Geometry of Adjoint-Invariant Submanifolds of SE(3)

Guanfeng Liu, Yisheng Guan*, Yong Yang, Xin Chen

Abstract—This paper aims to extend the theory of Lie subgroups and symmetric subspaces for studying an important class of submanifolds of the special Euclidean group SE(3) whose tangent space at each point on the submanifold relates to that at the identity by an adjoint map. These submanifolds, which we call adjoint-invariant submanifolds in this paper, are known in the literature as persistent submanifolds, since they are strictly related to the concept of persistent screw systems. The difference is that in this paper, just as Lie subgroups and symmetric subspaces, we put forward adjoint-invariant submanifolds as independent geometric objects from mechanisms and their associated local screw systems. Adjoint invariance relaxes the strict left and right invariance of Lie subgroups and the reflective invariance of symmetric subspaces by allowing generic moving reference frame in the aforementioned adjoint map. It turns out such adjoint invariance can be studied under the framework of distributions on manifolds, which allows us to explore global geometric properties of adjoint-invariant submanifolds. We classify adjoint-invariant submanifolds into reflective-type and product-type submanifolds, and derive the conditions for their adjoint invariance. We then propose geometric methods and algorithms for synthesizing the kinematic generators for reflective-type submanifolds, as demonstrated with a number of examples.

Keywords: rigid body motion, adjoint-invariant submanifold, distributions, kinematic generator

I. INTRODUCTION

Characterizing the motion pattern (or type) of robot task space is of vital importance to the analysis and synthesis of mechanisms [1]–[5]. It not only requires finding the right subset (usually a Lie subgroup or a submanifold) of SE(3), but also verifying that the mechanism does generate the desired motion pattern either locally or globally. For serial robot the problem is quite straightforward as their forward kinematics is given by the product of exponentials (POE) formula [6]. The case of parallel mechanisms is much more complex because of the nonlinear nature of the loop-closure constraints.

Despite the complexity of their topology, significant progresses have been made toward understanding the motion pattern of parallel mechanisms. The first major progress lies in the mechanisms with Lie subgroup motions [1]–[5]. Lie subgroups are both left and right invariant which imply a kind of rigidity about Lie subgroup motion types. In other words a mechanism exhibiting the instantaneous degrees of freedom (DoFs) of a Lie subgroup at a given non-singular configuration will keep the same motion pattern in a neighborhood of this configuration.

In addition to Lie subgroup motion types, submanifolds of SE(3) have also received lots of interests. Hervé and his colleagues proposed kinematic bonds as a fundamental tool for mechanism synthesis [2]. Most of the traditional aTbR parallel mechanisms [7], [8] are not kinematic generators of Lie subgroups, but of special submanifolds of SE(3). Although submanifolds of SE(3) lose the left and right translational symmetry of Lie subgroups, sometimes they still can satisfy so called inversion symmetry. This leads to the breakthroughs made by Wu and his colleagues [9], [10]. In a series of works a new type of submanifold, symmetric subspace is proposed along with a complete theoretical framework for analysis, classification, and mechanism synthesis of such motion type. 7 different classes of symmetric subspaces are identified in [9], and their corresponding kinematic generators are synthesized by a novel method in [10] that employs symmetric subchains as well as an interconnection scheme for generating correct constraints. The motion of constant velocity (CV) joints and various types of omni-wrists, which used to be studied using screw theory [11], [12], can now be completely explained under the framework of symmetric subspace.

It should be noted that SE(3) has infinite number of submanifolds. For most of these submanifolds the nature of their DoFs is hard to justify globally as it might change along with the task configuration. It is important to identify and classify submanifolds whose features have global meaning, while taking into account the non-commutative nature between rotational and translational DoFs. Carricato and his coauthors [13], [14] were among the first ones to explore mechanisms whose screw systems at different configurations are related by an adjoint map. In [13], mechanisms with such a nice property are said to have a persistent screw system (PSS) of the end-effector, since the end-effector screw system remains invariant up to a rigid displacement under arbitrary finite motions away from special configurations, namely it is adjoint-invariant. In [14], the submanifold of SE(3) “enveloped” by a persistent twist system was generally called a persistent manifold. It is important to emphasize that the notions

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of persistent manifolds and adjoint-invariant manifold- 
s coincide, though the latter name is preferred in this 
paper. Selig and Carricato [14] showed that the concept of 
1-dimensional persistent (or adjoint-invariant) motion is a 
slight generalization of a class of motions (called 
Ribaucour motions) that were already studied by Study 
[15]. In [16]–[18], Carricato and coauthors classified all 
persistent submanifolds of dimension smaller than 5 that 
can be generated by serial kinematic chains, namely that 
are products of Lie subgroups. The notion of persistence or 
adjoint-invariance applies to general chains generating 
submanifolds of $SE(3)$ with distinct geometries. In 
this paper, we study adjoint-invariant submanifolds as a 
generalization of Lie subgroups and symmetric subspaces. 
We employ the framework of distributions on manifolds 
for studying the global geometric properties of adjoint-

invariant submanifolds, from which we propose algorithms 
for synthesizing the kinematic generators for some adjoint-

invariant submanifolds. Our theory is demonstrated with 
a number of examples, among which some mechanisms, to 
the best of our knowledge, are first proposed.

This paper is organized as follows. In Section II, we 
propose the concept of adjoint-invariant submanifolds and 
analyze their geometric properties using the theory of dis-

tributions on manifolds. In Section III and IV, we classify 
adjoint-invariant submanifolds into two subcategories and 
deduce the conditions for their adjoint invariance. In Sec-

tion V, we propose tools and algorithms for synthesizing 
the kinematic generators along with a number of examples. 
We conclude our paper in Section VI.

II. ADJOINT-INVARIANT SUBMANIFOLDS OF $SE(3)$

In this section we study an important class of sub-

manifolds of the special Euclidean group $SE(3)$ which possess 
invariant properties. Throughout this paper we adopt the 
notations in [6], [9], [10], which are summarized in Table I.

A. Definition

Let $Q \subset SE(3)$ be an $n$-dimensional submanifold of 
$SE(3)$ passing through the identity $e$. The instantaneous 
spatial velocity space $V_gQ$ at $g \in Q$ is given by the 
following right translation map

$$V_gQ = R_{g^{-1}}T_gQ,$$

where $T_gQ$ is the tangent space of $Q$ at $g \in Q$. $Q$ is 
called locally adjoint-invariant if there exists an open 
neighborhood $U_e$ of $e$ on $Q$ such that $\forall g \in U_e \subset Q$ there 
exists $g_1(g) \in SE(3)$ satisfying

$$V_gQ = Ad_{g_1(g)}T_eQ.$$  

Intuitively Eqn. (2) means that $V_gQ$ is invariant with 
respect to a reference frame that is shifting the world 
frame through a rigid body motion $g_1(g)$. This refer-
ence frame is a moving frame as $g_1(g)$ might depend on 
g. $Q$ is globally adjoint-invariant (or persistent according 
to [13], [14]) if Eqn. (2) holds for all $g \in Q$.

Under the changing of the world frame through a rigid 
body motion $g_0$, a given adjoint-invariant submanifold $Q$ 
turns into another adjoint-invariant submanifold $I_{g_0}(Q)$ as

$$I_{g_1(g)}(I_{g_0}(Q)) = Ad_{g_0}V_gQ = Ad_{g_0g_1(g)}^{-1}(Ad_{g_0}T_eQ) = Ad_{I_{g_0}(g_1(g))}(I_{g_0}(Q))).$$

In other words, $Q$ and $Q$ belong to the same 
adjoint conjugate class, which are similar to the cases of Lie 
subgroups and symmetric subspaces. Therefore adjoint 
invariance of a submanifold of $SE(3)$ is coordinate-free, 
i.e., independent of the chosen world and tool frames.

B. Geometric Properties

In this subsection we derive basic geometric properties 
of adjoint-invariant submanifolds along with examples.

**Lemma 1.** If $Q \subset SE(3)$ is a locally or globally non-trivial 
(i.e. not Lie subgroups) adjoint-invariant submanifold, 
then $g_1(g)$ cannot be $e$ or $g$.

**Proof:** If $g_1(g) = e$, then $T_gQ = R_eT_eQ$; and if 
g_1(g) = g, then $T_gQ = L_gT_eQ$. Therefore $\cup_{g \in Q} T_gQ$ 
or $\cup_{g \in U_e} T_gQ$ forms a right-invariant or left-invariant

<table>
<thead>
<tr>
<th>Table I</th>
</tr>
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<tbody>
<tr>
<td><strong>Notation</strong></td>
</tr>
<tr>
<td>$SE(3)$</td>
</tr>
<tr>
<td>$se(3)$, $T_xSE(3)$</td>
</tr>
<tr>
<td>$\mathbb{R}^n$</td>
</tr>
<tr>
<td>$\mathcal{R}(z)$, $\mathcal{R}(P,\omega)$</td>
</tr>
<tr>
<td>$T(z)$, $T(\omega)$</td>
</tr>
<tr>
<td>$H_p(z)$, $H_p(P,\omega)$</td>
</tr>
<tr>
<td>$T_2(z)$, $T_2(\omega)$</td>
</tr>
<tr>
<td>$\mathbb{P}_L(z), \mathbb{P}_L(\omega)$</td>
</tr>
<tr>
<td>$\mathcal{V}_p(z)$, $\mathcal{V}_p(P,\omega)$</td>
</tr>
<tr>
<td>$\mathcal{X}(\omega)$</td>
</tr>
<tr>
<td>$M_n$, $M_n^a$, $M_n^b$</td>
</tr>
<tr>
<td>$\xi$, $\zeta$, $\eta$, $\epsilon_i$, $\zeta_i$</td>
</tr>
<tr>
<td>$Q$, $Q_i$</td>
</tr>
<tr>
<td>$T_gQ$</td>
</tr>
<tr>
<td>$V_gQ$</td>
</tr>
<tr>
<td>$\Delta$</td>
</tr>
<tr>
<td>$R_e$, $R_{ax}$</td>
</tr>
<tr>
<td>$\exp$, $\epsilon$</td>
</tr>
<tr>
<td>${e_i}$, ${\epsilon_i}$</td>
</tr>
<tr>
<td>$S, S_h$</td>
</tr>
<tr>
<td>$G, G_i, H$</td>
</tr>
<tr>
<td>${\cdot, \cdot}$</td>
</tr>
<tr>
<td>$I_g$</td>
</tr>
<tr>
<td>$Ad_{g}$</td>
</tr>
<tr>
<td>$ad_{X}$</td>
</tr>
<tr>
<td>$U_e, U_0, U_Q$</td>
</tr>
<tr>
<td>$\Theta, \alpha, \beta_i$</td>
</tr>
<tr>
<td>$\theta, \alpha, \beta_i$</td>
</tr>
<tr>
<td>$g_1(g) \in SE(3)$</td>
</tr>
<tr>
<td>$g_1(g)$, $g, \theta$</td>
</tr>
<tr>
<td>$x, y, z$</td>
</tr>
<tr>
<td>$v_1, v_2, v_3$</td>
</tr>
</tbody>
</table>
distribution. $Q$ is either a Lie subgroup or non-integrable based on whether or not $T_eQ$ is a Lie subalgebra of $se(3)$.

Conversely, to construct or enumerate feasible adjoint-invariant submanifolds one might start with a subspace $\Delta(e) = \{\xi_j \mid j = 1, \cdots , n\} \subset se(3)$ and a function $g_1(g) : SE(3) \to SE(3), g \to g_1(g)$, and then to each configuration $g \in SE(3)$ we assign a subspace

$$\Delta(g) = R_g \cdot Ad_{g_1(g)} \Delta(e) \subset T_g SE(3).$$

(3)

This yields a distribution $\Delta = \bigcup_g \Delta(g)$ on $SE(3)$. We refer to $\Delta$ as an adjoint-invariant distribution because it satisfies Eqn. (3). By comparing Eqn. (3) with Eqn. (2) it is easy to conclude that if $\Delta$ is integrable, then its integration $Q$ has the product structure $M \times G$. Notice that a basis of $\Delta$ is given by $\{g_1(\xi_i)g_1^{-1}(g)g \mid i = 1, \cdots , n\}$.

**Proposition 1.** If $\Delta$ is involutive, i.e., $\forall i, j \in (1, \cdots , n)$, we have

$$\begin{align*}
g_1(\xi_i)\hat{g}_1^{-1}(g)g_1(\xi_j)\hat{g}_1^{-1}(g)g & \in \text{span}\{g_1(\xi_1)\hat{g}_1^{-1}(g)g, \cdots , g_1(\xi_n)\hat{g}_1^{-1}(g)g\}, \quad (4) \\
\text{span}\{g_1(\xi_1)\hat{g}_1^{-1}(g)g, \cdots , g_1(\xi_n)\hat{g}_1^{-1}(g)g\}, \quad (5)
\end{align*}$$

then $\Delta$ is integrable, and the corresponding integration manifold $Q$ is adjoint-invariant.

Appendix A provides the calculation results about the Lie bracket $\begin{align*}
\left[g_1(\xi_i)\hat{g}_1^{-1}(g)g, g_1(\xi_j)\hat{g}_1^{-1}(g)g\right] = \left[g_1(\xi_i)\hat{g}_1^{-1}(g)g, g_1(\xi_j)\hat{g}_1^{-1}(g)g\right].
\end{align*}$ Moreover, it is possible to show that both Lie subgroups and symmetric subspaces are adjoint-invariant submanifolds (as proven in [9], [13], respectively).

**Example 1.** Lie subgroups and symmetric subspaces are adjoint-invariant submanifolds

Lie subgroups are obtained by integrating the distribution $\Delta(g) = R_{g} \cdot \Delta(e), g \in SE(3)$, with $\Delta(e) \subset se(3)$ a Lie subalgebra. $\Delta$ satisfies Eqn. (3) with $g_1(g) = e$. Then $\begin{align*}
g_1(\xi_i)\hat{g}_1^{-1}(g)g_1(\xi_j)\hat{g}_1^{-1}(g)g = \left[g_1(\xi_i)\hat{g}_1^{-1}(g)g, g_1(\xi_j)\hat{g}_1^{-1}(g)g\right] = \left[g_1(\xi_i)\hat{g}_1^{-1}(g)g, g_1(\xi_j)\hat{g}_1^{-1}(g)g\right].
\end{align*}$ Because $(\Delta(e)$ is a Lie subalgebra, $\Delta$ satisfies the involutive condition (5). So Lie subgroups are globally adjoint-invariant submanifolds [13].

Symmetric subspaces are in fact the integration manifold of the distribution $\Delta(g) = R_{g_1} \cdot \Delta(e), g \in SE(3)$ with $\Delta(e)$ a Lie triple system (i.e., closed under double Lie brackets) [9]. $\Delta$ satisfies Eqn. (3) with $g_1(g) = g_1^{1/2}$. The integrability of $\Delta$ can be verified by applying the general result of Appendix A. We have

$$\begin{align*}
g_1(\xi_i)\hat{g}_1^{-1}(g)g_1(\xi_j)\hat{g}_1^{-1}(g)g &= g_1(g)(\text{Ad}_{g_1^{1/2}(g)}[\xi_i, \xi_j] - \text{Ad}_{g_1^{1/2}(g)}[\hat{\xi}_i, \hat{\xi}_j])g_1(g)
\end{align*}$$

where $\hat{\xi}_i, \hat{\xi}_j \in \Delta(e)$. By expanding the formula we see that it only contains terms which are double Lie brackets of the elements in $\Delta(e)$. Therefore $\Delta$ satisfies the involutive condition, and its integration manifold is adjoint-invariant [9]. Symmetric subspaces have been classified in [19] (see Table II).

<table>
<thead>
<tr>
<th>Symmetric subspaces</th>
<th>$T_eQ$</th>
</tr>
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<tbody>
<tr>
<td>$M_p$</td>
<td>$m_p \triangleq {e_1, e_2, e_3, e_4, e_5}$</td>
</tr>
<tr>
<td>$M_1$</td>
<td>$m_1 \triangleq {e_1, e_2, e_3, e_4}$</td>
</tr>
<tr>
<td>$M_{1,2}$</td>
<td>$m_{1,2} \triangleq {e_1, e_2, e_4}$</td>
</tr>
<tr>
<td>$M_{3,4}$</td>
<td>$m_{3,4} \triangleq {e_1, e_3, e_4}$</td>
</tr>
<tr>
<td>$M_{3,5}$</td>
<td>$m_{3,5} \triangleq {e_1, e_3, e_5}$</td>
</tr>
<tr>
<td>$M_{2,5}$</td>
<td>$m_{2,5} \triangleq {e_2, e_5}$</td>
</tr>
<tr>
<td>$M_{2,4}$</td>
<td>$m_{2,4} \triangleq {e_2, e_4}$</td>
</tr>
<tr>
<td>$M_{2,3}$</td>
<td>$m_{2,3} \triangleq {e_2, e_3}$</td>
</tr>
<tr>
<td>$M_{2,4,5}$</td>
<td>$m_{2,4,5} \triangleq {e_2, e_4, e_5}$</td>
</tr>
</tbody>
</table>

It is expected that adjoint-invariant submanifolds of $SE(3)$ might exist in great abundance. We have the following existence and uniqueness result given a $\Delta$ satisfying (3).

**Proposition 2.** Let $\Delta$ be the distribution constructed from a given function $g_1(g)$ and $\Delta(e) \subset se(3)$ based upon (3). If $\Delta$ is involutive, then there exists a unique adjoint-invariant integration manifold $Q$ which is simply connected and maximal.

Proposition 2 implies that the solution manifolds in Example 1 are unique.

**Example 2.** Distributions whose integration manifold has the product structure

Let $g_1$ be the Lie subalgebras of an $m$-D Lie subgroup $G_1$ ($m < 6$), and $W$ be a different subspace of $se(3)$ such that $W \cap g_1 = \emptyset$. Suppose $\Delta(e) = g_1 \oplus W$ is an $n$-D subspace of $se(3)$ with $n < 6$. Any $g \in SE(3)$ can be written as $g = g_2 \cdot g_2^{-1} \cdot g_1$, where $g_2 \in G_1$ and $\tilde{g} = g_1^{-1} \cdot g$. Now construct the distribution $\Delta$ on $SE(3)$ as $\Delta(g) = R_g \cdot Ad_{g} \Delta(e)$, i.e. $g_1(g) = g_2$. We check the integrability of $\Delta$. The basis for $\Delta$ is given by $\text{span}(g_1(\tilde{\xi}_i) \tilde{g}) \mid i = 1, \cdots , n\}$, where $\tilde{\xi}_i$ is the basis of $\Delta(e)$. Then it is easy to verify that (see Appendix A)

$$\begin{align*}
g_2(\xi, \tilde{\xi}_i, \tilde{g}) &= g_2\left(\left[\xi, \tilde{\xi}_i\right] + \left[\tilde{\xi}_i, \tilde{\xi}_j\right]\right)\tilde{g}
\end{align*}$$

where $\tilde{\xi}_i$ (resp. $\tilde{\xi}_j$) is the projection of $\tilde{\xi}_i$ (resp. $\tilde{\xi}_j$) to the subspace $g_1$, while $\tilde{\xi}_i$ and $\tilde{\xi}_j$ are the projections of $\tilde{\xi}_i$ and $\tilde{\xi}_j$ onto $W$. As $g_1$ is a Lie subalgebra, $\left[\tilde{\xi}_i, \tilde{\xi}_j\right] \in g_1 \subset \Delta(e)$. So as long as $\left[\tilde{\xi}_i, \tilde{\xi}_j\right] \in \Delta(e)$, then $\Delta$ is involutive.

One solution is that $W = g_0$, the Lie subalgebra of another Lie subgroup $G_2$. The integration manifold of $\Delta$ is $G_1 \cdot G_2$. A more generic solution is that $W$ is the tangent space of a symmetric subspace $M$, whose completion algebra is contained in $\Delta(e)$. This leads to the integration submanifold $G_1 \cdot M$. The same results are rederived in Section IV with a different method.

The integration manifold depends on both $\Delta(e)$ and $g_1(g)$. Moreover, although finding integrable distributions $\Delta$ satisfying Eqn. (3) provides a general method for constructing adjoint-invariant submanifolds, it tends to be harder to check the involutive condition of the distribution $\Delta$ in Eqn. (3) as the function $g_1(g)$ becomes more and
more complex. In what follows we explore the methods that directly construct globally or locally adjoint-invariant submanifolds from the basic building blocks, Lie subgroups and symmetric subspaces. Through analytic extension of the tangent bundle $TQ = \cup_y T_y Q$ of these adjoint-invariant submanifolds $Q$, we obtain their corresponding integrable adjoint-invariant distribution $\Delta$ on $SE(3)$.

### III. REFLECTIVE-TYPE SUBMANIFOLDS

Let

$$S_h : SE(3) \to SE(3), h_0 \to hh_0^{-1}h \tag{6}$$

be the inversion map on $SE(3)$. $S_h$ has been extensively studied in [9], [10] for classifying and synthesizing symmetric subspaces motions. Here we extend the inversion map in (6) to define the inversion of a submanifold about a second one. Let $Q_i$, $i = 1, 2$, be $n_i$-dimensional submanifolds of $SE(3)$ such that $T_eQ_1 \cap T_eQ_2 = \emptyset$. Define $S_{Q_1}(Q_2) \triangleq \{S_{g_Q}(g) \mid g_Q \in Q_1, g \in Q_2\}$. Under suitable conditions $S_{Q_1}(Q_2)$ could be a local or global regular adjoint-invariant submanifold of $SE(3)$.

#### A. Conditions for adjoint invariance

We have the following result for the adjoint invariance of $S_{Q_1}(Q_2)$.

**Proposition 3.** Suppose $Q_i \neq M^p_{2A}$ are symmetric subspaces (including Lie subgroups as special cases). $S_{Q_1}(Q_2)$ is a globally adjoint-invariant submanifold if $\forall g_Q \in Q_1, g \in Q_2$, we have

$$(Ad_{g^{-1/2}_1g^{-1/2}_2})T_eQ_1 + T_eQ_2 = T_eQ_1 + T_eQ_2. \tag{7}$$

Under the same condition it is only locally adjoint-invariant if one of $Q_i$ is $M^p_{2A}$.

**Proof:** See Appendix B. $\Box$

The condition (7) can be replaced by a simplified condition.

**Corollary 1.** Suppose $Q_i$, $i = 1, 2$, are $n_i$-dimensional symmetric subspaces. If $\forall \xi_1 \in T_eQ_1, \xi_2 \in T_eQ_2$, and $\forall \eta \in [T_eQ_1, T_eQ_1]$,

$$ad_{\xi_1} \eta \in T_eQ_1 + T_eQ_2 \tag{8}$$

$$ad_{\xi_2} \xi_1 \in T_eQ_1 + T_eQ_2, \tag{9}$$

then $S_{Q_1}(Q_2)$ is a locally adjoint-invariant submanifold after excision out the region of singularities of measure 0, i.e. there exists an open neighborhood $U_e \in SE(3)$, such that $U_e \cap S_{Q_1}(Q_2)$ is an $(n_1 + n_2)$-dimensional adjoint-invariant submanifold.

**Proof:** See Appendix C. $\Box$

#### B. Two Important Sub-categories

We derive two important cases that satisfy Eqn. (8) and (9).

1. **Case 1:** $Q_1$ is a sub-6 DoF Lie subgroup: Obviously $Q_1$ cannot be $SO(3)$ for which $[T_eQ_1, T_eQ_1] = so(3)$ as there exists only empty-set $T_eQ_2$ which satisfies both $T_eQ_2 + T_eQ_1 \neq se(3)$ and Eqn. (8). For other Lie subgroups, $[T_eQ_1, T_eQ_1] = \emptyset$ or $\{e, e_2\}$ up to the adjoint map. The only non-trivial cases (i.e. neither Lie subgroups nor symmetric subspaces) which satisfy both Eqn. (8) and (9) are $S_{C(z)}(R(y))$ and $S_{C(z)}(I_{e_{1/2}}(M_{2A}))$.

2. **Case 2:** $Q_1$ is a symmetric subspace but not a Lie subgroup: In this case $T_eQ_1$, $T_eQ_1$ is the Lie subalgebra $h$ of an isotropy group $H$ of $Q_1$, and $g \triangleq T_eQ_1 + h$ is the completion Lie algebra of $T_eQ_1$. Notice $[h, h] \subset h$ and $[h, T_eQ_1] \subset T_eQ_1$ based upon the relation between $T_eQ_1$ and the Lie subalgebra $h$ [9]. We could chose $T_eQ_2$ as a subspace of $h$ (which already satisfies (9)) such that

$$ad_{\eta} T_eQ_2 \subset T_eQ_2, \forall \eta \in h \tag{10}$$

for satisfying (8), or choose $T_eQ_2 \subset g^1$ that satisfies both (8) and (9). Finally, we obtain $S_{M_4}(R(z)), S_{M_4}(H_{p}(z)), S_{M_{2A}}(R(y))$, and $S_{M^p_{2A}}(R(y))$.

**Example 3.** $S_{M_4}(R(z))$ is locally adjoint-invariant

Let $g_{1/2}^1 = \begin{bmatrix} R_{xy} & (R_{xy} + 1)P_{xy} \ 0 & 1 \end{bmatrix} \in M_4$, and $g_{1/2}^2 = \begin{bmatrix} z & 0 \ 0 & z^{-1} \end{bmatrix} \in R(z)$, where $z = e^{i\theta}, \theta \in \mathbb{R}, R_{xy} = e^{i\theta_1 + i\theta_2}, \theta_1, \theta_2 \in \mathbb{R}$, and $P_{xy} = \gamma_1 x + \gamma_2 y, \gamma_1, \gamma_2 \in \mathbb{R}$. We calculate $(Ad_{g_{1/2}^1g_{1/2}^2})T_eQ_1$ as

$$\begin{bmatrix} (R_{xy}^TR_{xy}^T + R_{xy}R_{xy})x & (R_{xy}^TR_{xy}^T + R_{xy}R_{xy})y \ 0 & 0 \ v_1 & v_2 \ (R_{xy}^TR_{xy}^T + R_{xy}R_{xy})x & (R_{xy}^TR_{xy}^T + R_{xy}R_{xy})y \ \end{bmatrix},$$

where it is easy to check that both $(R_{xy}^TR_{xy}^T + R_{xy}R_{xy})x \parallel \{x, y\}$ and $(R_{xy}^TR_{xy}^T + R_{xy}R_{xy})y \parallel \{x, y\}$, so do $v_1$, $v_2$.

So Eqn. (7) holds at least in an open neighborhood of $e$ without singularities.

The list of non-trivial reflective-type adjoint-invariant submanifolds is given in Table III. They are sub-6 dimensional adjoint-invariant submanifolds, which, to the best of our knowledge, have not been studied in the previous literatures.

### TABLE III

**Non-trivial reflective-type adjoint-invariant submanifolds**

<table>
<thead>
<tr>
<th>Submanifolds</th>
<th>$T_eQ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{M_4}(R(z))$</td>
<td>${e_1, e_2, e_3, e_4, e_5, e_6}$</td>
</tr>
<tr>
<td>$S_{M_4}(H_{p}(z))$</td>
<td>${e_1, e_2, e_4, e_5, e_6 + e_3}$</td>
</tr>
<tr>
<td>$S_{C(z)}(I_{e_{1/2}}(M_{2A}))$</td>
<td>${e_1, e_3, e_4, e_5}$</td>
</tr>
<tr>
<td>$S_{C(z)}(R(y))$</td>
<td>${e_1, e_4, e_5}$</td>
</tr>
<tr>
<td>$S_{M^p_{2A}}(R(y))$</td>
<td>${e_3, e_4, e_5}$</td>
</tr>
</tbody>
</table>
IV. Product-type submanifolds

In this section we study product-type submanifolds of the form $Q = Q_1 \cdot Q_2$, where $Q_i$ are symmetric subspaces (including Lie subgroups as special cases). The cases where both $Q_i$ are Lie subgroups have already been studied by [13], and therefore are considered trivial here.

**Proposition 4.** If $Q_i$ are $n_i$-dimensional symmetric subspaces, $T_c Q_1 \cap T_c Q_2 = \{0\}$, and $\forall g_a \in Q_1, g_b \in Q_2$ we have

$$Ad_{g_a^{-1}} T_c Q_1 + Ad_{g_b^{-1}} T_c Q_2 = T_c Q,$$

(11)

then $Q$ is a globally $(n_1 + n_2)$-dimensional adjoint-invariant submanifold.

**Proof:** Following the proof of Proposition 3 by using the fact that $V_{g_a g_b} (Q_1 \cdot Q_2) = Ad_{g_a} (Ad_{g_b^{-1}} T_c Q_1 + Ad_{g_a^{-1}} T_c Q_2)$.

A simpler sufficient condition for Eqn. (11) is given by

**Corollary 2.** Given two symmetric subspaces $Q_i$ with $T_c Q_1 \cap T_c Q_2 = \{0\}$. Let $g_i$ be the completion algebra of $T_c Q_i$ such that $g_i = T_c Q_i + [T_c Q_1, T_c Q_2]$. If

$$g_i \subset T_c Q_1 + T_c Q_2, i = 1, 2,$$

(12)

then $Q$ is a globally $(n_1 + n_2)$-dimensional adjoint-invariant submanifold after excising out the region of singularities of measure 0.

**Proof:** Following the proof of Corollary 1 by expanding $Ad_{g_a^{-1}} T_c Q_1 + Ad_{g_b^{-1}} T_c Q_2$. The special case that $Q_1$ is a Lie group is proved in Example 2. □

$Q_1 \cdot Q_2$ is fundamentally different from $S_{Q_1}(Q_2)$. The only exceptions are $M_5$ and $M_{3A}$, for which $Q_1 \cdot Q_2$ and $S_{Q_1}(Q_2)$ are sometimes equivalent, as proved in [10]. Table IV summarizes the list of non-trivial (i.e., excluding the products of two Lie subgroups) product-type adjoint-invariant submanifolds and their tangent spaces $T_Q$ at identity $e$. These submanifolds exhibit adjoint-invariant DoFs which, to the best of our knowledge, haven’t been adequately studied.

V. Kinematic Generators (KG) of Reflective-type adjoint-invariant Submanifolds

In this section we will focus on synthesizing kinematic generators of reflective-type adjoint-invariant submanifolds of $SE(3)$. Consider the maximal inscribing symmetric subspace $M_{max}$ and the minimal covering symmetric subspace $M_{min}$ with $M_{max} \subset Q \subset M_{min}$. $S_{Q_1}(Q_2)$ can be obtained by compressing $M_{min}$ or expanding $M_{max}$.

A. Compressing $M_{min}$

Some reflective-type adjoint-invariant submanifolds can be synthesized by assembling $Q_1 : Q_2 : Q_1$ chains with a $M_{min}$ generator in parallel.

**Example 4. KG for $S_{C}(\mathcal{R}(\mathcal{y}))$**

Notice that $S_{C}(\mathcal{R}(\mathcal{y})) \subset C^-(x) \cdot \mathcal{R}(\mathcal{y}) \cdot C^+(x)$ which is generated by cascading a pair of symmetric joints $(C^-(x), C^+(x))$ about the $x - y$ plane with a $\mathcal{R}(\mathcal{y})$ joint in the middle, as shown in Fig. 1-(a). On the other hand the minimal covering symmetric subspace $M_{min}$ of $S_{C}(\mathcal{R}(\mathcal{y}))$ is $M_4$. Then we show that $S_{C}(\mathcal{R}(\mathcal{y})) = \mathcal{C}^-(x) \cdot \mathcal{R}(\mathcal{y}) \cdot C^+(x)$ and $M_4$. First, $S_{C}(\mathcal{R}(\mathcal{y}))$ belongs to both $C^-(x) \cdot \mathcal{R}(\mathcal{y}) \cdot C^+(x)$ and $M_4$. Second, at home configuration $e$ the constraint forces of $M_4$ is $\{e_3, e_6\}$, while that of $C^-(x) \cdot \mathcal{R}(\mathcal{y}) \cdot C^+(x)$ is $e_2$. The feasible tangent space of the parallel mechanism at home configuration is simply $\{e_1, e_4, e_5\}$. According to Position 6 of [5], the parallel mechanism consisting of a $M_4$ KG and a $C^-(x) - \mathcal{R}(\mathcal{y}) - C^+(x)$ subchain is the KG for $S_{C}(\mathcal{R}(\mathcal{y}))$, as shown in Fig. 1-(b) (only one subchain of the $M_4$ KG is drawn here for clarity). A practical mechanism can be derived by replacing the full $M_4$ generator (e.g. Example 5 in [10]) by its subchains $M^s_4$, and interconnecting $M^s_4$ as well as the $C^-(x) - \mathcal{R}(\mathcal{y}) - C^+(x)$ chain in a similar manner. This reduce the number of $M^s_4$ subchains from 3 as required in the full $M_4$ generator to 2, as shown in Fig. 1-(c).

**Example 5. KG for $S_{C}(I_{e^{-\pi/2}}(M_{2A}))$**

It is easy to see that $S_{C}(I_{e^{-\pi/2}}(M_{2A}))$ is equivalent to $S_{C}(\mathcal{R}(M_{2A}))$ up to the conjugation map $I_{e^{-\pi/2}}$. Recall $S_{C}(\mathcal{R}(M_{2A})) = S_{\mathcal{R}(\mathcal{y})}(S_{\mathcal{R}(\mathcal{y})}(M_{2A})) \subset S_{\mathcal{R}(\mathcal{y})}(\mathcal{P}(x)) \subset \mathcal{R}(\mathcal{y})^{-} \cdot \mathcal{P}(x) \cdot \mathcal{R}(\mathcal{y})$, where $\mathcal{P}(x)$ is realized by cascading three revolute joints parallel to $x$, and $(\mathcal{R}(\mathcal{y}), \mathcal{R}(\mathcal{y}))$ are a pair of symmetric revolute joints about the $x - y$ plane. Combine the two distal revolute joints into a $\mathcal{U}(x, y^-)$ pair and a $\mathcal{U}(x, y^+)$ pair ($y^-$ and $y^+$ in the $y$ pairs are used to show that they are symmetric about the $x - y$ plane). This yields a $\mathcal{U}(x, y^-) - \mathcal{R}(x) - \mathcal{U}(x, y^+)$ mechanism, as shown in the left subchain in Fig. 2. Its constraint force space is given by $\{[x^T, (P_{xy} \times x)^T]^T\}$, where $P_{xy} \in \mathbb{R}^3$ is a point in the $x - y$ plane. On the
other hand $S_{c(y)}(M_{2A}) \subset M_{\text{min}} = M_5$. It is realized as the Delta - Omni-wrist mechanism (the right subchain of Fig. 2), which contributes the constraint force space $\{[0, z^T]^T\}$. The parallel mechanism formed by connecting these two subchains in parallel gives rise to the constraint force space $\{e_1, e_6\}$, and therefore it is a KG of $S_{c(z)}(I_{e,\pi/2}(M_{2A}))$.

![Fig. 1](image1.png)

**Fig. 1.** (a): Parallel Mechanism composed of a $C^-(x) - R(y) - C^+(x)$ subchain and an $M_4$ subchain; (b): $M_4$ subchain is realized by 3 pairs of symmetric $U - U$ chains interconnected through cylindrical joints as proposed in [10]; (c): Only 2 pairs of symmetric $U - U$ chains are required if we employ additional interconnection with the $C^-(x) - R(y) - C^+(x)$ subchain.

**B. Expanding $M_{\text{max}}$**

The reflective-type adjoint-invariant submanifolds, $S_{M_4}(R(z))$ and $S_{M_4}(H_p(z))$, can be synthesized by expanding the KG of its maximal inscribing symmetric subspace $M_{\text{max}}$.

**Proposition 5.** If $S_{Q_1}(Q_2)$ is a reflective-type adjoint-invariant submanifold with $Q_1$ a symmetric subspace ($\neq M_4$), and $Q_2$ a Lie subgroup satisfying $T_cQ_2 \subset h = [T_cQ_1, T_cQ_1]$ and $ad(T_cQ_2 \subset T_cQ_2, \forall \eta) \in h$, then a KG for $S_{Q_1}(Q_2)$ could be synthesized by inserting a $Q_2$ chain between each pair of symmetric sub-chains in the KG for $Q_1$, while reducing the corresponding DoFs in all interconnecting chains.

![Fig. 2](image2.png)

**Fig. 2.** KG for $S_{c(y)}(M_{2A})$ composed of a $U(x, y^-) - R(x) - U(x, y^+)$ subchain and an $M_5$ subchain whose wrist plane (the plane passing through the three spherical joints in the wrist) is parallel to the $x - y$ plane. The $y^-$ and $y^+$ axes of the pair of symmetric $U$ pairs of the $U(x, y^-) - R(x) - U(x, y^+)$ subchain intersect at a point $P_{xy}$ in the $x - y$ plane.

**Proof:** See Appendix D.

**Example 6. KG for $S_{M_4}(R(z))$**

It is easy to see that the maximal inscribing symmetric subspace $M_{\text{max}}$ of $S_{M_4}(R(z))$ is $M_4$. For $M_4$, we have $[m_4, m_4] = \{e_1, e_6\}$. We choose $Q_2 = R(z) = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$, which satisfies the condition in Proposition 5. Now we add this additional rotational DoF $R(z)$ to the middle of the original subchain $M_4^1$ of $M_4$. The new subchain is denoted as $N_4^1$. Assembling $3 N_4^1$ together, and interconnecting them with a prismatic joint (instead of the cylindrical pair in the original $M_4$ KG) yields a KG for $S_{M_4}(R(z))$, as shown in Fig. 3-(a). The KG for $S_{M_4}(H_p(z))$ can be synthesized in the same way.

![Fig. 3](image3.png)

**Fig. 3.** (a): A KG for $S_{M_4}(R(z))$ by adding a rotational DoF of $R(z)$ to the middle of the original subchain $M_4^1$ in a $M_4$ KG; (b): A KG for $S_{M_4}(R(z))$. (c)

**C. Compressing covering reflective-type submanifolds**

Some reflective-type adjoint-invariant submanifolds are contained in one or multiple reflective-type submanifolds (called covering reflective-type submanifolds). The KG for these covering reflective-type submanifolds can be used as the primitive subchains.
Example 7. KG for $S_{M_2}(R(y))$

Notice that $S_{M_2}(R(y)) \subset S_{M_1}(R(y))$, where $M_1 = I_{e+1/2}(M_2)$ is a 4-dimensional symmetric subspace satisfying $S_{M_1} \subset M_1$. $S_{M_2}(R(y))$ is equivalent to $I_{e+1/2}(S_{M_2}(R(y)))$. The KG for the latter reflective submanifold $S_{M_2}(R(y))$ is discussed in Example 6. On the other hand $S_{M_2}(R(y)) = \{ e^{\Omega} \theta_1 e^{\Omega} \theta_2 \}$ based on the facts that $M_2 = T(x) \cdot M_2$. Since $e^{\Omega} \theta_1 e^{\Omega} \theta_2 = e^{\Omega} \theta_1 e^{\Omega} \theta_2$, by direct computation, we have $S_{M_2}(R(y)) = S_{T(x)}(S_{M_2}(R(y))) \subset S_{T(x)}(M_2B) \subset T^-(x) \cdot M_2B \cdot T^+(x)$, where $T^-(x) \cdot M_2B \cdot T^+(x)$ can be generated by cascading a pair of symmetric translational pair ($T^-(x), T^+(x)$) with a KG (e.g. Example 4 in [10]) for $M_2B$ in between. Finally assembling the KG for $S_{M_2}(R(y))$ and that for $T^-(x) \cdot M_2B \cdot T^+(x)$ yields a KG for $S_{M_3}(R(y))$, as illustrated in Fig. 3-(b). This can be proved by recalling that at home configuration $e$ the constraint force of the former subchain is $e_{f_2}$, while that of the latter subchain is $e_{f_3}$, and therefore $T_c S_{M_2}(R(y)) \cap T_c (T^-(x) \cdot M_2B \cdot T^+(x)) = T_c S_{M_3}(R(y))$.

VI. CONCLUSION

In this paper we propose a class of submanifolds of $SE(3)$, the adjoint-invariant submanifolds, which extends the theory of Lie subgroups and symmetric subspaces by relaxing the symmetry requirements in these objects. We study global geometric properties as well as existence and uniqueness of adjoint-invariant submanifolds based on the theories of distributions on manifolds and their integrability. Then we classify adjoint-invariant submanifolds into reflective-type submanifolds and product-type submanifolds, and derive the conditions for adjoint invariance for each of the subcategory spaces. With the developed theory and methods we obtain the list of non-trivial reflective-type and product-type adjoint-invariant submanifolds. Finally we propose geometric tools and algorithms for constructing the kinematic generators for reflective-type adjoint-invariant submanifolds along with a number of examples.

APPENDIX A

Calculation of $[g_1(g) \xi_1 g_1^{-1}(g)g, g_1(g) \xi_2] g_1^{-1}(g)g]$

Let $\xi_1 = g_1(g) \xi_1 g_1^{-1}(g)g$. The integral curve of the vector field $\xi_1$ is simply $h_i(t) = g_1(g) e^{\xi_1 t} g_1^{-1}(g)g$. Then given a function $f$ on $SE(3)$ we calculate $[\xi_1, \xi_2] = [g_1(g) \xi_1] g_1^{-1}(g)g = [g_1(g) \xi_1 g_1^{-1}(g)g] f$ as

$$((\xi_1 g_1^{-1}(g)g) |_{t=0} \xi_2) - (\xi_2 g_1^{-1}(g)g) |_{t=0} \xi_1 = 0$$

and $+(g_1(g)(\xi_2 \xi_1 g_1^{-1}(g)g) |_{t=0} \xi_1 = 0$$

$+g_1(g) [\xi_1, \xi_2] g_1^{-1}(g)g],$

where $(\xi_1 g_1^{-1}(g)g) |_{t=0} = 0$ and $(\xi_2 g_1^{-1}(g)g) |_{t=0} = 0$ denote the directional derivative of $g_1(g)$ and $g_1^{-1}(g)g$ along the integral curve $h_i(t)$ of $\xi_1$ at $t = 0$.

If $g_1(g) = e$, then $[\xi_1, \xi_2] = g_1[\xi_1, \xi_2] g_1^{-1}(g)g$.

APPENDIX B

Proof of Proposition 3

Notice that we have a local parameterization for each open neighborhood $U_i$ of $e$ on $Q_i$, $i = 1, 2$, as they are all symmetric subspaces

$g_a \in U_1 = e^{\Sigma_{i=1}^{n-1} \xi_i e_i}, g_b \in U_2 = e^{\Sigma_{i=1}^{n+2} \xi_i e_i}$

where $T_c Q_1 = \{ \xi_1, \cdots, \xi_n \}$, and $T_c Q_2 = \{ \xi_{n+1}, \cdots, \xi_{n+n+2} \}$. At a generic point $g = g_a g_b$, we can assign a coordinate map $\phi_\hat{g}$ on an open neighborhood $U_\hat{g}$, namely, $\phi_\hat{g}(\Theta) = g_1^{1/2} e^{\Sigma_{i=1}^{n-1} \xi_i e_i} g_{a_1}^{1/2} g_{a_1}^{1/2} e^{\Sigma_{i=1}^{n+2} \xi_i e_i} g_{a_2}^{1/2} e^{\Sigma_{i=1}^{n+2} \xi_i e_i} g_{a_2}^{1/2}$, as $g_1^{1/2} e^{\Sigma_{i=1}^{n-1} \xi_i e_i} g_{a_1}^{1/2}$ is a local coordinate map in a neighborhood about $g$ on $Q_1$, and $g_{a_2}^{1/2} e^{\Sigma_{i=1}^{n+2} \xi_i e_i} g_{a_2}^{1/2}$ is a local coordinate map on $Q_2$. Notice that $\phi_\hat{g} : U_0 \rightarrow U_\hat{g}, \Theta \rightarrow \phi_\hat{g}(\Theta)$, where $U_0 \subset \mathbb{R}^{n+1}$. The Jacobian $J$ of $\phi_\hat{g}$ is given by $J = Ad g_b^{1/2} J_1$, where $J_1 = [A_1 \xi_1, \cdots, A_\xi_{n+1} \xi_1, \cdots, \xi_{n+n+2}]$, and $A = Ad \ g_{a_1}^{1/2} g_{a_2}^{1/2} + Ad \ g_{a_2}^{1/2} g_{a_2}^{1/2}$. The range space of $J_1$ is easy to see to be $(Ad \ g_{a_1}^{1/2} g_{a_1}^{1/2} + Ad \ g_{a_2}^{1/2} g_{a_2}^{1/2}) T_c Q_1 + T_c Q_2$. It is exactly $T_c Q_1 + T_c Q_2$, when $g_a = e$. Therefore if $g_{a_1} \in Q_1, g_b \in Q_2, Ad \ g_{a_2}^{1/2} g_{a_2}^{1/2} + Ad \ g_{a_2}^{1/2} g_{a_2}^{1/2} T_c Q_1 + T_c Q_2$ is exactly $T_c Q_1 + T_c Q_2$. $J_1$ is nonsingular at every combinations of $(g_a, g_b)$. In fact we can prove that $U_\hat{g}$ is a slice of $V_\hat{g}$, a local neighborhood of $\hat{g}$ on $SE(3)$. We have $\phi_\hat{g}(\Theta) = g_a g_b^{1/2} h_{\phi_\hat{g}^{-1}} g_a^{1/2} h_{\phi_\hat{g}^{-1}}$, where $h = e^{\Sigma_{i=1}^{n-1} A_{\phi_\hat{g}^{-1}} \xi_i e_i} e^{\Sigma_{i=1}^{n+2} \xi_i e_i}$, $\xi_{i} = (Ad \ g_{a_1}^{1/2} g_{a_1}^{1/2} + Ad \ g_{a_2}^{1/2} g_{a_2}^{1/2}) \xi_i$, and $g_a = e^{1/2} h e^{\Sigma_{i=1}^{n-1} a_i} \xi_i e^{1/2} h_e^{1/2} g_a$. $\phi_\hat{g}$ is a slice of $\psi_\hat{g}(\Theta, \alpha) = g_a g_b^{1/2} h e^{\Sigma_{i=1}^{n-1} a_i} \xi_i e^{1/2} h_e^{1/2} g_a$. 

If $g_1(g) = g^{1/2}$, then $[\xi_1, \xi_2] = g_1([g_1^{1/2} B_1 g_1^{1/2} B_j - [B_j g_1^{1/2} B_1 g_1 B_j, g_1^{1/2} B_j]) g_1$, where $B_i = \frac{d(g_1 g_1^{1/2} B_i g_1)+g_1 B_i}{dt} |_{t=0}$, and we use the fact that $g_1 \xi_1 g_1 = \frac{d(g_1 g_1^{1/2} B_i g_1)+g_1 B_i}{dt} |_{t=0} = B_i B_j B_1 + g_1 B_i$.
for which \( \alpha_j = 0, j = 1, \ldots, 6 - n_1 - n_2 \). Notice that 
\[ (\zeta_1, \ldots, \zeta_{n_1}, \xi_{n_1+1}, \ldots, \xi_{n_1+n_2}, \eta_1, \ldots, \eta_{6-n_1-n_2}) \]
is a basis of \( \mathfrak{se}(3) \); \( \psi_{\hat{g}} \) is a local coordinate map at \( \hat{g} \) on \( SE(3) \). Therefore \( \phi_g \) generates an atlas for \( S_{Q_1}(Q_2) \).

**Appendix C**

**Proof of Corollary 1**

As \( Q_1, i = 1, 2 \), are symmetric subspaces (including Lie subgroup as a special case), we can express \( g_a^{1/2} = e^{C_1} \) and \( g_b^{1/2} = e^{C_2} \), \( i \in T_2Q_1, i = 1, 2 \) (globally (except for \( M_2^g \) for which the expression only holds locally)). Given \( \xi_3 \in T_2Q_1 \), we can see that \( \xi_3 \in T_2Q_1 \) and \( \eta \in [T_2Q_1, T_2Q_1] \). Further calculation shows that 
\[ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{n_1}^{\infty} \sum_{n_2}^{\infty} \sum_{\xi_3}^{\eta} \]}
Notice that \( ad^2\xi_3 \in C_3 \).

**Appendix D**

**Proof of Proposition 5**

Recall that the \( n \)-D symmetric subspace \( Q_1 \) (except for \( M_2^g \)) can be constructed by assembling \( k \) subchains \( C_i, i = 1, \ldots, k \), which are in turn composed of a pair of \( n \)-D symmetric sub-subchains \( \{C_1^+, C_1^-\} \). Suitable interconnecting subchains are added that link the middle link of \( C_1 \). According to [10] the forward kinematic map of \( C_i^+ \) has the form \( e^{C_1 \xi} (\xi \in T_2Q_1, \eta \in [T_2Q_1, T_2Q_1]) \) and that of \( C_i^- \) is \( e^{-\eta} e^{C_1} \), and their combo-kinematic map of \( C_1 \) is \( e^{2C_1} e^{\xi} \) as long as \( \{C_1^+, C_1^-\} \) maintains a symmetric arrangement. Inserting a \( Q_2 \) chain to the middle of \( C_1 \) with \( Q_2 \) a Lie subgroup and \( T_1Q_2 \subset [T_2Q_1, T_2Q_1] \) yields a new subchain \( A_1 = C_1^+ - Q_2 - C_1^- \) whose combo-kinematic map (with \( \zeta_1 \in T_3Q_2 \) is \( e^{2C_1} e^{\xi} e^{-\eta} e^{\xi} = e^{e^{A_1}} e^{\xi} \in S_{Q_1}(Q_2) \), as long as \( e^{\xi} e^{-\eta} = e^{\xi} \). The latter is ensured by the condition in the proposition. The centers of pairs of subchains \( \{A_1, A_2\} \) are joined by a new interconnecting mechanism whose screws come from \([T_2Q_1, T_2Q_1]\) but excluding those from \( T_2Q_2 \).

Now we strictly prove that the task space of the mechanism \( \{A_1, \ldots, A_k\} \) is \( S_{Q_1}(Q_2) \) after applying the closedloop constraints by following the rigidity argument proposed in [10]. First, given an arbitrary motion of \( C_i^+ - Q_2 \) in \( A_1 \), there is only one feasible solution locally for \( C_i^+ - Q_2 \) of \( A_j \) (\( j \neq 1 \)) and the interconnecting mechanism between \( A_1 \) and \( A_j \). The remaining mechanism composed of \( C_i^+ \) for all \( j \) forms a motionless rigid mechanism. This yields a manifold of dimension exactly same as that of \( S_{Q_1}(Q_2) \) as \( S_{Q_1}(Q_2) \) has same degrees of freedom as \( C_i^+ - Q_2 \). Then we consider a submanifold \( Q_2 \) of the task space \( Q_2 \) of the entire mechanism. Each point of \( Q_2 \) is obtained by applying an arbitrary motion of \( C_i^+ \) first while freezing the motion of the \( Q_2 \) chain of all subchains to be identity \( e \). The remaining mechanism \( \{C_i^+, \ldots, C_i^+\} \) becomes rigid again so that \( (C_i^+, C_i^-) \) in \( A_j \) forms a symmetric arrangement exactly as the KG for \( Q_1 \). Then we move the entire top half of the mechanism relative to the entire bottom half, by an arbitrary motion in \( Q_2 \). As the result of the composition of these two motions, the motion of \( A_j \) is exactly given by \( e^{\psi_{Q_1}} \psi_{Q_2} \in S_{Q_1}(Q_2) \), and thus \( S_{Q_1}(Q_2) \subset Q_2 \subset Q_3 \) (at least locally). We just proved that \( Q_2 \) has the same dimension as \( S_{Q_1}(Q_2) \). They must match at least in an open neighborhood of \( e \).