Geometry of Adjoint-Invariant Submanifolds of SE(3)

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***Abstract*—This paper aims to extend the theory of Lie subgroups and symmetric subspaces for studying an important class of submanifolds of the special Euclidean group $SE(3)$ whose tangent space at each point on the submanifold relates to that at the identity by an adjoint map. These submanifolds, which we call adjoint-invariant submanifolds in this paper, are known in the literature as persistent submanifolds, since they are strictly related to the concept of persistent screw systems. The difference is that in this paper, just as Lie subgroups and symmetric subspaces, we put forward adjoint-invariant submanifolds as independent geometric objects from mechanisms and their associated local screw systems. Adjoint invariance relaxes the strict left and right invariance of Lie subgroups and the reflective invariance of symmetric subspaces by allowing generic moving reference frame in the aforementioned adjoint map. It turns out such adjoint invariance can be studied under the framework of distributions on manifolds, which allows us to explore global geometric properties of adjoint-invariant submanifolds. We classify adjoint-invariant submanifolds into reflective-type and product-type submanifolds, and derive the conditions for their adjoint invariance. We then propose geometric methods and algorithms for synthesizing the kinematic generators for reflective-type submanifolds, as demonstrated with a number of examples.**

Keywords: rigid body motion, adjoint-invariant submanifold, distributions, kinematic generator

I. Introduction

Characterizing the motion pattern (or type) of robot task space is of vital importance to the analysis and synthesis of mechanisms [1]–[5]. It not only requires finding the right subset (usually a Lie subgroup or a submanifold) of $SE(3)$, but also verifying that the mechanism does generate the desired motion pattern either locally or globally. For serial robot the problem is quite straightforward as their forward kinematics is given by the product of exponentials (POE) formula [6]. The case of parallel mechanisms is much more complex because of the nonlinear nature of the loop-closure constraints.

Despite the complexity of their topology, significant progresses have been made toward understanding the motion pattern of parallel mechanisms. The first major progress lies in the mechanisms with Lie subgroup motions [1]–[5]. Lie subgroups are both left and right invariant which imply a kind of rigidity about Lie subgroup motion types. In other words a mechanism exhibiting the instantaneous degrees of freedom (DoFs) of a Lie subgroup at a given non-singular configuration will keep the same motion pattern in a neighborhood of this configuration.

In addition to Lie subgroup motion types, submanifolds of $SE(3)$ have also received lots of interests. Hervé and his colleagues proposed kinematic bonds as a fundamental tool for mechanism synthesis [2]. Most of the traditional $aTbR$ parallel mechanisms [7], [8] are not kinematic generators of Lie subgroups, but of special submanifolds of $SE(3)$. Although submanifolds of $SE(3)$ lose the left and right translational symmetry of Lie subgroups, sometimes they still can satisfy so called inversion symmetry. This leads to the breakthroughs made by Wu and his colleagues [9], [10]. In a series of works a new type of submanifold, symmetric subspace is proposed along with a complete theoretical framework for analysis, classification, and mechanism synthesis of such motion type. 7 different classes of symmetric subspaces are identified in [9], and their corresponding kinematic generators are synthesized by a novel method in [10] that employs symmetric subchains as well as an interconnection scheme for generating correct constraints. The motion of constant velocity (CV) joints and various types of omni-wrists, which used to be studied using screw theory [11], [12], can now be completely explained under the framework of symmetric subspace.

It should be noted that $SE(3)$ has infinite number of submanifolds. For most of these submanifolds the nature of their DoFs is hard to justify globally as it might change along with the task configuration. It is important to identify and classify submanifolds whose features have global meaning, while taking into account the noncommutative nature between rotational and translational DoFs. Carricato and his coauthors [13], [14] were among the first ones to explore mechanisms whose screw systems at different configurations are related by an adjoint map. In [13], mechanisms with such a nice property are said to have a persistent screw system (PSS) of the end-effector, since the end-effector screw system remains invariant up to a rigid displacement under arbitrary finite motions away from special configurations, namely it is adjoint-invariant. In [14], the submanifold of SE(3) “enveloped” by a persistent twist system was generally called a persistent manifold. It is important to emphasize that the notions

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of persistent manifolds and adjoint-invariant manifolds coincide, though the latter name is preferred in this paper. Selig and Carricato [14] showed that the concept of 1-dimensional persistent (or adjojnt-invariant) motion is a slight generalization of a class of motions (called Ribaucour motions) that were already studied by Study [15]. In [16]–[18], Carricato and coauthors classified all persistent submanifolds of dimension smaller than 5 that can be generated by serial kinematic chains, namely that are products of Lie subgroups. The notion of persistence or adjoint-invariance applies to general chains generating submanifolds of $SE(3)$ with distinct geometries. In this paper, we study adjoint-invariant submanifolds as a generalization of Lie subgroups and symmetric subspaces. We employ the framework of distributions on manifolds for studying the global geometric properties of adjoint-invariant submanifolds, from which we propose algorithms for synthesizing the kinematic generators for some adjoint-invariant submanifolds. Our theory is demonstrated with a number of examples, among which some mechanisms, to the best of our knowledge, are first proposed.

This paper is organized as follows. In Section II, we propose the concept of adjoint-invariant submanifolds and analyze their geometric properties using the theory of distributions on manifolds. In Section III and IV, we classify adjoint-invariant submanifolds into two subcategories and deduce the conditions for their adjoint invariance . In Section V, we propose tools and algorithms for synthesizing the kinematic generators along with a number of examples. We conclude our paper in Section VI.

II. Adjoint-Invariant Submanifolds of $SE(3)$

In this section we study an important class of submanifolds of the special Euclidean group $SE(3)$ which possess invariant properties. Throughout this paper we adopt the notations in [6], [9], [10], which are summarized in Table I .

TABLE I
List of notations used in this paper

Notation	Explanation
$SE(3)$	special Euclidean group
$se(3)$, $T_eSE(3)$	Lie algebra of $SE(3)$
$\mathbb{R}^n$	n-dimensional real vector space
$\mathcal{R}(z)$, $\mathcal{R}(P,\omega)$	group of a revolute joint
$\mathcal{T}(z)$, $\mathcal{T}(\omega)$	group of a prismatic joint
$\mathcal{H}_p(z)$, $\mathcal{H}_p(P,\omega)$	group of a helical joint with pitch p
$\mathcal{T}_2(z)$, $\mathcal{T}_2(\omega)$	2-D translational group
$\mathcal{PL}(z)$, $\mathcal{PL}(\omega)$	planar group
$\mathcal{Y}_p(z)$, $\mathcal{Y}_p(P,\omega)$	planar group with pitch p
$\mathcal{X}(\omega)$	Schönflies group
M_n, M^p_{nA}, M_{nB}	n-D symmetric subspace
$\mathfrak{m}_n$, $\mathfrak{m}^p_{nA}$, $\mathfrak{m}_{nB}$	basis of symmetric subspaces
$\hat{\xi}_i$, ξ_i, η, η_i, ζ, ζ_i	twists in $se(3)$
Q, Q_i	submanifold of $SE(3)$
T_gQ	Tangent space of Q at g
$\mathcal{V}_gQ$	spatial velocity space of Q at g
Δ	distribution on $SE(3)$
R_g, $R_{g\star}$	right translation map on $SE(3)$
EXP, e	exponential map on $SE(3)$
$\{e_i\}$, $\{\hat{e}_i\}$	canonical basis of $se(3)$
S, S_h	reflection map
G, G_i, H	Lie subgroup of $SE(3)$
$[\cdot,\cdot]$	Lie bracket
I_g	conjugate map
Ad_g	adjoint matrix of $g \in SE(3)$
ad_X	adjoint representation of a twist X
U_e, U_0, U_Q	open neighborhood
θ,θ_i, α_i, β_i	joint angles
Θ, α	joint angle vector
g,g_a, g_b, g_0, h, h_0	element of $SE(3)$
$g_1(g) \in SE(3)$	a function of g
$\mathfrak{g}_i$, $\mathfrak{g}$, $\mathfrak{h}$	Lie algebra
x,y,z	canonical basis of the Cartesian space
v_1,v_2,v_i	3-D vectors in the Cartesian space

A. Definition

Let $Q \subset SE(3)$ be an n-dimensional submanifold of $SE(3)$ passing through the identity e. The instantaneous spatial velocity space $\mathcal{V}_gQ$ at $g \in Q$ is given by the following right translation map

$$\mathcal{V}_gQ = R_{g^{-1}\star}T_gQ, \tag{1}$$

where T_gQ is the tangent space of Q at $g \in Q$. Q is called locally adjoint-invariant if there exists an open neighborhood U_e of e on Q such that $\forall g \in U_e \subset Q$ there exists $g_1(g) \in SE(3)$ satisfying

$$\mathcal{V}_gQ = Ad_{g_1(g)}T_eQ. \tag{2}$$

Intuitively Eqn. (2) means that $\mathcal{V}_gQ$ is invariant with respect to a reference frame that is given by shifting the world frame through a rigid body motion $g_1(g)$. This reference frame is a moving frame as $g_1(g)$ might depend on g. Q is globally adjoint-invariant (or persistent according to [13], [14]) if Eqn. (2) holds for all $g \in Q$.

Under the changing of the world frame through a rigid body motion g_0, a given adjoint-invariant submanifold Q turns into another adjoint-invariant submanifold $I_{g_0}(Q)$ as $\mathcal{V}_{I_{g_0}(g)}(I_{g_0}(Q)) = Ad_{g_0}\mathcal{V}_gQ = Ad_{g_0g_1(g)g_0^{-1}}(Ad_{g_0}T_eQ) = Ad_{I_{g_0}(g_1(g))}(T_e(I_{g_0}(Q)))$. $I_{g_0}(Q)$ and Q belong to the same conjugate class, which are similar to the cases of Lie subgroups and symmetric subspaces. Therefore adjoint invariance of a submanifold of $SE(3)$ is coordinate-free, i.e., independent of the chosen world and tool frames.

B. Geometric Properties

In this subsection we derive basic geometric properties of adjoint-invariant submanifolds along with examples.

Lemma 1. *If $Q \subset SE(3)$ is a locally or globally non-trivial (i.e. not Lie subgroups) adjoint-invariant submanifold, then $g_1(g)$ cannot be e or g.*

Proof: If $g_1(g) = e$, then $T_gQ = R_{g\star}T_eQ$; and if $g_1(g) = g$, then $T_gQ = L_{g\star}T_eQ$. Therefore $\cup_{g\in Q}T_gQ$ (or $\cup_{g\in U_e}T_gQ$) forms a right-invariant or left-invariant distribution. Q is either a Lie subgroup or non-integrable based upon whether or not T_eQ is a Lie subalgebra of $se(3)$.

Conversely, to construct or enumerate feasible adjoint-invariant submanifolds one might start with a subspace $\Delta(e) = \{\hat{\xi}_j \mid j = 1, \cdots, n\} \subset se(3)$ and a function $g_1(g) : SE(3) \to SE(3), g \to g_1(g)$, and then to each configuration $g \in SE(3)$ we assign a subspace

$$\Delta(g) = R_{g\star} Ad_{g_1(g)} \Delta(e) \subset T_g SE(3). \tag{3}$$

This yields a distribution $\Delta = \cup_g \Delta(g)$ on $SE(3)$. We refer to Δ as an adjoint-invariant distribution because it satisfies Eqn. (3). By comparing Eqn. (3) with Eqn. (2) it is easy to conclude that if Δ is integrable, then its integration manifold Q is an adjoint-invariant submanifold of $SE(3)$. Notice that a basis of Δ is given by $\{g_1(g)\hat{\xi}_i g_1^{-1}(g)g \mid i = 1, \cdots, n\}$.

Proposition 1. *If Δ is involutive, i.e., $\forall i, j \in (1, \cdots, n)$, we have*

$$\left[g_1(g)\hat{\xi}_i g_1^{-1}(g)g, g_1(g)\hat{\xi}_j g_1^{-1}(g)g\right] \in \tag{4}$$

$$span\{g_1(g)\hat{\xi}_1 g_1^{-1}(g)g, \cdots, g_1(g)\hat{\xi}_n g_1^{-1}(g)g\}, \tag{5}$$

then Δ is integrable, and the corresponding integration manifold Q is adjoint-invariant.

Appendix A provides the calculation results about the Lie bracket $\left[g_1(g)\hat{\xi}_i g_1^{-1}(g)g, g_1(g)\hat{\xi}_j g_1^{-1}(g)g\right]$. Moreover, it is possible to show that both Lie subgroups and symmetric subspaces are adjoint-invariant submanifolds (as proven in [9], [13], respectively).

Example 1. Lie subgroups and symmetric subspaces are adjoint-invariant submanifolds
Lie subgroups are obtained by integrating the distribution $\Delta(g) = R_{g\star}\Delta(e), g \in SE(3)$, with $\Delta(e) \subset se(3)$ a Lie subalgebra. Δ satisfies Eqn. (3) with $g_1(g) = e$. Then $\left[g_1(g)\hat{\xi}_i g_1^{-1}(g)g, g_1(g)\hat{\xi}_j g_1^{-1}(g)g\right] = \left[\hat{\xi}_j, \hat{\xi}_i\right] g = g_1(g)\left[\hat{\xi}_j, \hat{\xi}_i\right] g_1^{-1}(g)g$. Because $\Delta(e)$ is a Lie subalgebra, Δ satisfies the involutive condition (5). So Lie subgroups are globally adjoint-invariant submanifolds [13].

Symmetric subspaces are in fact the integration manifold of the distribution $\Delta(g) = R_{g\star} Ad_{g^{1/2}}\Delta(e), g \in SE(3)$ with $\Delta(e)$ a Lie triple system (i.e. closed under double Lie brackets) [9]. Δ satisfies Eqn. (3) with $g_1(g) = g^{1/2}$. The integrability of Δ can be verified by applying the general result of Appendix A. We have

$$\left[g_1(g)\hat{\xi}_i g_1^{-1}(g)g, g_1(g)\hat{\xi}_j g_1^{-1}(g)g\right] =$$
$$g_1(g)(Ad_{g_1^{-1/2}(g)}\left[\hat{\zeta}_i, \hat{\zeta}_j\right] - Ad_{g_1^{1/2}(g)}\left[\hat{\zeta}_i, \hat{\zeta}_j\right])g_1(g)$$

where $\hat{\zeta}_i, \hat{\zeta}_j \in \Delta(e)$. By expanding the formula we see that it only contains terms which are double Lie brackets of the elements in $\Delta(e)$. Therefore Δ satisfies the involutive condition, and its integration manifold is adjoint-invariant [9]. Symmetric subspaces have been classified in [19] (see Table II).

TABLE II
7 Symmetric subspaces and their basis [9], [10]

Symmetric subspaces	T_eQ
M_5	$\mathfrak{m}_5 \triangleq \{e_1, e_2, e_3, e_4, e_5\}$
M_4	$\mathfrak{m}_4 \triangleq \{e_1, e_2, e_4, e_5\}$
M_{3B}	$\mathfrak{m}_{3B} \triangleq \{e_3, e_4, e_5\}$
M_{3A}	$\mathfrak{m}_{3A} \triangleq \{e_1, e_3, e_4\}$
M_{2B}	$\mathfrak{m}_{2B} \triangleq \{e_4, e_5\}$
M_{2A}	$\mathfrak{m}_{2A} \triangleq \{e_3, e_4\}$
M_{2A}^p	$\mathfrak{m}_{2A}^p \triangleq \{e_3, pe_1 + e_4\}$

It is expected that adjoint-invariant submanifolds of $SE(3)$ might exist in great abundance. We have the following existence and uniqueness result given a Δ satisfying (3).

Proposition 2. *Let Δ be the distribution constructed from a given function $g_1(g)$ and $\Delta(e) \subset se(3)$ based upon (3). If Δ is involutive, then there exists a unique adjoint-invariant integration manifold Q which is simply connected and maximal.*

Proposition 2 implies that the solution manifolds in Example 1 are unique.

Example 2. Distributions whose integration manifold has the product structure
Let $\mathfrak{g}_1$ be the Lie subalgebras of an m-D Lie subgroup G_1 ($m < 6$), and W be a different subspace of $se(3)$ such that $W \cap \mathfrak{g}_1 = \emptyset$. Suppose $\Delta(e) = \mathfrak{g}_1 \oplus W$ is an n-D subspace of $se(3)$ with $n < 6$. Any $g \in SE(3)$ can be written as $g_a\tilde{g}$, where $g_a \in G_1$ and $\tilde{g} = g_a^{-1}g$. Now construct the distribution Δ on $SE(3)$ as $\Delta(g) = R_{g^\star} Ad_{g_a}\Delta(e)$, i.e. $g_1(g) = g_a$. We check the integrability of Δ. The basis for Δ is given by $span\{g_a\hat{\xi}_i\tilde{g} \mid i = 1, \cdots, n\}$, where $\hat{\xi}_i$ is the basis of $\Delta(e)$. Then it is easy to verify that (see Appendix A)

$$\left[g_a\hat{\xi}_i\tilde{g}, g_a\hat{\xi}_j\tilde{g}\right] = g_a(\left[\hat{\xi}_i^1, \hat{\xi}_j^1\right] + \left[\hat{\xi}_j^2, \hat{\xi}_i^2\right])\tilde{g}$$

where $\hat{\xi}_i^1$ (resp. $\hat{\xi}_j^1$) is the projection of $\hat{\xi}_i$ (resp. $\hat{\xi}_j$) to the subspace $\mathfrak{g}_1$, while $\hat{\xi}_i^2$ and $\hat{\xi}_j^2$ are the projections of $\hat{\xi}_i$ and $\hat{\xi}_j$ onto W. As $\mathfrak{g}_1$ is a Lie subalgebra, $\left[\hat{\xi}_i^1, \hat{\xi}_j^1\right] \in \mathfrak{g}_1 \subset \Delta(e)$. So as long as $\left[\hat{\xi}_j^2, \hat{\xi}_i^2\right] \in \Delta(e)$, then Δ is involutive. One solution is that $W = \mathfrak{g}_2$, the Lie subalgebra of another Lie subgroup G_2. The integration manifold of Δ is $G_1 \cdot G_2$. A more generic solution is that W is the tangent space of a symmetric subspace M, whose completion algebra is contained in $\Delta(e)$. This leads to the integration submanifold $G_1 \cdot M$. The same results are rederived in Section IV with a different method.

The integration manifold depends on both $\Delta(e)$ and $g_1(g)$. Moreover, although finding integrable distributions Δ satifying Eqn. (3) provides a general method for constructing adjoint-invariant submanifolds, it tends to be harder to check the involutive condition of the distribution Δ in Eqn. (3) as the function $g_1(g)$ becomes more and more complex. In what follows we explore the methods that directly construct globally or locally adjoint-invariant submanifolds from the basic building blocks, Lie subgroups and symmetric subspaces. Through analytic extension of the tangent bundle $TQ = \cup_g T_g Q$ of these adjoint-invariant submanifolds Q, we obtain their corresponding integrable adjoint-invariant distribution Δ on $SE(3)$.

III. Reflective-type submanifolds

Let

$$S_h : SE(3) \to SE(3), h_0 \to h h_0^{-1} h \tag{6}$$

be the inversion map on $SE(3)$. S_h has been extensively studied in [9], [10] for classifying and synthesizing symmetric subspace motions. Here we extend the inversion map in (6) to define the inversion of a submanifold about a second one. Let Q_i, $i = 1, 2$, be n_i-dimensional submanifolds of $SE(3)$ such that $T_e Q_1 \cap T_e Q_2 = \emptyset$. Define $S_{Q_1}(Q_2) \triangleq \{S_{g_a}(g_b) \mid g_a \in Q_1, g_b \in Q_2\}$. Under suitable conditions $S_{Q_1}(Q_2)$ could be a local or global regular adjoint-invariant submanifold of $SE(3)$.

A. Conditions for adjoint invariance

We have the following result for the adjoint invariance of $S_{Q_1}(Q_2)$.

Proposition 3. *Suppose $Q_i \neq M_{2A}^p$ are symmetric subspaces (including Lie subgroups as special cases). $S_{Q_1}(Q_2)$ is a globally adjoint-invariant submanifold if $\forall g_a \in Q_1, g_b \in Q_2$, we have*

$$(Ad_{g_b^{-1/2} g_a^{-1/2}} + Ad_{g_b^{1/2} g_a^{1/2}}) T_e Q_1 + T_e Q_2 = T_e Q_1 + T_e Q_2. \tag{7}$$

Under the same condition it is only locally adjoint-invariant if one of Q_i is M_{2A}^p.

Proof: See Appendix B. □

The condition (7) can be replaced by a simplified condition.

Corollary 1. *Suppose Q_i, $i = 1, 2$, are n_i-dimensional symmetric subspaces. If $\forall \xi_i \in T_e Q_i$, $i = 1, 2$, and $\forall \eta \in [T_e Q_1, T_e Q_1]$,*

$$\begin{aligned} ad_{\xi_2} \eta &\in T_e Q_1 + T_e Q_2 && (8) \\ ad_{\xi_2}^2 \xi_1 &\in T_e Q_1 + T_e Q_2, && (9) \end{aligned}$$

then $S_{Q_1}(Q_2)$ is a locally adjoint-invariant submanifold after excision out the region of singularities of measure 0, i.e. there exists an open neighborhood $U_e \in SE(3)$, such that $U_e \cap S_{Q_1}(Q_2)$ is an $(n_1 + n_2)$-dimensional adjoint-invariant submanifold.

Proof: See Appendix C. □

B. Two Important Sub-categories

We derive two important cases that satisfy Eqn. (8) and (9).

1) Case 1: Q_1 is a sub-6 DoF Lie subgroup: Obviously Q_1 cannot be $SO(3)$ for which $[T_e Q_1, T_e Q_1] = so(3)$ as there exists only empty-set $T_e Q_2$ which satisfies both $T_e Q_2 + T_e Q_1 \neq se(3)$ and Eqn. (8). For other Lie subgroups, $[T_e Q_1, T_e Q_1] = \emptyset$ or $\{e_1, e_2\}$ up to the adjoint map. The only non-trivial cases (i.e. neither Lie subgroups nor symmetric subspaces) which satisfy both Eqn. (8) and (9) are $S_{\mathcal{C}(x)}(\mathcal{R}(y))$ and $S_{\mathcal{C}(x)}(I_{e^{\hat{z}\pi/2}}(M_{2A}))$.

2) Case 2: Q_1 is a symmetric subspace but not a Lie subgroup: In this case $[T_e Q_1, T_e Q_1]$ is the Lie subalgebra $\mathfrak{h}$ of an isotropy group H of Q_1, and $\mathfrak{g} \triangleq T_e Q_1 + \mathfrak{h}$ is the completion Lie algebra of $T_e Q_1$. Notice $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ and $[\mathfrak{h}, [\mathfrak{h}, T_e Q_1]] \subset [\mathfrak{h}, T_e Q_1] \subset T_e Q_1$ based upon the relation between $T_e Q_1$ and the Lie subalgebra $\mathfrak{h}$ [9]. We could chose $T_e Q_2$ as a subspace of $\mathfrak{h}$ (which already satisfies (9)) such that

$$ad_\eta T_e Q_2 \subset T_e Q_2, \forall \eta \in \mathfrak{h} \tag{10}$$

for satisfying (8), or choose $T_e Q_2 \subset \mathfrak{g}^\perp$ that satisfies both (8) and (9). Finally, we obtain $S_{M_4}(\mathcal{R}(z))$, $S_{M_4}(\mathcal{H}_p(z))$, $S_{M_{3A}}(\mathcal{R}(y))$, and $S_{M_{2A}^p}(\mathcal{R}(y))$.

Example 3. $S_{M_4}(\mathcal{R}(z))$ **is locally adjoint-invariant** Let $g_a^{1/2} = \begin{bmatrix} R_{xy} & (R_{xy} + I) P_{xy} \\ 0 & 1 \end{bmatrix} \in M_4$, and $g_b^{1/2} = \begin{bmatrix} R_z & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{R}(z)$, where $R_z = e^{\hat{z}\theta}, \theta \in \mathbb{R}$, $R_{xy} = e^{\hat{x}\theta_1 + \hat{y}\theta_2}, \theta_1, \theta_2 \in \mathbb{R}$, and $P_{xy} = \gamma_1 x + \gamma_2 y, \gamma_i \in \mathbb{R}$. We calculate $(Ad_{g_b^{-1/2} g_a^{-1/2}} + Ad_{g_b^{1/2} g_a^{1/2}}) T_e Q_1$ as

$$\left\{ \begin{bmatrix} (R_z^T R_{xy}^T + R_z R_{xy}) x \\ 0 \end{bmatrix}, \begin{bmatrix} (R_z^T R_{xy}^T + R_z R_{xy}) y \\ 0 \end{bmatrix}, \begin{bmatrix} v_1 \\ (R_z^T R_{xy}^T + R_z R_{xy}) x \end{bmatrix}, \begin{bmatrix} v_2 \\ (R_z^T R_{xy}^T + R_z R_{xy}) y \end{bmatrix} \right\}$$

where it is easy to check that both $(R_z^T R_{xy}^T + R_z R_{xy}) x \parallel \{x, y\}$ and $(R_z^T R_{xy}^T + R_z R_{xy}) y \parallel \{x, y\}$, so do v_1, v_2. So Eqn. (7) holds at least in an open neighborhood of e without singularities.

The list of non-trivial reflective-type adjoint-invariant submanifolds is given in Table III. They are sub-6 dimensional adjoint-invariant submanifolds, which, to the best of our knowledge, have not been studied in the previous literatures.

TABLE III
Non-trivial reflective-type adjoint-invariant submanifolds

Submanifolds	$T_e Q$
$S_{M_4}(\mathcal{R}(z))$	$\{e_1, e_2, e_4, e_5, e_6\}$
$S_{M_4}(\mathcal{H}_p(z))$	$\{e_1, e_2, e_4, e_5, p e_3 + e_6\}$
$S_{\mathcal{C}(x)}(I_{e^{\hat{z}\pi/2}}(M_{2A}))$ $S_{M_{3A}}(\mathcal{R}(y))$	$\{e_1, e_3, e_4, e_5\}$
$S_{\mathcal{C}(x)}(\mathcal{R}(y))$	$\{e_1, e_4, e_5\}$
$S_{M_{2A}^p}(\mathcal{R}(y))$	$\{e_3, e_4 + p e_1, e_5\}$

IV. Product-type submanifolds

In this section we study product-type submanifolds of the form $Q = Q_1 \cdot Q_2$, where Q_i are symmetric subspaces (including Lie subgroups as special cases). The cases where both Q_i are Lie subgroups have already been studied by [13], and therefore are considered trivial here.

Proposition 4. *If Q_i are n_i-dimensional symmetric subspace, $T_eQ_1 \cap T_eQ_2 = \emptyset$, and $\forall g_a \in Q_1, g_b \in Q_2$ we have*

$$Ad_{g_a{}^{-1/2}} T_eQ_1 + Ad_{g_b{}^{1/2}} T_eQ_2 = T_eQ, \tag{11}$$

then Q is a globally (n_1+n_2)-dimensional adjoint-invariant submanifold.

Proof: Following the proof of Proposition 3 by using the fact that $\mathcal{V}_{g_a g_b}(Q_1 \cdot Q_2) = Ad_{g_a}(Ad_{g_a^{-1/2}} T_eQ_1 + Ad_{g_b^{1/2}} T_eQ_2)$. $\square$

A simpler sufficient condition for Eqn. (11) is given by

Corollary 2. *Given two symmetric subspaces Q_i with $T_eQ_1 \cap T_eQ_2 = \emptyset$. Let $\mathfrak{g}_i$ be the completion algebra of T_eQ_i such that $\mathfrak{g}_i = T_eQ_i + [T_eQ_i, T_eQ_i]$. If*

$$\mathfrak{g}_i \subset T_eQ_1 + T_eQ_2, i = 1, 2, \tag{12}$$

then Q is a globally (n_1+n_2)-dimensional adjoint-invariant submanifold after excision out the region of singularities of measure 0.

Proof: Following the proof of Corollary 1 by expanding $Ad_{g_a{}^{-1/2}} T_eQ_1 + Ad_{g_b{}^{1/2}} T_eQ_2$. The special case that Q_1 is a Lie group is proved in Example 2. $\square$

$Q_1 \cdot Q_2$ is fundamentally different from $S_{Q_1}(Q_2)$. The only exceptions are M_5 and M_{3A}, for which $Q_1 \cdot Q_2$ and $S_{Q_1}(Q_2)$ are sometimes equivalent, as proved in [10]. Table IV summarizes the list of non-trivial (i.e., excluding the products of two Lie subgroups) product-type adjoint-invariant submanifolds and their tangent spaces T_eQ at identity e. These submanifolds exhibit adjoint-invariant DoFs which, to the best of our knowledge, haven't been adequately studied.

TABLE IV
Product-type submanifolds

Products	T_eQ
$M_{2B} \cdot \mathcal{PL}(z)$	$\{e_1, e_2, e_4, e_5, e_6\}$
$\mathcal{PL}(z) \cdot M_{2B}$	$\{e_1, e_2, e_4, e_5, e_6\}$
$M_{2B} \cdot \mathcal{Y}_p(z)$	$\{e_1, e_2, e_4, e_5, pe_3 + e_6\}$
$\mathcal{Y}_p(z) \cdot M_{2B}$	$\{e_1, e_2, e_4, e_5, pe_3 + e_6\}$
$M_{3A} \cdot \mathcal{C}(y)$	$\{e_1, e_2, e_3, e_4, e_5\}$
$\mathcal{C}(y) \cdot M_{3A}$	$\{e_1, e_2, e_3, e_4, e_5\}$
$M_4 \cdot \mathcal{T}(z)$ $\mathcal{T}(z) \cdot M_4$ $M_{3B} \cdot \mathcal{T}_2(z)$ $\mathcal{T}_2(z) \cdot M_{3B}$ $M_{2B} \cdot \mathcal{T}_3$ $\mathcal{T}_3 \cdot M_{2B}$	$\{e_1, e_2, e_3, e_4, e_5\}$
$M_{2B} \cdot \mathcal{C}(z)$	$\{e_3, e_4, e_5, e_6\}$
$\mathcal{C}(z) \cdot M_{2B}$	$\{e_3, e_4, e_5, e_6\}$
$\mathcal{C}(y) \cdot M_{2A}$	$\{e_2, e_3, e_4, e_5\}$
$M_{2A} \cdot \mathcal{C}(y)$	$\{e_2, e_3, e_4, e_5\}$
$M_{2A} \cdot \mathcal{T}(x)$ $\mathcal{T}(x) \cdot M_{2A}$	$\{e_1, e_3, e_4\}$

V. Kinematic Generators (KG) of Reflective-type adjoint-invariant Submanifolds

In this section we will focus on synthesizing kinematic generators of reflective-type adjoint-invariant submanifolds of $SE(3)$. Consider the maximal inscribing symmetric subspace $M_{\max}$ and the minimal covering symmetric subspace $M_{\min}$ with $M_{\max} \subset Q \subset M_{\min}$. $S_{Q_1}(Q_2)$ can be obtained by compressing $M_{\min}$ or expanding $M_{\max}$.

A. Compressing $M_{\min}$

Some reflective-type adjoint-invariant submanifolds can be synthesized by assembling $Q_1 \cdot Q_2 \cdot Q_1$ chains with a $M_{\min}$ generator in parallel.

Example 4. KG for $S_{\mathcal{C}(x)}(\mathcal{R}(y))$
Notice that $S_{\mathcal{C}(x)}(\mathcal{R}(y)) \subset \mathcal{C}^-(x) \cdot \mathcal{R}(y) \cdot \mathcal{C}^+(x)$ which is generated by cascading a pair of symmetric joints $(\mathcal{C}^-(x), \mathcal{C}^+(x))$ about the $x-y$ plane with a $\mathcal{R}(y)$ joint in the middle, as shown in Fig. 1-(a). On the other hand the minimal covering symmetric subspace $M_{\min}$ of $S_{\mathcal{C}(x)}(\mathcal{R}(y))$ is M_4. Then we show that $S_{\mathcal{C}(x)}(\mathcal{R}(y)) = (\mathcal{C}^-(x) \cdot \mathcal{R}(y) \cdot \mathcal{C}^+(x)) \cap M_4$. First, $S_{\mathcal{C}(x)}(\mathcal{R}(y))$ belongs to both $\mathcal{C}^-(x) \cdot \mathcal{R}(y) \cdot \mathcal{C}^+(x)$ and M_4. Second, at home configuration e the constraint forces of M_4 is $\{e_3, e_6\}$, while that of $\mathcal{C}^-(x) \cdot \mathcal{R}(y) \cdot \mathcal{C}^+(x)$ is $\{e_2\}$. The feasible tangent space of the parallel mechanism at home configuration is simply $\{e_1, e_4, e_5\}$. According to Position 6 of [5], the parallel mechanism consisting of a M_4 KG and a $\mathcal{C}^-(x) - \mathcal{R}(y) - \mathcal{C}^+(x)$ subchain is the KG for $S_{\mathcal{C}(x)}(\mathcal{R}(y))$, as shown in Fig. 1-(b) (only one subchain of the M_4 KG is drawn here for clarity) . A practical mechanism can be derived by replacing the full M_4 generator (e.g. Example 5 in [10]) by its subchains M_4^j, and interconnecting M_4^j as well as the $\mathcal{C}^-(x) - \mathcal{R}(y) - \mathcal{C}^+(x)$ chain in a similar manner. This reduce the number of M_4^j subchains from 3 as required in the full M_4 generator to 2, as shown in Fig. 1-(c).

Example 5. KG for $S_{\mathcal{C}(x)}(I_{e^{\hat{z}\pi/2}}(M_{2A}))$
It is easy to see that $S_{\mathcal{C}(x)}(I_{e^{\hat{z}\pi/2}}(M_{2A}))$ is equivalent to $S_{\mathcal{C}(y)}(M_{2A})$ up to the conjugation map $I_{e^{-\hat{z}\pi/2}}$. Recall $S_{\mathcal{C}(y)}(M_{2A}) = S_{\mathcal{R}(y)}(S_{\mathcal{T}(y)}(M_{2A})) \subset S_{\mathcal{R}(y)}(\mathcal{PL}(x)) \subset \mathcal{R}(y)^- \cdot \mathcal{PL}(x) \cdot \mathcal{R}^+(y)$, where $\mathcal{PL}(x)$ is realized by cascading three revolute joints parallel to x, and $(\mathcal{R}^-(y), \mathcal{R}^+(y))$ are a pair of symmetric revolute joints about the $x-y$ plane. Combine the two distal revolute joints into a $\mathcal{U}(x, y^-)$ pair and a $\mathcal{U}(x, y^+)$ pair (y^- and y^+ in the $\mathcal{U}$ pairs are used to show that they are symmetric about the $x-y$ plane). This yields a $\mathcal{U}(x, y^-) - \mathcal{R}(x) - \mathcal{U}(x, y^+)$ mechanism, as shown in the left subchain in Fig. 2. Its constraint force space is given by $\{[x^T, (P_{xy} \times x)^T]^T\}$, where $P_{xy} \in \mathbb{R}^3$ is a point in the $x-y$ plane. On the

other hand $S_{\mathcal{C}(y)}(M_{2A}) \subset M_{\min} = M_5$. It is realized as the Delta - Omni-wrist mechanism (the right subchain of Fig. 2), which contributes the constraint force space $\{\left[0, z^T\right]^T\}$. The parallel mechanism formed by connecting these two subchains in parallel gives rise to the constraint force space $\{e_1, e_6\}$, and therefore it is a KG of $S_{\mathcal{C}(x)}(I_{e^{\hat{z}\pi/2}}(M_{2A}))$.

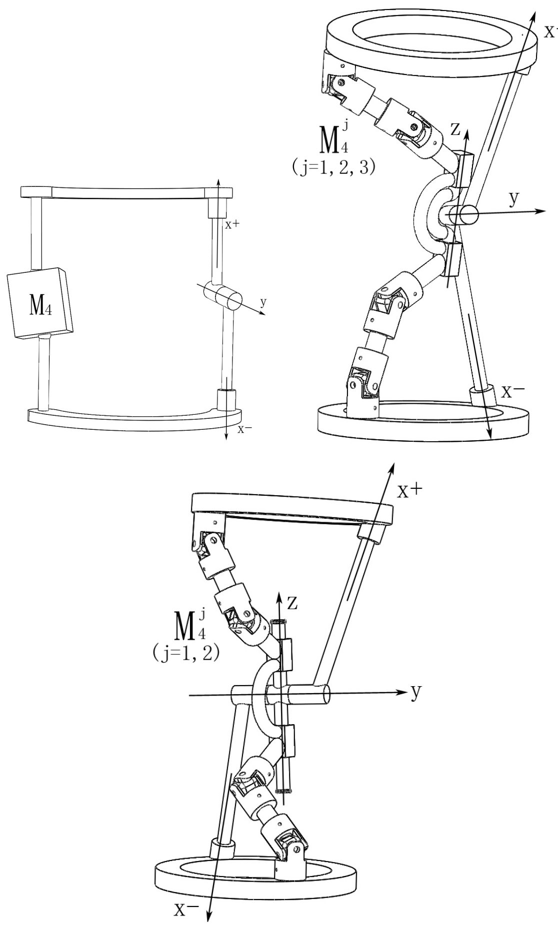

Fig. 1. (a): Parallel Mechanism composed of a $\mathcal{C}^-(x)-\mathcal{R}(y)-\mathcal{C}^+(x)$ subchain and an M_4 subchain; (b): M_4 subchain is realized by 3 pairs of symmetric $\mathcal{U}-\mathcal{U}$ chains interconnected through cylindrical joints as proposed in [10]; (c): Only 2 pairs of symmetric $\mathcal{U}-\mathcal{U}$ chains are required if we employ additional interconnection with the $\mathcal{C}^-(x) - \mathcal{R}(y) - \mathcal{C}^+(x)$ subchain.

B. Expanding $M_{\max}$

The reflective-type adjoint-invariant submanifolds, $S_{M_4}(\mathcal{R}(z))$ and $S_{M_4}(\mathcal{H}_p(z))$, can be synthesized by expanding the KG of its maximal inscribing symmetric subspace $M_{\max}$.

Proposition 5. *If $S_{Q_1}(Q_2)$ is a reflective-type adjoint-invariant submanifold with Q_1 a symmetric subspace ($\neq M_5$), and Q_2 a Lie subgroup satisfying $T_eQ_2 \subset \mathfrak{h} = [T_eQ_1, T_eQ_1]$ and $ad_\eta T_eQ_2 \subset T_eQ_2, \forall \eta \in \mathfrak{h}$, then a KG for $S_{Q_1}(Q_2)$ could be synthesized by inserting a Q_2 chain between each pair of symmetric sub-subchains in the KG for Q_1, while reducing the corresponding DoFs in all interconnecting chains.*

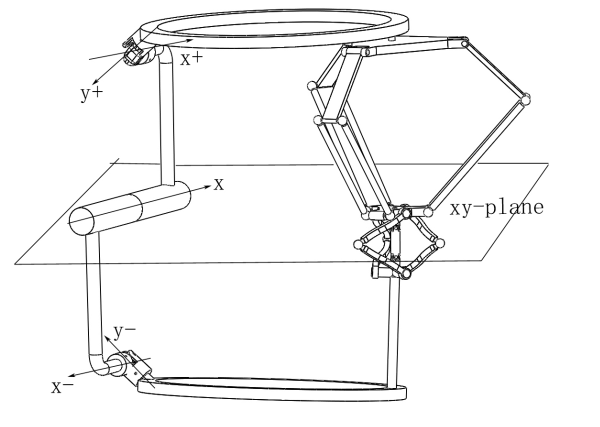

Fig. 2. KG for $S_{\mathcal{C}(y)}(M_{2A})$ composed of a $\mathcal{U}(x,y^-)-\mathcal{R}(x)-\mathcal{U}(x,y^+)$ subchain and an M_5 subchain whose wrist plane (the plane passing through the three spherical joints in the wrist) is parallel to the $x-y$ plane. The y^- and y^+ axes of the pair of symmetric $\mathcal{U}$ pairs of the $\mathcal{U}(x,y^-) - \mathcal{R}(x) - \mathcal{U}(x,y^+)$ subchain intersect at a point P_{xy} in the $x-y$ plane.

Proof: See Appendix D. □

Example 6. KG for $S_{M_4}(\mathcal{R}(z))$

It is easy to see that the maximal inscribing symmetric subspace $M_{\max}$ of $S_{M_4}(\mathcal{R}(z))$ is M_4. For M_4, we have $[\mathfrak{m}_4, \mathfrak{m}_4] = \{e_3, e_6\}$. We choose $Q_2 = \mathcal{R}(z) = \{e^{\hat{z}\theta} \mid \theta \in \mathbb{R}\}$, which satisfies the condition in Proposition 5. Now we add this additional rotational DoF $\mathcal{R}(z)$ to the middle of the original subchain M_4^j of M_4. The new subchain is denoted as N_4^j. Assembling 3 N_4^j together, and interconnecting them with a prismatic joint (instead of the cylindrical pair in the original M_4 KG) yields a KG for $S_{M_4}(\mathcal{R}(z))$, as shown in Fig. 3-(a). The KG for $S_{M_4}(\mathcal{H}_p(z))$ can be synthesized in the same way.

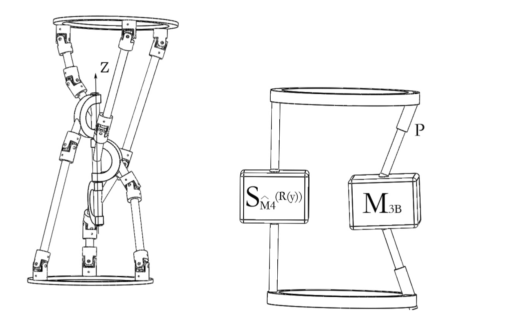

Fig. 3. (a): A KG for $S_{M_4}(\mathcal{R}(z))$ by adding a rotational DoF of $\mathcal{R}(z)$ to the middle of the original subchain M_4^j in a M_4 KG; (b):A KG for $S_{M_{3A}}(\mathcal{R}(y))$.

C. Compressing covering reflective-type submanifolds

Some reflective-type adjoint-invariant submanifolds are contained in one or multiple reflective-type submanifolds (called covering reflective-type submanifolds). The KG for these covering reflective-type submanifolds can be used as the primitive subchains.

Example 7. KG for $S_{M_{3A}}(\mathcal{R}(y))$
Notice that $S_{M_{3A}}(\mathcal{R}(y)) \subset S_{\hat{M}_4}(\mathcal{R}(y))$, where $\hat{M}_4 = I_{e^{\hat{x}\pi/2}}(M_4)$ is a 4-dimensional symmetric subspace satisfying $M_{3A} \subset \hat{M}_4$. $S_{\hat{M}_4}(\mathcal{R}(y))$ is equivalent to $I_{e^{\hat{x}\pi/2}}(S_{M_4}(\mathcal{R}(z)))$. The KG for the latter reflective submanifold $S_{M_4}(\mathcal{R}(z))$ is discussed in Example 6. On the other hand $S_{M_{3A}}(\mathcal{R}(y)) = \{e^{\hat{e}_1\theta_1}e^{\hat{e}_3\theta_2+\hat{e}_4\theta_3}e^{\hat{y}\theta_4}e^{\hat{e}_1\theta_1}e^{\hat{e}_3\theta_2+\hat{e}_4\theta_3}\}$ based on the facts that $M_{3A} = \mathcal{T}(x)\cdot M_{2A}$. Since $e^{\hat{e}_1\theta_1}e^{\hat{e}_3\theta_2+\hat{e}_4\theta_3} = e^{\hat{e}_3\theta_2+\hat{e}_4\theta_3}e^{\hat{e}_1\theta_1}$ by direct computation, we have $S_{M_{3A}}(\mathcal{R}(y)) = S_{\mathcal{T}(x)}(S_{M_{2A}}(\mathcal{R}(y))) \subset S_{\mathcal{T}(x)}(M_{3B}) \subset \mathcal{T}^-(x)\cdot M_{3B}\cdot\mathcal{T}^+(x)$, where $\mathcal{T}^-(x)\cdot M_{3B}\cdot\mathcal{T}^+(x)$ can be generated by cascading a pair of symmetric translational pair $(\mathcal{T}^-(x),\mathcal{T}^+(x))$ with a KG (e.g. Example 4 in [10]) for M_{3B} in between. Finally assembling the KG for $S_{\hat{M}_4}(\mathcal{R}(y))$ and that for $\mathcal{T}^-(x)\cdot M_{3B}\cdot\mathcal{T}^+(x)$ yields a KG for $S_{M_{3A}}(\mathcal{R}(y))$, as illustrated in Fig. 3-(b). This can be proved by recalling that at home configuration e the constraint force of the former subchain is $\{e_2\}$, while that of the latter subchain is $\{e_6\}$, and therefore $T_eS_{\hat{M}_4}(\mathcal{R}(y))\cap T_e(\mathcal{T}^-(x)\cdot M_{3B}\cdot\mathcal{T}^+(x)) = T_eS_{M_{3A}}(\mathcal{R}(y))$.

VI. Conclusion

In this paper we propose a class of submanifolds of $SE(3)$, the adjoint-invariant submanifolds, which extends the theory of Lie subgroups and symmetric subspaces by relaxing the symmetry requirements in these objects. We study global geometric properties as well as existence and uniqueness of adjoint-invariant submanifolds based on the theories of distributions on manifolds and their integrability. Then we classify adjoint-invariant submanifolds into reflective-type submanifolds and product-type submanifolds, and derive the conditions for adjoint invariance for each of the subcategory spaces. With the developed theory and methods we obtain the list of non-trivial reflective-type and product-type adjoint-invariant submanifolds. Finally we propose geometric tools and algorithms for constructing the kinematic generators for reflective-type adjoint-invariant submanifolds along with a number of examples.

Appendix A
Calculation of $\left[g_1(g)\hat{\xi}_i g_1^{-1}(g)g, g_1(g)\hat{\xi}_j g_1^{-1}(g)g\right]$

Let $\tilde{\xi}_i = g_1(g)\hat{\xi}_i g_1^{-1}(g)g$. The integral curve of the vector field $\tilde{\xi}_i$ is simply $h_i(t) = g_1(g)e^{\hat{\xi}_i t}g_1^{-1}(g)g$. Then given a function f on $SE(3)$ we calculate $\left[\tilde{\xi}_i,\tilde{\xi}_j\right]f = \left[g_1\hat{\xi}_i g_1^{-1}g, g_1\hat{\xi}_j g_1^{-1}g\right]f$ as

$$\begin{gathered}(((\tilde{\xi}_i g_1(g))\mid_{t=0}\hat{\xi}_j - (\tilde{\xi}_j g_1(g))\mid_{t=0}\hat{\xi}_i)g_1^{-1}(g)g \\ +g_1(g)(\hat{\xi}_j(\tilde{\xi}_i g_1^{-1}(g))\mid_{t=0} - \hat{\xi}_i(\tilde{\xi}_j g_1^{-1}(g))\mid_{t=0})g \\ +g_1(g)\left[\hat{\xi}_j,\hat{\xi}_i\right]g_1^{-1}(g)g))f,\end{gathered}$$

where $(\tilde{\xi}_i g_1(g))\mid_{t=0}$ and $(\tilde{\xi}_i g_1^{-1}(g))\mid_{t=0}$ denote the directional derivative of $g_1(g)$ and $g_1^{-1}(g)$ along the integral curve $h_i(t)$ of $\tilde{\xi}_i$ at $t=0$.
If $g_1(g) = e$, then $\left[\tilde{\xi}_i,\tilde{\xi}_j\right] = g_1\left[\hat{\xi}_j,\hat{\xi}_i\right]g_1^{-1}g$.
If $g_1(g) = g^{1/2}$, then $\left[\tilde{\xi}_i,\tilde{\xi}_j\right] = g_1(\left[g_1^{-1}B_i, g_1^{-1}B_j\right] - \left[B_i g_1^{-1}, B_j g_1^{-1}\right])g_1$, where $B_i = \frac{d(g_1 e^{\hat{\xi}_i t}g_1)^{1/2}}{dt}\mid_{t=0}$, and we use the fact that $g_1\hat{\xi}_i g_1 = \frac{d(g_1 e^{\hat{\xi}_i t}g_1)}{dt}\mid_{t=0} = B_i g_1 + g_1 B_i$.

Because B_i is a tangent vector based at g_1, and recall that now the distribution Δ at g_1 is given by $\Delta(g_1) = R_{g_1 *}Ad_{g_1^{1/2}}\Delta(e)$, we have $B_i g_1^{-1} = Ad_{g_1^{1/2}}\zeta_i$ and $B_j g_1^{-1} = Ad_{g_1^{1/2}}\zeta_j$ for some $\zeta_i,\zeta_j \in \Delta(e)$. Therefore

$$\begin{gathered}g_1(\left[g_1^{-1}B_i, g_1^{-1}B_j\right] - \left[B_i g_1^{-1}, B_j g_1^{-1}\right])g_1 = g_1(\widehat{W})g_1 \\ W = Ad_{g_1^{-1/2}}\left[\zeta_i,\zeta_j\right] - Ad_{g_1^{1/2}}\left[\zeta_i,\zeta_j\right]\end{gathered}$$

which is valid for all $g \in SE(3)$.

If $g = g_a\tilde{g}$ and $g_1(g) = g_a$, where g_a is an element of a Lie subgroup G_1, then the integral curve of $\tilde{\xi}_i$ is $h_i(t) = g_a e^{\hat{\xi}_i t}\tilde{g}$. So we have

$$\begin{aligned}\left[\tilde{\xi}_i,\tilde{\xi}_j\right]f &= g_a(\tfrac{d(e^{\hat{\xi}_i^1 t}\hat{\xi}_j e^{\hat{\xi}_i^2 t} - e^{\hat{\xi}_j^1 t}\hat{\xi}_i e^{\hat{\xi}_j^2 t})}{dt}\tilde{g} \\ &= g_a(\left[\hat{\xi}_i^1,\hat{\xi}_j^1\right] + \left[\hat{\xi}_j^2,\hat{\xi}_i^2\right])\tilde{g}\end{aligned}$$

where ξ_i^1 (resp. ξ_j^1) is the projection of ξ_i (resp. ξ_j) onto $\mathfrak{g}_1$, the Lie algebra of G_1, while ξ_i^2,ξ_j^2 are the corresponding projections onto W.

Appendix B
Proof of Proposition 3

Notice that we have a local parameterization for each open neighborhood U_i of e on Q_i, $i=1,2$, as they are all symmetric subspaces

$$g_a \in U_1 = e^{\Sigma_{i=1}^{n_1}\hat{\xi}_i\theta_i}, g_b \in U_2 = e^{\Sigma_{i=n_1+1}^{n_1+n_2}\hat{\xi}_i\theta_i}$$

where $T_eQ_1 = \{\xi_1,\cdots,\xi_{n_1}\}$, and $T_eQ_2 = \{\xi_{n_1+1},\cdots,\xi_{n_1+n_2}\}$. At a generic point $\tilde{g} = g_a g_b g_a$, we can assign a coordinate map $\phi_{\tilde{g}}$ on an open neighborhood $U_{\tilde{g}}$ about $\tilde{g}$, namely, $\phi_{\tilde{g}}(\Theta) = g_a^{1/2}e^{\Sigma_{i=1}^{n_1}\hat{\xi}_i\theta_i}g_a^{1/2}g_b^{1/2}e^{\Sigma_{i=n_1+1}^{n_1+n_2}\hat{\xi}_i\theta_i}g_b^{1/2}g_a^{1/2}e^{\Sigma_{i=1}^{n_1}\hat{\xi}_i\theta_i}g_a^{1/2}$ as $g_a^{1/2}e^{\Sigma_{i=1}^{n_1}\hat{\xi}_i\theta_i}g_a^{1/2}$ is a local coordinate map in a neighborhood about g_a on Q_1, and $g_b^{1/2}e^{\Sigma_{i=n_1+1}^{n_1+n_2}\hat{\xi}_i\theta_i}g_b^{1/2}$ is a local coordinate map on Q_2. Notice that $\phi_{\tilde{g}}: U_0 \to U_{\tilde{g}}, \Theta \to \phi_{\tilde{g}}(\Theta)$, where $U_0 \subset \mathbb{R}^{n_1+n_2}$. The Jacobian J of $\phi_{\tilde{g}}$ is given by $J = Ad_{g_a g_b^{1/2}}J_1$, where $J_1 = [A\xi_1,\cdots,A\xi_{n_1},\xi_{n_1+1},\cdots,\xi_{n_1+n_2}]$, and $A = Ad_{g_b^{-1/2}g_a^{-1/2}} + Ad_{g_b^{1/2}g_a^{1/2}}$. The range space of J_1 is easy to see to be $(Ad_{g_b^{-1/2}g_a^{-1/2}} + Ad_{g_b^{1/2}g_a^{1/2}})T_eQ_1 + T_eQ_2$. It is exactly $T_eQ_1 + T_eQ_2$ when $g_a = g_b = e$. Therefore if $\forall g_a \in Q_1, g_b \in Q_2, Ad_{g_b^{-1/2}g_a^{-1/2}} + Ad_{g_b^{1/2}g_a^{1/2}})T_eQ_1 + T_eQ_2 = T_eQ_1 + T_eQ_2$, J_1 is nonsingular at every combinations of (g_a,g_b). In fact we can prove that $U_{\tilde{g}}$ is a slice of $V_{\tilde{g}}$, a local neighborhood of $\tilde{g}$ on $SE(3)$. We have $\phi_{\tilde{g}}(\Theta) = g_a g_b^{1/2}hg_b^{1/2}g_a$, where $h = e^{\Sigma_{i=n_1+1}^{n_1+n_2}\widehat{Ad_{g_c}\xi_i}\theta_i}e^{\Sigma_{i=1}^{n_1}\hat{\zeta}_i\theta_i}$, $\zeta_i = (Ad_{g_b^{-1/2}g_a^{-1/2}} + Ad_{g_b^{1/2}g_a^{1/2}})\xi_i$, and $g_c = e^{\Sigma_{i=1}^{n_1}\widehat{Ad_{g_b^{-1/2}g_a^{-1/2}}\xi_i}\theta_i}$. $\phi_{\tilde{g}}$ is a slice of

$$\psi_{\tilde{g}}(\Theta,\alpha) = g_a g_b^{1/2}he^{\Sigma_{j=1}^{6-n_1-n_2}\hat{\eta}_j\alpha_j}g_b^{1/2}g_a$$

for which $\alpha_j = 0, j = 1, \cdots, 6-n_1-n_2$. Notice that $(\zeta_1, \cdots, \zeta_{n_1}, \xi_{n_1+1}, \cdots, \xi_{n_1+n_2}, \eta_1, \cdots, \eta_{6-n_1-n_2})$ is a basis of $se(3)$. $\psi_{\tilde{g}}$ is a local coordinate map at $\tilde{g}$ on $SE(3)$. Therefore $\phi_{\tilde{g}}$ generates an atlas for $S_{Q_1}(Q_2)$.

Appendix C
Proof of Corollary 1

As Q_i, $i = 1, 2$, are symmetric subspaces (including Lie subgroup as a special case), we can express $g_a^{1/2} = e^{\hat{\xi}_1}$ and $g_b^{1/2} = e^{\hat{\xi}_2}$, $\xi_i \in T_eQ_i$, $i = 1, 2$, globally (except for M_{2A}^p for which the expression only holds locally). Given $\xi_3 \in T_eQ_1$, we calculate $Ad_{g_a^{1/2}}\xi_3 = \xi + \eta$ and $Ad_{g_a^{-1/2}}\xi_3 = \xi - \eta$, where $\xi = \sum_{i=0}^{\infty} \frac{ad_{\xi_1}^{2i}}{(2i)!}\xi_3$, $\eta = \sum_{i=0}^{\infty} \frac{ad_{\xi_1}^{2i+1}}{(2i+1)!}\xi_3$. We can see that $\xi \in T_eQ_1$ and $\eta \in [T_eQ_1, T_eQ_1]$. Further calculation shows that $(Ad_{g_b^{-1/2}}Ad_{g_a^{-1/2}} + Ad_{g_b^{1/2}}Ad_{g_a^{1/2}})\xi_3 = 2\sum_{i=0}^{\infty} \frac{ad_{\xi_2}^{2i}}{(2i)!}\xi + 2\sum_{i=0}^{\infty} \frac{ad_{\xi_2}^{2i+1}}{(2i+1)!}\eta$. Notice that $ad_{\xi_2}^2\tilde{\xi}_2 \in T_eQ_2$, $\forall \tilde{\xi}_2 \in T_eQ_2$ as Q_2 is a symmetric subspace. As long as $ad_{\xi_2}^2\xi \in T_eQ_1 + T_eQ_2$ and $ad_{\xi_2}\eta \in T_eQ_1 + T_eQ_2$, all items in the summation are contained in $T_eQ_1 + T_eQ_2$ using induction.

Appendix D
Proof of Proposition 5

Recall that the n-D symmetric subspace Q_1 (except for M_5) can be constructed by assembling k subchains C_i, $i = 1, \cdots, k$, which are in turn composed of a pair of n-D symmetric sub-subchains $\{C_i^+, C_i^-\}$. Suitable interconnecting subchains are added that link the middle link of C_i. According to [10] the forward kinematic map of C_i^+ has the form $e^{\hat{\xi}}e^{\hat{\eta}}$ ($\xi \in T_eQ_1$, $\eta \in [T_eQ_1, T_eQ_1]$), while that of C_i^- is $e^{-\hat{\eta}}e^{\hat{\xi}}$, and their combo-kinematic map of C_i is $e^{2\hat{\xi}} \in Q_1$ as long as (C_i^+, C_i^-) maintains a symmetric arrangement. Inserting a Q_2 chain to the middle of C_i with Q_2 a Lie subgroup and $T_eQ_2 \subset [T_eQ_1, T_eQ_1]$ yields a new subchain $A_i = C_i^+ - Q_2 - C_i^-$ whose combo-kinematic map (with $\zeta, \zeta_1 \in T_eQ_2$) is $e^{\hat{\xi}}e^{\hat{\eta}}e^{\hat{\zeta}}e^{-\hat{\eta}}e^{\hat{\xi}} = e^{\hat{\xi}}e^{\hat{\zeta}_1}e^{\hat{\xi}} \in S_{Q_1}(Q_2)$, as long as $e^{\hat{\eta}}e^{\hat{\zeta}}e^{-\hat{\eta}} = e^{\hat{\zeta}_1}$. The latter is ensured by the condition in the proposition. The centers of pairs of subchains (A_1, A_j) are joined by a new interconnecting mechanism whose screws come from $[T_eQ_1, T_eQ_1]$ but excluding those from T_eQ_2.

Now we strictly prove that the task space of the mechanism $\{A_1, \cdots, A_k\}$ is $S_{Q_1}(Q_2)$ after applying the closed-loop constraints by following the rigidity argument proposed in [10]. First, given an arbitrary motion of $C_1^+ - Q_2$ in A_1, there is only one feasible solution locally for $C_j^+ - Q_2$ of A_j ($j \neq 1$) and the interconnecting mechanism between A_1 and A_j. The remaining mechanism composed of C_j^- for all j forms a motionless rigid mechanism. This yields a manifold of dimension exactly same as that of $S_{Q_1}(Q_2)$ as $S_{Q_1}(Q_2)$ has same degrees of freedom as $C_j^+ - Q_2$. Then we consider a submanifold Q_s of the task space Q of the entire mechanism. Each point of Q_s is obtained by applying an arbitrary motion of C_1^+ first while freezing the motion of the Q_2 chain of all subchains to be identity e. The remaining mechanism $\{C_1^-, \cdots, C_k^-\}$ becomes rigid again so that (C_j^+, C_j^-) in A_j forms a symmetric arrangement exactly as the KG for Q_1. Then we move the entire top half of the mechanism relative to the entire bottom half, by an arbitrary motion in Q_2. As the result of the composition of these two motions, the motion of A_j is exactly given by $e^{\hat{\xi}}e^{\hat{\zeta}_1}e^{\hat{\xi}} \in S_{Q_1}(Q_2)$, and thus $S_{Q_1}(Q_2) \subset Q_s \subset Q$ (at least locally). We just proved that Q has the same dimension as $S_{Q_1}(Q_2)$. They must match at least in an open neighborhood of e.

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