N') \]

2) for each goal node G:

\[ h(G) = 0 \]

A consistent heuristic is also admissible.
Admissibility and Consistency

- A consistent heuristic is also admissible
- An admissible heuristic may not be consistent, but many admissible heuristics are consistent

8-Puzzle

- $h_1(N) = \text{number of misplaced tiles}$
- $h_2(N) = \text{sum of the (Manhattan) distances of every tile to its goal position}$

Robotic Navigation

Result #2

If $h$ is consistent, then whenever $A^*$ expands a node, it has already found an optimal path to this node’s state.

Proof (1/2)

1) Consider a node $N$ and its child $N’$.
   Since $h$ is consistent: $h(N) \leq c(N,N’)+h(N’)$
   
   $f(N) = g(N)+h(N) \leq g(N)+c(N,N’)+h(N’)$ = $f(N’)$
   
   So, $f$ is non-decreasing along any path

Proof (2/2)

2) If a node $K$ is selected for expansion, then any other node $N$ in the fringe verifies $f(N) \geq f(K)$

If one node lies on another path to the state of $K$, the cost of this other path is no smaller than that of the path to $K$: $f(N) \geq f(N') \geq f(K)$ and $h(N') = h(K)$

So, $g(N) \geq g(K)$
If a node K is selected for expansion, then any other node N in the fringe verifies $f(N) \geq f(K)$.

**Proof (2/2)**

If $h$ is consistent, then whenever A* expands a node, it has already found an optimal path to this node's state.

Result #2

If one node N lies on another path to the state of K, the cost of this other path is no smaller than that of the path to K:

$f(N) \geq f(K)$

So, $g(N) \geq g(K)$

**Implication of Result #2**

The path to N is the optimal path to S.

If one node N lies on another path to the state of K, the cost of this other path is no smaller than that of the path to K:

$f(N') \geq f(N) \geq f(K) \\
\text{and} \\
h(N') = h(K)$

So, $g(N') \geq g(K)$

**Revisited States with Consistent Heuristic**

- When a node is expanded, store its state into CLOSED.
- When a new node N is generated:
  - If STATE(N) is in CLOSED, discard N.
  - If there exists a node N' in the fringe such that STATE(N') = STATE(N), discard the node - N or N' - with the largest $f$ (or, equivalently, $g$).

**Is A* with some consistent heuristic all that we need?**

No!

There are very dumb consistent heuristic functions.

**Heuristic Accuracy**

Let $h_1$ and $h_2$ be two consistent heuristics such that for all nodes N:

$h_1(N) \leq h_2(N)$

$h_2$ is said to be more accurate (or more informed) than $h_1$.

- $h_2(N) =$ number of misplaced tiles
- $h_2(N) =$ sum of distances of every tile to its goal position
- $h_2$ is more accurate than $h_1$.
Result #3

- Let $h_2$ be more accurate than $h_1$
- Let $A_1^*$ be A* using $h_1$
- and $A_2^*$ be A* using $h_2$
- Whenever a solution exists, all the nodes expanded by $A_2^*$, except possibly for some nodes such that $f_1(N) = f_2(N) = C^*$ (cost of optimal solution) are also expanded by $A_1^*$

Proof

- $C^* = h^*(\text{initial-node})$ (cost of optimal solution)
- Every node $N$ such that $f(N) < C^*$ is eventually expanded. No node $N$ such that $f(N) > C^*$ is ever expanded
- Every node $N$ such that $h(N) < C^* - g(N)$ is eventually expanded. So, every node $N$ such that $h_1(N) < C^* - g(N)$ is expanded by $A_2^*$. Since $h_1(N) \leq h_2(N)$, $N$ is also expanded by $A_1^*$
- If there are several nodes $N$ such that $f_1(N) = f_2(N) = C^*$ (such nodes include the optimal goal nodes, if there exists a solution), $A_1^*$ and $A_2^*$ may or may not expand them in the same order (until one goal node is expanded)

Effective Branching Factor

- It is used as a measure the effectiveness of a heuristic
- Let $n$ be the total number of nodes expanded by A* for a particular problem and $d$ the depth of the solution
- The effective branching factor $b^*$ is defined by $n = 1 + b^* + (b^*)^2 + ... + (b^*)^d$

Experimental Results

- 8-puzzle with:
  - $h_1 =$ number of misplaced tiles
  - $h_2 =$ sum of distances of tiles to their goal positions
- Random generation of many problem instances
- Average effective branching factors (number of expanded nodes):

<table>
<thead>
<tr>
<th>$d$</th>
<th>$A_1^*$</th>
<th>$A_2^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2.45</td>
<td>1.79</td>
</tr>
<tr>
<td>6</td>
<td>2.73</td>
<td>1.34</td>
</tr>
<tr>
<td>12</td>
<td>2.78 (3,644,035)</td>
<td>1.42 (227)</td>
</tr>
<tr>
<td>16</td>
<td></td>
<td>1.45</td>
</tr>
<tr>
<td>20</td>
<td></td>
<td>1.47</td>
</tr>
<tr>
<td>24</td>
<td></td>
<td>1.48 (39,135)</td>
</tr>
</tbody>
</table>

Can we do better?

- For example, we could consider two more complex relaxed problems:
  - $d_{2134} =$ length of the shortest path to move tiles 1, 2, 3, and 4 to their goal positions, ignoring the other tiles, e.g., $d_2 = 2$
  - $h_2 = \Sigma_{i=1..8} d_i$
  - $\Rightarrow h = d_{2134} + d_{678}$ [disjoint pattern heuristic]
For example, we could consider two more complex relaxed problems:

\[ d_{1234} = \text{length of the shortest path to move tiles 1, 2, 3, and 4 to their goal positions, ignoring the other tiles} \]

\[ d_{5678} \]

\[ h = d_{1234} + d_{5678} \] \[ \text{[disjoint pattern heuristic]} \]

How to compute \( d_{1234} \) and \( d_{5678} \)?

Several order-of-magnitude speedups for the 15- and 24-puzzle (see R&N)

\[ h = d_{1234} + d_{5678} \] \[ \text{[disjoint pattern heuristic]} \]

These distances are pre-computed and stored

(Each requires generating a tree of 3,024 nodes/states (breadth-first search))

On Completeness and Optimality

A* with a consistent heuristic function has nice properties: completeness, optimality, no need to revisit states

Theoretical completeness does not mean “practical” completeness if you must wait too long to get a solution (remember the time limit issue)

So, if one can’t design an accurate consistent heuristic, it may be better to settle for a non-admissible heuristic that “works well in practice”, even through completeness and optimality are no longer guaranteed

Iterative Deepening A* (IDA*)

Idea: Reduce memory requirement of A* by applying cutoff on values of \( f \)

Consistent heuristic function \( h \)

Algorithm IDA*:
1. Initialize cutoff to \( f(\text{initial-node}) \)
2. Repeat:
   a. Perform depth-first search by expanding all nodes \( N \) such that \( f(N) \leq \text{cutoff} \)
   b. Reset cutoff to smallest value \( f \) of non-expanded (leaf) nodes

8-Puzzle

\[ f(N) = g(N) + h(N) \]

with \( h(N) = \text{number of misplaced tiles} \)
8-Puzzle
f(N) = g(N) + h(N)
with h(N) = number of misplaced tiles

Cutoff=4

8-Puzzle
f(N) = g(N) + h(N)
with h(N) = number of misplaced tiles

Cutoff=4

8-Puzzle
f(N) = g(N) + h(N)
with h(N) = number of misplaced tiles

Cutoff=4

8-Puzzle
f(N) = g(N) + h(N)
with h(N) = number of misplaced tiles

Cutoff=5

8-Puzzle
f(N) = g(N) + h(N)
with h(N) = number of misplaced tiles

Cutoff=5
8-Puzzle

\[ f(N) = g(N) + h(N) \]

with \( h(N) = \) number of misplaced tiles

Cutoff=5

Advantages/Drawbacks of IDA*

**Advantages:**
- Still complete and optimal
- Requires less memory than A*
- Avoid the overhead to sort the fringe

**Drawbacks:**
- Can't avoid revisiting states not on the current path
- Available memory is poorly used
  (Often memory-bounded search, see R&N p. 101-104)
Local Search

- Light-memory search method
- No search tree; only the current state is represented!
- Only applicable to problems where the path is irrelevant (e.g., 8-queen), unless the path is encoded in the state
- Many similarities with optimization techniques

Steepest Descent

1) $S \leftarrow$ initial state
2) Repeat:
   a) $S' \leftarrow \arg \min_{S' \in \text{SUCCESSORS}(S)} \{h(S')\}$
   b) if GOAL?(S') return S'
   c) if $h(S') < h(S)$ then $S \leftarrow S'$ else return failure

Similar to:
- hill climbing with $-h$
- gradient descent over continuous space

Application: 8-Queen

1) Pick an initial state $S$ at random with one queen in each column
2) Repeat $k$ times:
   a) If GOAL?(S) then return S
   b) Pick an attacked queen Q at random
   c) Move Q in its column to minimize the number of attacking queens $\rightarrow$ new S [min-conflicts heuristic]
3) Return failure

Why does it work ???

1) There are many goal states that are well-distributed over the state space
2) If no solution has been found after a few steps, it's better to start it all over again. Building a search tree would be much less efficient because of the high branching factor
3) Running time almost independent of the number of queens

Steepest Descent

1) $S \leftarrow$ initial state
2) Repeat:
   a) $S' \leftarrow \arg \min_{S' \in \text{SUCCESSORS}(S)} \{h(S')\}$
   b) if GOAL?(S') return S'
   c) if $h(S') < h(S)$ then $S \leftarrow S'$ else return failure

may easily get stuck in local minima
$\rightarrow$ Random restart (as in n-queen example)
$\rightarrow$ Monte Carlo descent

Monte Carlo Descent

1) $S \leftarrow$ initial state
2) Repeat $k$ times:
   a) If GOAL?(S) then return S
   b) $S' \leftarrow$ successor of $S$ picked at random
   c) if $h(S') \leq h(S)$ then $S \leftarrow S'$
   d) else $\Delta h = h(S') - h(S)$
      with probability $\sim \exp(-\Delta h/T)$, where $T$ is called the "temperature", do: $S \leftarrow S'$ [Metropolis criterion]
3) Return failure

Simulated annealing lowers $T$ over the $k$ iterations. It starts with a large $T$ and slowly decreases $T$
“Parallel” Local Search Techniques
They perform several local searches concurrently, but not independently:
- Beam search
- Genetic algorithms

See R&N, pages 115-119

When to Use Search Techniques?
1) The search space is small, and
   - No other technique is available, or
   - Developing a more efficient technique is not worth the effort
2) The search space is large, and
   - No other available technique is available, and
   - There exist “good” heuristics