Constraint Satisfaction Problems (CSP)

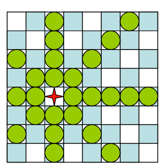
(Where we postpone making difficult decisions until they become easy to make)

R&N: Chap. 5

What we will try to do ...

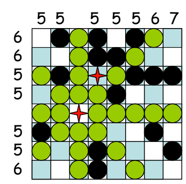
- Search techniques make choices in an often arbitrary order. Often little information is available to make each of them
- In many problems, the same states can be reached independent of the order in which choices are made ("commutative" actions)
- Can we solve such problems more efficiently by picking the order appropriately? Can we even avoid making any choice?

Constraint Propagation



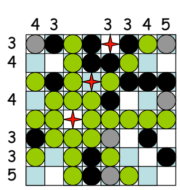
- Place a queen in a square
- Remove the attacked squares from future consideration

Constraint Propagation



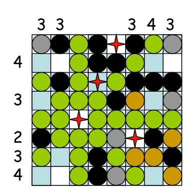
- Count the number of non-attacked squares in every row and column
- Place a queen in a row or column with minimum number
- Remove the attacked squares from future consideration

Constraint Propagation

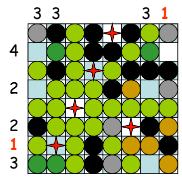


- Repeat

Constraint Propagation

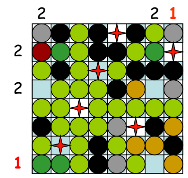


Constraint Propagation



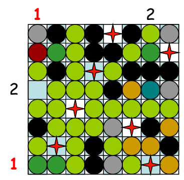
7

Constraint Propagation



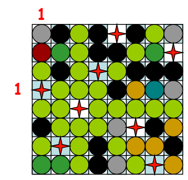
8

Constraint Propagation



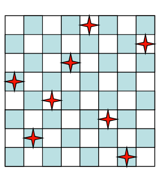
9

Constraint Propagation



10

Constraint Propagation



11

What do we need?

- More than just a successor function and a goal test
- We also need:
  - A means to propagate the constraints imposed by one queen's position on the positions of the other queens
  - An early failure test

→ Explicit representation of constraints
→ Constraint propagation algorithms

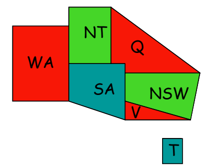
12

Constraint Satisfaction Problem (CSP)

- Set of variables $\{X_1, X_2, \ldots, X_n\}$
- Each variable X_i has a domain D_i of possible values. Usually, D_i is finite
- Set of constraints $\{C_1, C_2, \ldots, C_p\}$
- Each constraint relates a subset of variables by specifying the valid combinations of their values
- Goal: Assign a value to every variable such that all constraints are satisfied

13

Map Coloring



- 7 variables {WA,NT,SA,Q,NSW,V,T}
- Each variable has the same domain: {red, green, blue}
- No two adjacent variables have the same value:
  WA≠NT, WA≠SA, NT≠SA, NT≠Q, SA≠Q, SA≠NSW, SA≠V, Q≠NSW, NSW≠V

14

8-Queen Problem

- 8 variables X_i, i = 1 to 8
- The domain of each variable is: {1,2,…,8}
- Constraints are of the forms:
  - $X_i = k \rightarrow X_j \neq k$ for all j = 1 to 8, j≠i
  - Similar constraints for diagonals

All constraints are binary

15

Street Puzzle

1 2 3 4 5

N_i = {English, Spaniard, Japanese, Italian, Norwegian}
C_i = {Red, Green, White, Yellow, Blue}
D_i = {Tea, Coffee, Milk, Fruit-juice, Water}
J_i = {Painter, Sculptor, Diplomat, Violinist, Doctor}
A_i = {Dog, Snails, Fox, Horse, Zebra}

Who owns the Zebra?
Who drinks Water?

The Englishman lives in the Red house
The Spaniard has a Dog
The Japanese is a Painter
The Italian drinks Tea
The Norwegian lives in the first house on the left
The owner of the Green house drinks Coffee
The Green house is on the right of the White house
The Sculptor breeds Snails
The Diplomat lives in the Yellow house
The owner of the middle house drinks Milk
The Norwegian lives next door to the Blue house
The Violinist drinks Fruit juice
The Fox is in the house next to the Doctor's
The Horse is next to the Diplomat's

16

Street Puzzle

1 2 3 4 5

N_i = {English, Spaniard, Japanese, Italian, Norwegian}
C_i = {Red, Green, White, Yellow, Blue}
D_i = {Tea, Coffee, Milk, Fruit-juice, Water}
J_i = {Painter, Sculptor, Diplomat, Violinist, Doctor}
A_i = {Dog, Snails, Fox, Horse, Zebra}

$\forall i,j \in [1,5], i \neq j, N_i \neq N_j$
$\forall i,j \in [1,5], i \neq j, C_i \neq C_j$
...

The Englishman lives in the Red house
The Spaniard has a Dog
The Japanese is a Painter
The Italian drinks Tea
The Norwegian lives in the first house on the left
The owner of the Green house drinks Coffee
The Green house is on the right of the White house
The Sculptor breeds Snails
The Diplomat lives in the Yellow house
The owner of the middle house drinks Milk
The Norwegian lives next door to the Blue house
The Violinist drinks Fruit juice
The Fox is in the house next to the Doctor's
The Horse is next to the Diplomat's

17

Street Puzzle

1 2 3 4 5

N_i = {English, Spaniard, Japanese, Italian, Norwegian}
C_i = {Red, Green, White, Yellow, Blue}
D_i = {Tea, Coffee, Milk, Fruit-juice, Water}
J_i = {Painter, Sculptor, Diplomat, Violinist, Doctor}
A_i = {Dog, Snails, Fox, Horse, Zebra}

The Englishman lives in the Red house → $(N_i = \text{English}) \Leftrightarrow (C_i = \text{Red})$
The Spaniard has a Dog
The Japanese is a Painter → $(N_i = \text{Japanese}) \Leftrightarrow (J_i = \text{Painter})$
The Italian drinks Tea
The Norwegian lives in the first house on the left → $(N_1 = \text{Norwegian})$
The owner of the Green house drinks Coffee
The Green house is on the right of the White house → $(C_i = \text{White}) \Leftrightarrow (C_{i+1} = \text{Green})$, $(C_5 \neq \text{White})$, $(C_1 \neq \text{Green})$
The Sculptor breeds Snails
The Diplomat lives in the Yellow house
The owner of the middle house drinks Milk
The Norwegian lives next door to the Blue house → left as an exercise
The Violinist drinks Fruit juice
The Fox is in the house next to the Doctor's
The Horse is next to the Diplomat's

18

Street Puzzle

1 2 3 4 5

N_i = {English, Spaniard, Japanese, Italian, Norwegian}
C_i = {Red, Green, White, Yellow, Blue}
D_i = {Tea, Coffee, Milk, Fruit-juice, Water}
J_i = {Painter, Sculptor, Diplomat, Violinist, Doctor}
A_i = {Dog, Snails, Fox, Horse, Zebra}

The Englishman lives in the Red house → $(N_i = \text{English}) \Leftrightarrow (C_i = \text{Red})$
The Spaniard has a Dog
The Japanese is a Painter → $(N_i = \text{Japanese}) \Leftrightarrow (J_i = \text{Painter})$
The Italian drinks Tea
The Norwegian lives in the first house on the left → $(N_1 = \text{Norwegian})$
The owner of the Green house drinks Coffee
The Green house is on the right of the White house → $(C_i = \text{White}) \Leftrightarrow (C_{i+1} = \text{Green})$, $(C_5 \neq \text{White})$, $(C_1 \neq \text{Green})$
The Sculptor breeds Snails
The Diplomat lives in the Yellow house
The owner of the middle house drinks Milk
The Norwegian lives next door to the Blue house
The Violinist drinks Fruit juice
The Fox is in the house next to the Doctor's
The Horse is next to the Diplomat's

unary constraints

19

Street Puzzle

1 2 3 4 5

N_i = {English, Spaniard, Japanese, Italian, Norwegian}
C_i = {Red, Green, White, Yellow, Blue}
D_i = {Tea, Coffee, Milk, Fruit-juice, Water}
J_i = {Painter, Sculptor, Diplomat, Violinist, Doctor}
A_i = {Dog, Snails, Fox, Horse, Zebra}

The Englishman lives in the Red house
The Spaniard has a Dog
The Japanese is a Painter
The Italian drinks Tea
The Norwegian lives in the first house on the left → N_1 = Norwegian
The owner of the Green house drinks Coffee
The Green house is on the right of the White house
The Sculptor breeds Snails
The Diplomat lives in the Yellow house
The owner of the middle house drinks Milk → D_3 = Milk
The Norwegian lives next door to the Blue house
The Violinist drinks Fruit juice
The Fox is in the house next to the Doctor's
The Horse is next to the Diplomat's

20

Street Puzzle

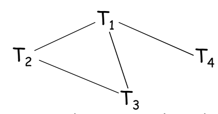
1 2 3 4 5

N_i = {English, Spaniard, Japanese, Italian, Norwegian}
C_i = {Red, Green, White, Yellow, Blue}
D_i = {Tea, Coffee, Milk, Fruit-juice, Water}
J_i = {Painter, Sculptor, Diplomat, Violinist, Doctor}
A_i = {Dog, Snails, Fox, Horse, Zebra}

The Englishman lives in the Red house → $C_1 \neq$ Red
The Spaniard has a Dog → $A_1 \neq$ Dog
The Japanese is a Painter
The Italian drinks Tea
The Norwegian lives in the first house on the left → N_1 = Norwegian
The owner of the Green house drinks Coffee
The Green house is on the right of the White house
The Sculptor breeds Snails
The Diplomat lives in the Yellow house
The owner of the middle house drinks Milk → D_3 = Milk
The Norwegian lives next door to the Blue house
The Violinist drinks Fruit juice → $J_3 \neq$ Violinist
The Fox is in the house next to the Doctor's
The Horse is next to the Diplomat's

21

Task Scheduling



Four tasks T_1, T_2, T_3, and T_4 are related by time constraints:

- T_1 must be done during T_3
- T_2 must be achieved before T_1 starts
- T_2 must overlap with T_3
- T_4 must start after T_1 is complete

- Are the constraints compatible?
- What are the possible time relations between two tasks?
- What if the tasks use resources in limited supply?

How to formulate this problem as a CSP?

22

3-SAT

- n Boolean variables $u_1, \ldots, u_n$
- p constraints of the form
  $$u_i^* \vee u_j^* \vee u_k^* = 1$$
  where u* stands for either u or $\neg u$
- Known to be NP-complete

23

Finite vs. Infinite CSP

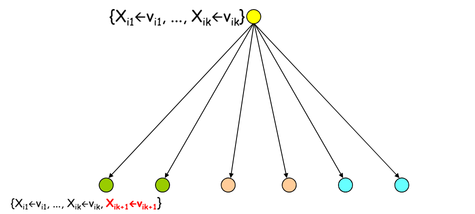
- Finite CSP: each variable has a finite domain of values
- Infinite CSP: some or all variables have an infinite domain
  E.g., linear programming problems over the reals:
  for i = 1, 2, ..., p : $a_{i,1}x_1 + a_{i,2}x_2 + \ldots + a_{i,n}x_n = a_{i,0}$
  for j = 1, 2, ..., q : $b_{j,1}x_1 + b_{j,2}x_2 + \ldots + b_{j,n}x_n \leq b_{j,0}$
- We will only consider finite CSP

24

CSP as a Search Problem

- n variables $X_1, \ldots, X_n$
- Valid assignment: $\{X_{i1} \leftarrow v_{i1}, \ldots, X_{ik} \leftarrow v_{ik}\}$, $0 \le k \le n$, such that the values $v_{i1}, \ldots, v_{ik}$ satisfy all constraints relating the variables $X_{i1}, \ldots, X_{ik}$
- Complete assignment: one where k = n
  [if all variable domains have size d, there are $O(d^n)$ complete assignments]
- States: valid assignments
- Initial state: empty assignment {}, i.e. k = 0
- Successor of a state:
  $\{X_{i1}\leftarrow v_{i1}, \ldots, X_{ik}\leftarrow v_{ik}\} \rightarrow \{X_{i1}\leftarrow v_{i1}, \ldots, X_{ik}\leftarrow v_{ik}, X_{ik+1}\leftarrow v_{ik+1}\}$
- Goal test: k = n

25



r = n–k variables with s values → r×s branching factor

26

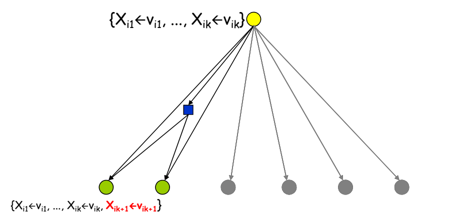
A Key property of CSP: Commutativity

The order in which variables are assigned values has no impact on the reachable complete valid assignments

Hence:

1) One can expand a node N by first selecting **one** variable X not in the assignment A associated with N and then assigning every value v in the domain of X
   [→ big reduction in branching factor]

27



r = n–k variables with s values → s branching factor

The depth of the solutions in the search tree is un-changed (n)

28

- 4 variables $X_1, \ldots, X_4$
- Let the valid assignment of N be:
  $A = \{X_1 \leftarrow v_1, X_3 \leftarrow v_3\}$
- For example pick variable X_4
- Let the domain of X_4 be $\{v_{4,1}, v_{4,2}, v_{4,3}\}$
- The successors of A are all the valid assignments among:
  $\{X_1 \leftarrow v_1, X_3 \leftarrow v_3 , X_4 \leftarrow v_{4,1}\}$
  $\{X_1 \leftarrow v_1, X_3 \leftarrow v_3 , X_4 \leftarrow v_{4,2}\}$
  $\{X_1 \leftarrow v_1, X_3 \leftarrow v_3 , X_4 \leftarrow v_{4,2}\}$

29

A Key property of CSP: Commutativity

The order in which variables are assigned values has no impact on the reachable complete valid assignments

Hence:

1) One can expand a node N by first selecting **one** variable X not in the assignment A associated with N and then assigning every value v in the domain of X
   [→ big reduction in branching factor]
2) One need not store the path to a node
   → Backtracking search algorithm

30

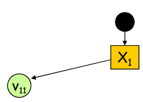
Backtracking Search

Essentially a simplified depth-first algorithm using recursion

Backtracking Search
(3 variables)

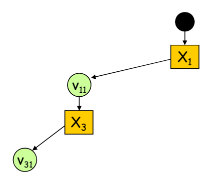
Assignment = {}

Backtracking Search
(3 variables)



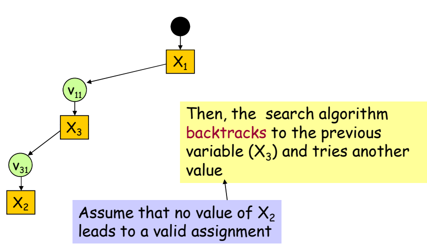
Assignment = $\{(X_1,v_{11})\}$

Backtracking Search
(3 variables)



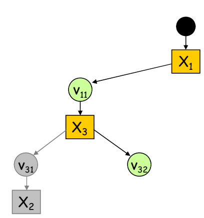
Assignment = $\{(X_1,v_{11}), (X_3,v_{31})\}$

Backtracking Search
(3 variables)



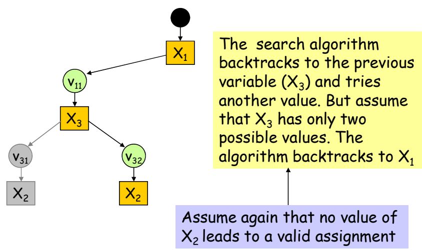
Assignment = $\{(X_1,v_{11}), (X_3,v_{31})\}$

Backtracking Search
(3 variables)



Assignment = $\{(X_1,v_{11}), (X_3,v_{32})\}$

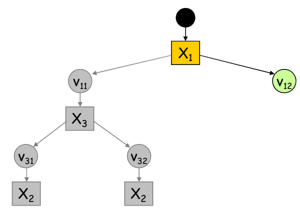
Backtracking Search
(3 variables)



Assignment = $\{(X_1,v_{11}), (X_3,v_{32})\}$

37

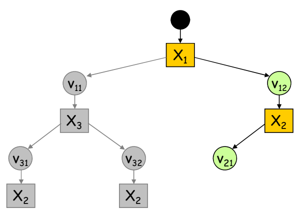
Backtracking Search
(3 variables)



Assignment = $\{(X_1,v_{12})\}$

38

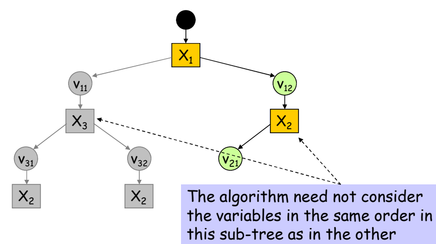
Backtracking Search
(3 variables)



Assignment = $\{(X_1,v_{12}), (X_2,v_{21})\}$

39

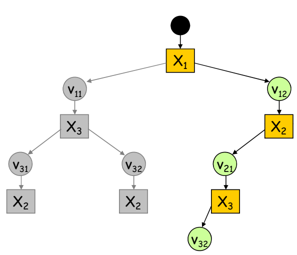
Backtracking Search
(3 variables)



Assignment = $\{(X_1,v_{12}), (X_2,v_{21})\}$

40

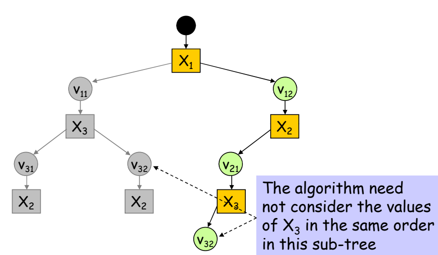
Backtracking Search
(3 variables)



Assignment = $\{(X_1,v_{12}), (X_2,v_{21}), (X_3,v_{32})\}$

41

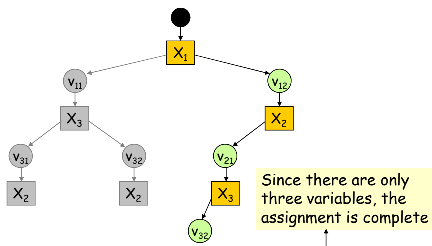
Backtracking Search
(3 variables)



Assignment = $\{(X_1,v_{12}), (X_2,v_{21}), (X_3,v_{32})\}$

42

Backtracking Search
(3 variables)



Assignment = {(X_1,v_{12}), (X_2,v_{21}), (X_3,v_{32})}

43

Backtracking Algorithm

CSP-BACKTRACKING(A)

1. If assignment A is complete then return A
2. X ← select a variable not in A
3. D ← select an ordering on the domain of X
4. For each value v in D do
   a. Add (X←v) to A
   b. If A is valid then
      i. result ← CSP-BACKTRACKING(A)
      ii. If result ≠ failure then return result
   c. Remove (X←v) from A
5. Return failure

Call CSP-BACKTRACKING({})

[This recursive algorithm keeps too much data in memory. An iterative version could save memory (left as an exercise)]

44

Critical Questions for the Efficiency of CSP-Backtracking

CSP-BACKTRACKING(A)

1. If assignment A is complete then return A
2. X ← **select** a variable not in A
3. D ← **select** an ordering on the domain of X
4. For each value v in D do
   a. Add (X←v) to A
   b. If a is valid then
      i. result ← CSP-BACKTRACKING(A)
      ii. If result ≠ failure then return result
   c. Remove (X←v) from A
5. Return failure

45

Critical Questions for the Efficiency of CSP-Backtracking

1) Which variable X should be assigned a value next?

2) In which order should X's values be assigned?

46

Critical Questions for the Efficiency of CSP-Backtracking

1) Which variable X should be assigned a value next?
The current assignment may not lead to any solution, but the algorithm does not know it yet. Selecting the right variable X may help discover the contradiction more quickly

2) In which order should X's values be assigned?

47

Critical Questions for the Efficiency of CSP-Backtracking

1) Which variable X should be assigned a value next?
The current assignment may not lead to any solution, but the algorithm does not know it yet. Selecting the right variable X may help discover the contradiction more quickly

2) In which order should X's values be assigned?
The current assignment may be part of a solution. Selecting the right value to assign to X may help discover this solution more quickly

48

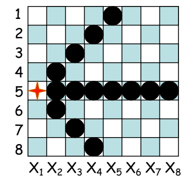
Critical Questions for the Efficiency of CSP-Backtracking

1) Which variable X should be assigned a value next?
   The current assignment may not lead to any solution, but the algorithm does not know it yet. Selecting the right variable X may help discover the contradiction more quickly
2) In which order should X's values be assigned?
   The current assignment may be part of a solution. Selecting the right value to assign to X may help discover this solution more quickly

More on these questions very soon ...

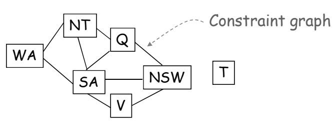
Forward Checking

A simple constraint-propagation technique:



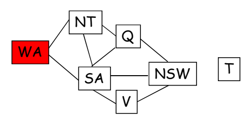
Assigning the value 5 to X_1 leads to removing values from the domains of $X_2, X_3, ..., X_8$

Forward Checking in Map Coloring



WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB

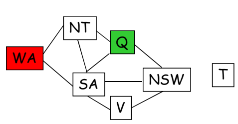
Forward Checking in Map Coloring



WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	~~R~~GB	RGB	RGB	RGB	~~R~~GB	RGB

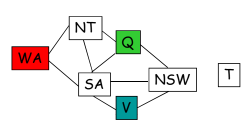
Forward checking removes the value Red of NT and of SA

Forward Checking in Map Coloring



WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	GB	RGB	RGB	RGB	GB	RGB
R	~~G~~B	G	R~~G~~B	RGB	~~G~~B	RGB

Forward Checking in Map Coloring



WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	GB	RGB	RGB	RGB	GB	RGB
R	B	G	RB	RGB	B	RGB
R	B	G	R~~B~~	B	~~B~~	RGB

Forward Checking in Map Coloring

Empty set: the current assignment
{(WA ← R), (Q ← G), (V ← B)}
does not lead to a solution

WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	GB	RGB	RGB	RGB	GB	RGB
R	B	G	RB	RGB	B	RGB
R	B	G	R~~B~~	B	~~B~~	RGB

Forward Checking (General Form)

Whenever a pair (X←v) is added to assignment A do:

For each variable Y not in A do:

For every constraint C relating Y to the variables in A do:

Remove all values from Y's domain that do not satisfy C

Modified Backtracking Algorithm

CSP-BACKTRACKING(A, var-domains)

1. If assignment A is complete then return A
2. X ← select a variable not in A
3. D ← select an ordering on the domain of X
4. For each value v in D do
   a. Add (X←v) to A
   b. var-domains ← forward checking(var-domains, X, v, A)
   c. If no variable has an empty domain then
      (i) result ← CSP-BACKTRACKING(A, var-domains)
      (ii) If result ≠ failure then return result
   d. Remove (X←v) from A
5. Return failure

Modified Backtracking Algorithm

CSP-BACKTRACKING(A, var-domains)

1. If assignment A is complete then return A
2. X ← select a variable not in A
3. D ← select an ordering on the domain of X
4. For each value v in D do
   a. Add (X←v) to A
   b. var-domains ← forward checking(var-domains, X, v, A)
   c. If no variable has an empty domain then
      (i) result ← CSP-BACKTRACKING(A, var-domains)
      (ii) If result ≠ failure then return result
   d. Remove (X←v) from A
5. Return failure

No need any more to verify that A is valid

Modified Backtracking Algorithm

CSP-BACKTRACKING(A, var-domains)

1. If assignment A is complete then return A
2. X ← select a variable not in A
3. D ← select an ordering on the domain of X
4. For each value v in D do
   a. Add (X←v) to A
   b. var-domains ← forward checking(var-domains, X, v, A)
   c. If no variable has an empty domain then
      (i) result ← CSP-BACKTRACKING(A, var-domains)
      (ii) If result ≠ failure then return result
   d. Remove (X←v) from A
5. Return failure

Need to pass down the updated variable domains

Modified Backtracking Algorithm

CSP-BACKTRACKING(A, var-domains)

1. If assignment A is complete then return A
2. X ← select a variable not in A
3. D ← select an ordering on the domain of X
4. For each value v in D do
   a. Add (X←v) to A
   b. var-domains ← forward checking(var-domains, X, v, A)
   c. If no variable has an empty domain then
      (i) result ← CSP-BACKTRACKING(A, var-domains)
      (ii) If result ≠ failure then return result
   d. Remove (X←v) from A
5. Return failure

1) Which variable X_i should be assigned a value next?
   → Most-constrained-variable heuristic
   → Most-constraining-variable heuristic
2) In which order should its values be assigned?
   → Least-constraining-value heuristic

These heuristics can be quite confusing

Keep in mind that **all** variables must eventually get a value, while only **one** value from a domain must be assigned to each variable

61

Most-Constrained-Variable Heuristic

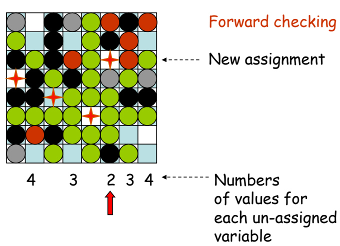
1) Which variable X_i should be assigned a value next?

Select the variable with the smallest remaining domain

[Rationale: Minimize the branching factor]

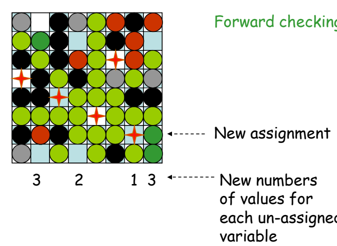
62

8-Queens



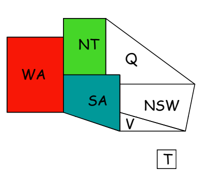
63

8-Queens



64

Map Coloring



- SA's remaining domain has size 1 (value Blue remaining)
- Q's remaining domain has size 2
- NSW's, V's, and T's remaining domains have size 3

→ Select SA

65

Most-Constraining-Variable Heuristic

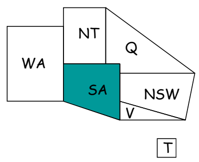
1) Which variable X_i should be assigned a value next?

Among the variables with the smallest remaining domains (ties with respect to the most-constrained-variable heuristic), select the one that appears in the largest number of constraints on variables not in the current assignment

[Rationale: Increase future elimination of values, to reduce future branching factors]

66

Map Coloring



- Before any value has been assigned, all variables have a domain of size 3, but SA is involved in more constraints (5) than any other variable

→ Select SA and assign a value to it (e.g., Blue)

Least-Constraining-Value Heuristic

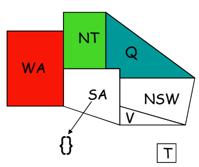
2) In which order should X's values be assigned?

Select the value of X that removes the smallest number of values from the domains of those variables which are not in the current assignment

[Rationale: Since only one value will eventually be assigned to X, pick the least-constraining value first, since it is the most likely not to lead to an invalid assignment]

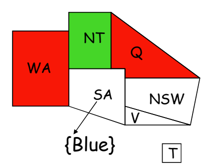
[Note: Using this heuristic requires performing a forward-checking step for every value, not just for the selected value]

Map Coloring



- Q's domain has two remaining values: Blue and Red
- Assigning Blue to Q would leave 0 value for SA, while assigning Red would leave 1 value

Map Coloring



- Q's domain has two remaining values: Blue and Red
- Assigning Blue to Q would leave 0 value for SA, while assigning Red would leave 1 value

→ So, assign Red to Q

Modified Backtracking Algorithm

CSP-BACKTRACKING(A, var-domains)

1. If assignment A is complete then return A
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4. For each value v in D do
   a. Add (X←v) to A
   b. var-domains ← forward checking(var-domains, X, v, A)
   c. If no variable has an empty domain then
      (i) result ← CSP-BACKTRACKING(A, var-domains)
      (ii) If result ≠ failure then return result
   d. Remove (X←v) from A
5. Return failure

1) Most-constrained-variable heuristic
2) Most-constraining-variable heuristic

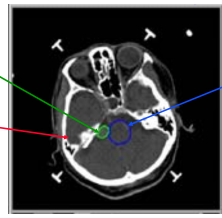
3) Least-constraining-value heuristic

Applications of CSP

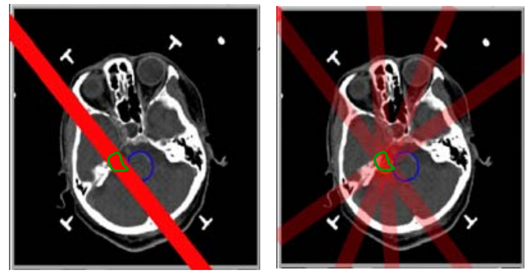
- CSP techniques are widely used
- Applications include:
  - Crew assignments to flights
  - Management of transportation fleet
  - Flight/rail schedules
  - Job shop scheduling
  - Task scheduling in port operations
  - Design, including spatial layout design
  - Radiosurgical procedures

Radiosurgery

Minimally invasive procedure that uses a beam of radiation as an ablative surgical instrument to destroy tumors

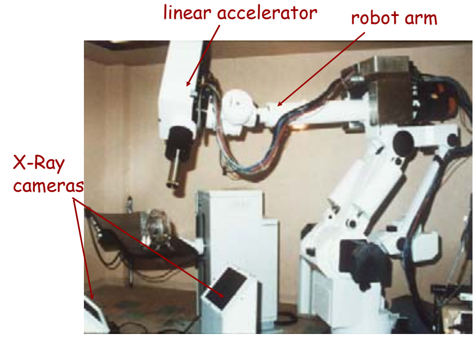


Problem



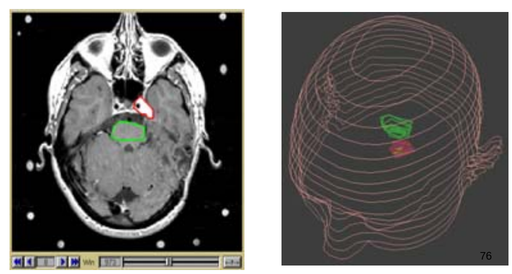
Burn tumor without damaging healthy tissue

The CyberKnife



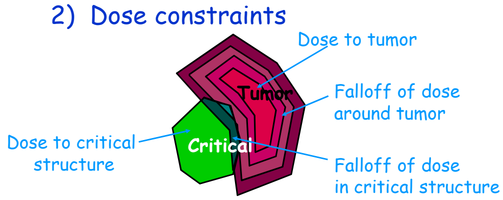
Inputs

1) Regions of interest

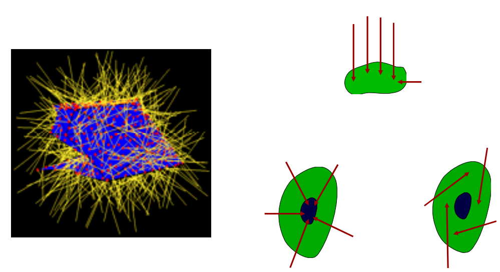


Inputs

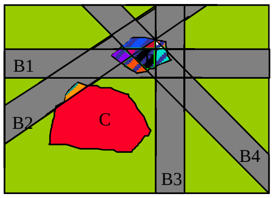
2) Dose constraints



Beam Sampling



Constraints



- 2000 ≤ Tumor ≤ 2200
  - 2000 ≤ B2 + B4 ≤ 2200
  - 2000 ≤ B4 ≤ 2200
  - 2000 ≤ B3 + B4 ≤ 2200
  - 2000 ≤ B3 ≤ 2200
  - 2000 ≤ B1 + B3 + B4 ≤ 2200
  - 2000 ≤ B1 + B4 ≤ 2200
  - 2000 ≤ B1 + B2 + B4 ≤ 2200
  - 2000 ≤ B1 ≤ 2200
  - 2000 ≤ B1 + B2 ≤ 2200
- 0 ≤ Critical ≤ 500
  - 0 ≤ B2 ≤ 500

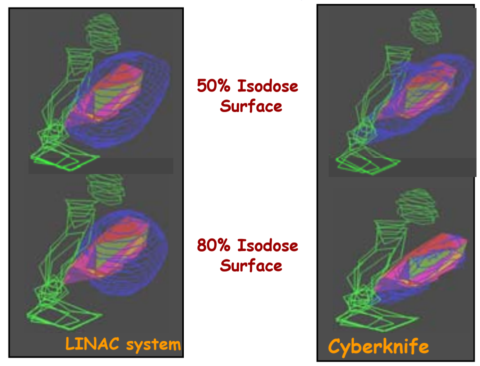
2000 < Tumor < 2200
2000 < B2 + B4 < 2200
2000 < B4 < 2200
2000 < B3 + B4 < 2200
2000 < B3 < 2200
2000 < B1 + B3 + B4 < 2200
2000 < B1 + B4 < 2200
2000 < B1 + B2 + B4 < 2200
2000 < B1 < 2200
2000 < B1 + B2 < 2200

⇨

2000 < Tumor < 2200
2000 < B4
2000 < B3
B1 + B3 + B4 < 2200
B1 + B2 + B4 < 2200
2000 < B1

79

Case Results



80