On Quality Functions for Grasp Synthesis, Fixture Planning, and Coordinated Manipulation

Guanfeng Liu, Jijie Xu, Xin Wang, and Zexiang Li

***Abstract*—Planning a proper set of contact points on a given object/workpiece so as to satisfy a certain optimality criterion is a common problem in grasp synthesis for multifingered robotic hands and in fixture planning for manufacturing automation. In this paper, we formulate the grasp planning problem as optimization problems with respect to three grasp quality functions. The physical significance and properties of each quality function are explained, and computation of the corresponding gradient flows is provided. One noticeable property of some of these quality functions is that the optimal solutions are also force-closure grasps if they do exist for the given object. Furthermore, when specialized to two-fingered or three-fingered grasps on a spherical object, the optimal solutions become the familiar antipodal grasp, or the symmetric grasp, respectively. Thus, by following the gradient flows with arbitrary initial conditions, the optimal grasp synthesis problem is solved for objects with smooth geometries manipulated by hands with any number of fingers. Also, note that our solutions do not involve linearization of the friction cones. We discuss two simplified versions of these problems when real-time solutions are needed, e.g., coordinated manipulation of a robotic hand with contact points servoing. We give simulation and experimental results illustrating validity of the proposed approach for optimal grasp planning.**

***Note to Practitioners:* This paper presents three new quality functions for comparing and planning grasps and fixtures. These measures improve on the traditional measure of force closure. We propose a method for computing the optimal solutions of these functions, and a method for reducing their computation time through reasonable simplification/approximation. Preliminary experiments with a three-fingered robotic hand demonstrate that the proposed functions can be used to optimize the grasp quality during manipulation/manufacturing, and keep the optimal grasp configuration once it is reached. However, we only obtain the local optimal solutions for the functions without simplification except for some special cases. We also assume that the object/workpiece is ideally rigid in all three functions. In future research, we will improve these limitations through a compliance model.**

***Index Terms*—Grasp synthesis, max-transfer problem, max-normal-grasping-force problem, min-analytic-center problem, gradient computation.**

I. INTRODUCTION

PLANNING a proper set of contact points on a given object/workpiece so as to satisfy a certain optimality criterion is a common problem in grasp synthesis for multifingered robotic hands, and in fixture planning for manufacturing automation. During a full multifingered manipulation cycle, grasp planning arises in several occasions, such as when an object is first picked up from say, a table top; or when the object is manipulated from an initial to a final grasp configuration through a continuum of force-closure grasps in order to not dropping the object (dextrous manipulation); or when the object is coordinatively manipulated to execute a given task (e.g., scribing) with contact points servoing (so as to maintain the object in an optimal grasp configuration). Research on grasp planning centers on two broad categories: *grasp analysis* and *grasp synthesis*.

Early work on grasp analysis includes that of Reulaux [1], who introduced the notion of force-closure and form-closure grasps; that of Salisbury [2], who developed mathematical models of contact and grasp, and provided necessary and sufficient conditions for force-closure grasps; that of Mishra *et al.* [3] for FPCs, who showed that a grasp is force closure if and only if the origin of the wrench space lies in the interior of the convex hull of the primitive wrenches. Several force-closure tests based on these conditions were developed by Chen and Burdick [4], Nguyen [5], and Trinkle [6]. Bicchi [7] translated the force-closure problem into the stability of an ordinary differential equation. Recently, by linearizing the friction cones, Liu [8] introduced a ray-shooting problem (LP) and proposed a clean-cut test for force-closure grasps. Han *et al.* [9] observed that the nonlinear friction cone constraints can be represented as linear matrix inequalities (LMIs) and the force-closure problem can be reformulated as the feasibility problem of a semi-definite or max-det problem, for which efficient algorithms are now available. Thus, the general problem of determining if a grasp is force closure is considered to be completely solved. Furthermore, the problem of computing optimal finger forces within the limits of the friction cones to balance a given external wrench is also solved with the work of [10], [11], [9].

Research on optimal grasp synthesis consists of: 1) determination of the optimality criteria and 2) derivation of methods and algorithms for computing contact locations with respect to the optimality criteria and subject to accessibility constraints. Early work in this area includes synthesis of grasps for polygonal and polyhedral objects which are force closure. Ji and Roth [12] derived conditions on contact positions and surface normals that guarantee a grasp to be force closure. Nguyen [5] gave conditions for constructing planar two-fingered force-closure grasps, which was generalized by Ponce and Faverjon [13] to three-fingered case, and by Ponce *et al.* [14] to four-fingered case. Mishra *et al.* [3] proposed an algorithm for computing force-closure grasps for polyhedral objects under FPCs. Ding *et al.* [15] pro-

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posed heuristics for searching an eligible set of grasping surfaces of a polyhedra and a quadratic programming approach for selecting an optimal form-closure grasp that minimizes the positioning errors. Liu [16] proposed an algorithm for computing all form-closure grasps of polygonal objects with arbitrary number of fingers. Apparently, to a given object there exist in general a large set of grasps which are force closure. In other words, force closure is too coarse a criterion to be used for grasp synthesis. More refined criteria are needed to define the notion of grasp optimality. Cutkosky [17] and Li and Sastry [18] proposed the use of task requirement for grasp selection. For general two-fingered grasps, Hong *et al.* [19] used the distance between two fingers as a grasp quality function, of which antipodal grasps are the optimal solutions. Using this function, Chen and Burdick [4] developed gradient algorithms for grasp planning. Similar works could also be found in [20]–[22]. A great deal of difficulties exist when one aims to extend this approach to grasps with more than two fingers except the particular case, a three-fingered hand grasping a spherical object, for which the area of the triangle formed by the three contact points is used as a physically meaningful quality function. The optimal solution for this function turns out to be the symmetric grasp where the three fingers locate at three symmetric points of a big circle. To develop a general approach to grasp synthesis that is not confined to objects with specific geometries, Kirkpatrick *et al.* [23] proposed a quality measure based on the capability of the grasp in resisting external wrenches. They further translated the problem to the computation of the radius of the largest L_2 ball contained in the convex hull of the primitive wrenches. The same idea was also adopted by Ferrari and Canny [24]. To avoid the ambiguity arising in defining physically meaningful norms for external wrenches, Mirtich and Canny [25] proposed two quality functions via decoupling the force and moment components of a wrench. Based on these two functions, they computed several examples and obtained the well known optimal grasps by other approaches. Zhu *et al.* [26], [27] introduced the Q distance and adopted the radius of the largest Q ball contained in the convex hull of the primitive wrenches as a quality measure.

To summarize, a complete solution to the general optimal grasp synthesis problem rests on derivation of grasp quality functions which: 1) incorporate the force-closure condition, i.e., optimal solutions are also force-closure grasps and 2) have easily computable gradients. In other words, an optimal grasp can be attained by following the gradient flows of the quality functions starting from some initial conditions which may not be force closure. Based on our review of previous works, this problem remains largely unsolved. The aim of this paper is to develop solutions to this problem that: 1) have clean senses of optimality; 2) do not involve approximation of the friction cones; and 3) can be applied to objects with smooth geometries grasped by hands with any number of fingers.

First, we will introduce several candidate grasp quality functions and formulate the grasp synthesis problem as a max-transfer, a max-normal-grasping-force, and a min-analytic-center problem. The physical meaning of each quality function will be explained. Each problem will assume the form of $\max$–$\min$–$\max$ or $\min$–$\max$–$\min$ type. Then, we will develop algorithms for computing the gradients of these quality

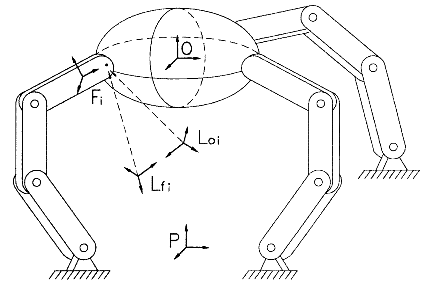

Fig. 1. A k-fingered hand grasping an object.

functions. When real-time solutions are needed for applications such as contact points servoing in coordinated manipulation [28], [20], we introduce two simplified quality functions, along with several examples. Note that the optimal solutions of the simplified problems coincide with previous results obtained using heuristic approaches, demonstrating again generality of our current methods. Finally, we perform experimental studies on the Hong Kong University of Science and Technology (HKUST) three-fingered hand using real-time optimization of the simplified quality functions.

The paper is organized as follows. In Section II, we briefly review the kinematic model of a multifingered hand manipulation system and the friction cone constraints. In Section III, we discuss several classical grasping examples and their optimal solutions by heuristic approaches. In Section IV, we show how to compute the gradients for $\max$–$\min$–$\max$ and $\min$–$\max$–$\min$ problems and propose numerical algorithms for grasp planning. In Section V, we introduce three new candidate quality functions for grasp synthesis, along with simulation results of a three-fingered hand grasping an ellipsoid. In Section VI, we derive two simplified quality functions for real-time grasp planning. Several examples are studied showing that the optimal solutions of the simplified problems coincide with those using heuristic approaches. In Section VII, we perform experimental studies on the HKUST three-fingered hand with real-time optimization of the simplified functions. In Section VIII, we end this paper with a short discussion of future work.

II. Grasp Models and Friction Constraints

In this section, we review the kinematic model of a multifingered hand manipulation system and the friction cone constraints.

A. Grasp Models

Consider a k-fingered hand grasping an object as shown in Fig. 1. Assume that all fingers make contacts of constant types with the object. Three contact models, frictionless point contact (FPC), point contact with friction (PCWF), and soft finger contact with elliptic approximation (SFCE) are considered in our analysis. Following the notations in [29], [30], we attach an object frame O to the center of mass of the object, finger frame F_i $(i = 1, \ldots, k)$ to the fingertip of the ith finger, and local frames L_{oi} and $L_{\mathrm{fi}}(i = 1, \ldots, k)$ to the object and finger i, respectively, at the point of contact. A configuration of contact is described by contact coordinates $\eta_i = (\alpha_{oi}^T, \alpha_{\mathrm{fi}}^T, \psi_i)^T \in \mathbb{R}^5$, where $\alpha_{oi} = (u_{oi}, v_{oi})^T \in \mathbb{R}^2$ are the local coordinates of contact relative to the object, $\alpha_{\rm fi} = (u_{\rm fi}, v_{\rm fi})^T \in \mathbb{R}^2$ the coordinates of contact relative to the fingertip, and ψ_i the angle of contact. Collectively, a contact configuration of the system is described in local coordinates by $\eta = (\eta_1^T, \ldots, \eta_k^T)^T \in \mathbb{R}^{5k}$. In this paper, we represent a grasp as

$$\vec{\alpha}_o = \left[\alpha_{o1}^T, \ldots, \alpha_{ok}^T\right]^T \in \mathbb{R}^{2k}.$$

The relation between the applied finger forces and the resulting object wrench is given by the grasp map, $G \in \mathbb{R}^{6\times n}$

$$w_o = Gx \tag{1}$$

where $x = [x_1{}^T \cdots x_k{}^T]^T \in \mathbb{R}^n$, with $x_i \in \mathbb{R}^{n_i}, i = 1, \ldots, k$ and $n = \sum_{i=1}^k n_i$, is the vector of finger forces. The finger force is constrained to the friction cone

$$\mathcal{FC}_i = \{x_i \in \mathbb{R}^{n_i} \,|\, x_{i,n} \geq 0, \|x_{i,t}\|_s \leq x_{i,n}\}$$

or collectively to

$$\mathcal{FC} = \mathcal{FC}_1 \times \cdots \times \mathcal{FC}_k = \{x \in \mathbb{R}^n \,|\, x_i \in \mathcal{FC}_i\}$$

with $x_{i,n}$ and $x_{i,t}$ being, respectively, the normal and the tangential components of the finger forces at the ith point of contact. Here, $x_{i,n} = x_{i,3}$ for PCWF and SFCE models and $x_{i,n} = x_i$ for FPC. $\|x_{i,t}\|_s$ denote vector norms described for each of the contact models by

$$\text{FPC}: \|x_{i,t}\|_s = 0 \tag{2}$$

$$\text{PCWF}: \|x_{i,t}\|_s = \frac{1}{\mu_i}\sqrt{x_{i,1}{}^2 + x_{i,2}{}^2} \tag{3}$$

$$\text{SFCE}: \|x_{i,t}\|_s = \sqrt{\frac{1}{\mu_i^2}\left(x_{i,1}{}^2 + x_{i,2}{}^2\right) + \frac{1}{\mu_{i,t}^2} x_{i,4}{}^2} \tag{4}$$

with $x_{i,1}$ and $x_{i,2}$ being the friction force components in the tangential plane, $x_{i,4}$ the moment along the contact normal, μ_i the Coulomb friction coefficient, and $\mu_{i,t}$ the coefficient of torsional friction.

A grasp $(G(\vec{\alpha}_o), \mathcal{FC})$ is said to be force closure if and only if $G(\mathcal{FC}) = \mathbb{R}^6$.

B. Friction Cones as Semi-Definite Constraints

By refining the results of [10], [11], Helmke *et al.* [31] showed that the friction-cone constraint (3) is equivalent to positive semi-definiteness of the following 2×2 symmetric matrix:

$$P_i = \begin{bmatrix} \mu_i x_{i,3} + x_{i,1} & x_{i,2} \\ x_{i,2} & \mu_i x_{i,3} - x_{i,1} \end{bmatrix} \geq 0.$$

Equation (4) is equivalent to

$$P_i = \begin{bmatrix} x_{i,3} + \frac{1}{\mu_i} x_{i,1} & \frac{1}{\mu_i} x_{i,2} - j\frac{1}{\mu_{i,t}} x_{i,4} \\ \frac{1}{\mu_i} x_{i,2} + j\frac{1}{\mu_{i,t}} x_{i,4} & x_{i,3} - \frac{1}{\mu_i} x_{i,1} \end{bmatrix} \geq 0$$

where $j = \sqrt{-1}$; and the friction constraints of the hand is equivalent to

$$P \in \mathbb{R}^{N\times N} = \text{diag}(P_1, \ldots, P_k) \geq 0, \qquad N = 2k. \tag{5}$$

Han *et al.* [9] further observed that (5) can be reformulated as LMIs of the form

$$P = A_1 x_1 + \cdots + A_n x_n \geq 0$$

with a reordering of the finger force indices. The force balance (1) is also translated into

$$\text{Tr}(B_i P) = w_{o,i}, \qquad i = 1, \ldots, 6 \tag{6}$$

where $B_i = B_i^T \in \mathbb{R}^{N\times N}$ are coefficient matrices.

III. Grasp Planning: Review of Classical Examples and Heuristic Approaches

In this section, we discuss previous heuristic approaches used in two classical examples of grasp planning.

Let us briefly review the conditions for a two-fingered force-closure grasp and that of a two-fingered antipodal grasp. We assume that the object is devoid of holes and has a closed surface which is homeomorphic to S^2. Following the notation of Do Carmo [32], we parameterize the surface of the object by

$$X(\alpha_o) = [x(\alpha_o)\, y(\alpha_o)\, z(\alpha_o)]^T, \qquad \alpha_o = [u_o, v_o]^T \in \mathbb{R}^2.$$

X_{u_o}, X_{v_o}, and $n(\alpha_o) = (X_{u_o} \times X_{v_o})/(\|X_{u_o} \times X_{v_o}\|)$ are, respectively, two tangent vectors and the outward normal vector at $\alpha_o = (u_o, v_o)^T$. It is well known [19], [33], [4], [5] that a two-fingered grasp with contact points $\alpha_{o1} = [u_{o1}, v_{o1}]^T$ and $\alpha_{o2} = [u_{o2}, v_{o2}]^T$ is force closure if and only if

$$n(\alpha_{o1}) \cdot \frac{X(\alpha_{o1}) - X(\alpha_{o2})}{\|X(\alpha_{o1}) - X(\alpha_{o2})\|} > c_{f1}$$
$$n(\alpha_{o2}) \cdot \frac{X(\alpha_{o2}) - X(\alpha_{o1})}{\|X(\alpha_{o2}) - X(\alpha_{o1})\|} > c_{f2}$$

for squeezing grasps

$$n(\alpha_{o1}) \cdot \frac{X(\alpha_{o1}) - X(\alpha_{o2})}{\|X(\alpha_{o1}) - X(\alpha_{o2})\|} < -c_{f1}$$
$$n(\alpha_{o2}) \cdot \frac{X(\alpha_{o2}) - X(\alpha_{o1})}{\|X(\alpha_{o2}) - X(\alpha_{o1})\|} < -c_{f2}$$

for expanding grasps. Here, $c_{\rm fi} = \cos(\tan^{-1}\mu_i), i = 1, 2$. In general, two-fingered force-closure grasps are not unique and force-closure regions can be identified for polygonal objects [5] and curved two-dimensional (2-D) objects [34]. An important problem naturally arises as which grasp in this region is the best. Hong *et al.* [19] first introduced the concept of antipodal grasps and proposed the following distance function:

$$E(\alpha_{o1}, \alpha_{o2}) = \frac{1}{2}\|X(\alpha_{o1}) - X(\alpha_{o2})\|^2 \tag{7}$$

whose critical points give candidates of antipodal configurations. Antipodal grasps are necessarily force-closure grasps and are regarded as the best among all two-fingered grasps, as shown in Fig. 2. Antipodal grasps can be synthesized by planning the contact points to follow the ascent gradient of E or other equivalent cost functions [4], [20].

In general, the above heuristic approach can not be extended to three-fingered grasps on objects of arbitrary geometries.

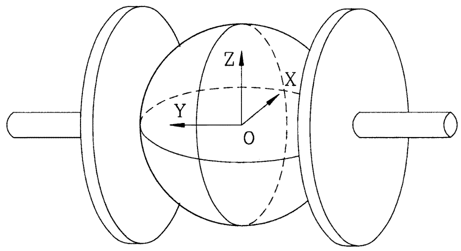

Fig. 2. Two-finger antipodal grasp.

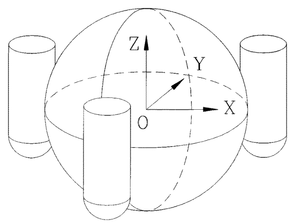

Fig. 3. Three-finger symmetric grasp.

However, for a spherical object, we can let the square of the area formed by the three contact points to be the objective function

$$E(\vec{\alpha}_o) = \frac{1}{4}(\|X(\alpha_{o3}) - X(\alpha_{o1})\|^2 \|X(\alpha_{o2}) - X(\alpha_{o1})\|^2 - ((X(\alpha_{o3}) - X(\alpha_{o1})) \cdot (X(\alpha_{o2}) - X(\alpha_{o1})))^2)$$

and the optimal solutions are symmetric grasps with three contact points located uniformly on a big circle, as shown in Fig. 3. To solve generally the grasp synthesis problem, in the sections follows we will introduce several grasp quality functions which assumes the form of min–max–min or max–min–max, for which gradient algorithms can be developed.

IV. Theory and Algorithms for min–max–min and max–min–max Problems

In this section, we first review the general theory of min–max, max–min, min–max–min, and max–min–max problems. Then, we propose algorithms for solving these problems.

A. Theory of Gradient Computation

Consider the min–max problem

$$\min_Y \max_Z F_0(Y, Z), Y \in \Omega_1 \subset \mathbb{R}^{N_1}, Z \in \Omega_2 \subset \mathbb{R}^{N_2} \quad (8)$$

where Ω_1 is an open set and Ω_2 a bounded closed subset. We shall assume that $F_0(Y,Z)$ and $(\partial F_0(Y,Z))/(\partial Y)$ is continuous on $\Omega_1 \times \Omega_2$. Let

$$F_1(Y) = \max_{Z \in \Omega_2} F_0(Y, Z).$$

$F_1(Y)$ possesses the following properties.

1) $F_1(Y)$ is continuous on Ω_1.

2) Suppose $\tilde{\Omega}_1 \subset \Omega_1$ and for some $Y_0 \in \tilde{\Omega}_1$ the set

$$\{Y \in \tilde{\Omega}_1 \,|\, F_1(Y) \le F_1(Y_0)\}$$

is bounded. Then, there exists a point $Y^* \in \tilde{\Omega}_1$ such that

$$F_1(Y^*) = \inf_{Y \in \tilde{\Omega}_1} F_1(Y)$$
$$\max_{Z \in \Omega_2} F_0(Y^*, Z) = \inf_{Y \in \tilde{\Omega}_1} \max_{Z \in \Omega_2} F_0(Y, Z).$$

If $\tilde{\Omega}_1$ is chosen as a local set, then $Y^\star$ is a local minimum, otherwise, it is a global optimum.

It is often impossible to find analytic solutions for the min–max and max–min problems, and thereby those for the min–max–min and max–min–max problems. Seeking a possible numerical solution requires us to compute the gradients of those quality functions in an efficient way. We first consider computation of the gradient of the following problem:

$$\nabla_Y F_1(Y) = \frac{\partial F_1(Y)}{\partial Y}.$$

For fixed $Y \in \Omega_1$, we define

$$R(Y) = \{Z \in \Omega_2 \,|\, F_0(Y, Z) = \max_Z F_0(Y, Z)\}.$$

Obviously, $R(Y) \subset \Omega_2$ is a bounded closed set. The following theorem [35] states how to compute the directional derivative of $F_1(Y)$.

Theorem 1: $F_1(Y)$ is a differentiable function with its directional derivative at $Y \in \Omega_1$ along $v \in \mathbb{R}^{N_1}$, $\|v\| = 1$, given by

$$\left\langle \frac{\partial F_1(Y)}{\partial Y}, v \right\rangle = \max_{Z \in R(Y)} \left\langle \frac{\partial F_0(Y, Z)}{\partial Y}, v \right\rangle.$$

From this theorem, we conclude that

$$\nabla_Y F_1(Y) = \left. \frac{\partial F_0(Y, Z)}{\partial Y} \right|_{Z^*}$$

if Z^* is the unique optimal solution for $\max_Z F_0(Y, Z)$. We can derive similar results for the max–min problem.

Second, let us consider the following min–max–min problem:

$$\min_{Y \in \Omega_1} \max_{Z \in \Omega_2} F_0(Y, Z) = \min_{Y \in \Omega_1} \max_{Z \in \Omega_2} \min_{W \in \Omega_3} F(Y, Z, W) \quad (9)$$

where Ω_3 is an open or close set. Given Y and Z, we assume that $W^*(Y, Z)$ is an optimal solution for

$$\min_{W \in \Omega_3} F(Y, Z, W). \quad (10)$$

Then, $F_0(Y, Z) = F(Y, Z, W^*(Y, Z))$.

Theorem 2: If the following three conditions are satisfied:

1) there is a unique solution $W^*(Y, Z)$ to (10);

2)

$$\frac{\partial W^*(Y, Z)}{\partial Y} = 0 \quad (11)$$

3) Z^* is the unique optimal solution for

$$\max_{Z\in\Omega_2} F(Y,Z,W^*(Y,Z)) = \max_{Z\in\Omega_2}\min_{W\in\Omega_3} F(Y,Z,W) \quad (12)$$

then,

$$\nabla_Y F_1(Y) = \left.\frac{\partial F(Y,Z,W)}{\partial Y}\right|_{Z^*,W^*(Y,Z^*)}. \quad (13)$$

Proof: Since Z^* is the unique optimal solution to (12), we have from Theorem 1 that

$$\nabla_Y F_1(Y) = \left.\frac{\partial F(Y,Z,W^*)}{\partial Y}\right|_{Z^*} + \left.\frac{\partial F(Y,Z,W^*)}{\partial W^*}\frac{\partial W^*}{\partial Y}\right|_{Z^*}. \quad (14)$$

The second term on the right-hand side is equal to zero because of (11). □

This approach can also be applied to max–min–max problems satisfying the similar conditions.

B. Numerical Algorithms

In this section, we will develop an algorithm for the min–max problem, and an algorithm for the min–max–min problem.

For problem (8), we assume that

$$\max_Z F_0(Y,Z)$$

has a unique solution and can be solved using some algorithm (called Algorithm A). Then we develop the following algorithm for (8):

Algorithm 1: **Algorithm for the** min–max **problem**

Input: initial value $Y(0)$, step size $\gamma_k > 0$, and tolerance $\epsilon > 0$;
Output: optimal value Y^*;
Step 1: set $k = 0$;
Step 2: solve $Z^*(k) = \max_Z F_0(Y(k),Z)$ using Algorithm A, and calculate $F_1(k) = F_0(Y(k),Z^*(k))$;
Step 3: calculate the gradient

$$\nabla_Y F_1(Y)\,|_{Y(k)} = \left.\frac{\partial F_0(Y,Z)}{\partial Y}\right|_{Y(k),Z^*(k)};$$

Step 4: set

$$Y(k+1) = Y(k) - \gamma_k \nabla_Y F_1(Y)\,|_{Y(k)};$$

Step 5: solve $Z^*(k+1) = \max_Z F_0(Y(k+1),Z)$ using Algorithm A, and calculate $F_1(k+1) = F_0(Y(k+1),Z^*(k+1))$;
Step 6: if $|F_1(k+1) - F_1(k)| \le \epsilon$, output $Y^* = Y(k+1)$; else set $k = k+1$ and go to Step 3.

The Algorithm A used in Step 5 depends on the properties of the cost function $F_0(Y,Z)$. It could be linear programming algorithms, semi-definite programming algorithms, or interior point algorithms. The uniqueness of the solution for $\max_Z F_0(Y,Z)$ is often satisfied.

To solve the min–max–min problem (9), we assume that all three conditions in Theorem 2 are satisfied. We design the following algorithm.

Algorithm 2: **Algorithm for the** min–max–min **problem**

Input: initial value $Y(0)$, step size $\gamma_k > 0$, and tolerance $\epsilon > 0$;
Output: optimal value Y^*;
Step 1: set $k = 0$;
Step 2: solve $Z^*(k)$ and $W^*(Y(k),Z^*(k))$ using Algorithm 1 for

$$\max_{Z\in\Omega_2}\min_{W\in\Omega_3} F(Y(k),Z,W),$$

and calculate $F_1(k) = F_0(Y(k),Z^*(k))$;
Step 3: calculate the gradient $\nabla_Y F_1(Y)\,|_{Y(k)}$ as (13);
Step 4: set

$$Y(k+1) = Y(k) - \gamma_k \nabla_Y F_1(Y)|_{Y(k)};$$

Step 5: solve $Z^*(k+1)$ and $W^*(Y(k+1),Z^*(k+1))$, and calculate $F_1(k+1) = F_0(Y(k+1),Z^*(k+1))$ as Step 2;
Step 6: if $|F_1(k+1) - F_1(k)| \le \epsilon$, output $Y^* = Y(k+1)$; else set $k = k+1$ and go to Step 3.

Remark 1: In the grasp synthesis problems that will be introduced in the section follows, condition (1) in Theorem 2 is often satisfied. However, condition (2) may not be true. Here, we adopt numerical approximation in (14)

$$\frac{\partial W^*(Y,Z)}{\partial Y} = \frac{W^*(Y+\delta Y,Z) - W^*(Y,Z)}{\delta Y}.$$

In general, we can only find local optimum in condition (3), which means that our algorithms can only be used to find local optimum for the min–max–min and max–min–max problems. Moreover, the efficiency of the algorithms relies on the chosen step sizes, please refer to [4] for more details.

V. Several Grasp Quality Functions and Simulation Examples

In this section, we introduce three new grasp quality functions and formulate the corresponding optimal grasp synthesis problems.

A. Max-Transfer Problem

Grasp map can be regarded as a transfer function taking finger forces to object wrenches with a domain being the friction cones. Planning of optimal grasps amounts to finding a set of contact points which optimize, in some sense, the transfer function.

Kirkpatrick *et al.* [23] utilized the *quantitative Steinitz's theorem* to evaluate a grasp, where the radius of the largest ball, centered in the origin of the wrench space and contained in the convex hull spanned by unit primitive forces, measures the quality of a grasp. Ferrari and Canny [24] proposed a global grasp quality measure by minimizing the maximal proportion between the norm of external forces and that of its respective finger forces. The same idea was further developed by Mirtich and Canny [25], where force and moment transfer functions were treated independently and optimized sequentially. By doing so, the ambiguity arising in specifying a physically meaningful norm for the external wrench space can be avoided. Since all the problems discussed in these works consider the optimal grasp planning from the ability of the system in resisting external wrenches, we call them the max-transfer problem.

It is well known that $\mathbb{R}^n = R(G^T) \oplus N(G)$, any finger force x can be uniquely decomposed into two components

$$x = x^{\perp} + x^{\parallel}, \quad x^{\perp} \in R(G^T), \quad x^{\parallel} \in N(G)$$

where $x^{\perp} = G^T(GG^T)^{-1}Gx$ can be interpreted as the manipulation force [36], and $x^{\parallel} = (I - G^T(GG^T)^{-1}G)x$ the internal grasping force. The manipulation force is determined as long as the external wrench is given. The undetermined component is the internal grasping force. Considering that finger forces with large magnitude are not allowed during manipulation, the grasp quality at $\vec{\alpha}_o$ can be measured as the minimal proportion (worst case) between the norm of the external wrench and that of the finger forces while fixing the norm of the manipulation force

$$x^{\perp T} x^{\perp} = x^T G^T (GG^T)^{-1} Gx = w_o^T (GG^T)^{-1} w_o = 1.$$

Problem 1: Max-Transfer Problem: Find $\vec{\alpha}_o$ such that

$$g_1(\vec{\alpha}_o) = \min_{w_o^T(GG^T)^{-1}w_o=1} \max_{Gx=w_o, P(x)\geq 0} \frac{w_o^T A w_o}{x^T x} \tag{15}$$

is maximal.

Since it is impossible to endow a bi-invariant metric on the space of external wrenches, we usually assign a left invariant metric

$$\|w_o\|_{w_o}^2 = w_o^T A w_o, \qquad A > 0.$$

Note that the problem in the current form is slightly different from that of Ferrari and Canny, and Mirtich and Canny in that the constraints $w_o^T(GG^T)^{-1}w_o = 1$ is position dependent.

Example 1. Planning of Optimal Grasps Using g_1 for a Three-Fingered Hand Manipulating an Ellipsoid: Consider the case of a three-fingered hand manipulating an ellipsoid through frictional point contacts. We parameterize the ellipsoid by the longitude and latitude coordinates

$$\alpha_o = \begin{bmatrix} u_o \\ v_o \end{bmatrix} \rightarrow \begin{bmatrix} a\cos u_o \cos v_o \\ b\cos u_o \sin v_o \\ c\sin u_o \end{bmatrix}$$

with $a = b = 1$ and $c = 3$. Initially, the three fingers are arbitrarily placed at the three points $\alpha_{o1} = (0,0)^T, \alpha_{o2} = (0, (\pi/4))^T$, and $\alpha_{o3} = ((\pi/8), -(\pi/4))^T$. We use g_1 to plan trajectories of the three fingers so that $(1/g_1)$ is minimized. A is chosen to be I. To apply Algorithm 1 and 2 of Section IV, we need to calculate $\nabla_{w_o}(x^T x)/(w_o^T w_o)$ and $\nabla_{\vec{\alpha}_o}(x^T x)/(w_o^T w_o)$. Note that

$$x^T x = w_o^T (GG^T)^{-1} w_o + y^T V^T V y,$$

we have

$$\frac{\partial \frac{x^T x}{w_o^T w_o}}{\partial w_o} = -\frac{2x^T x}{(w_o^T w_o)^2} w_o + 2\frac{(GG^T)^{-1} w_o}{w_o^T w_o}$$

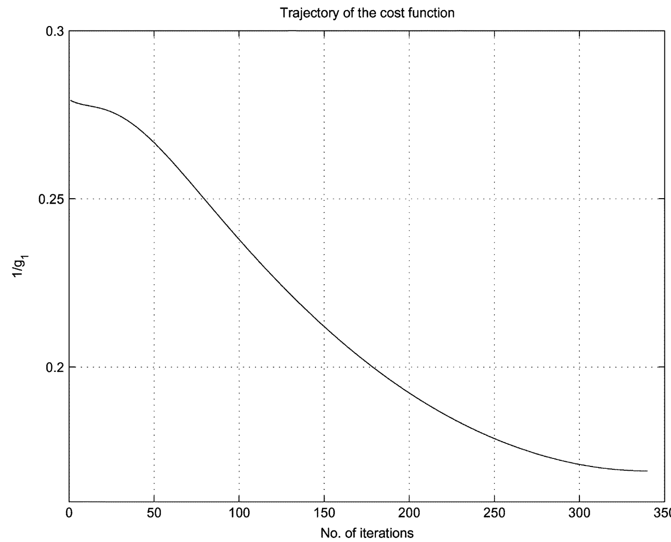

Fig. 4. View 3: Trajectory of cost function.

and $\nabla_{w_o}(x^T x)/(w_o^T w_o)$ is its projection to the constraint subspace $w_o^T(GG^T)^{-1}w_o = 1$. $\nabla_{\vec{\alpha}_o}(x^T x)/(w_o^T w_o)$ is calculated as

$$\nabla_{\vec{\alpha}_o} \frac{x^T x}{w_o^T w_o} = \frac{\nabla_{\vec{\alpha}_o} x^T x}{w_o^T w_o} = \frac{w_o^T \frac{\partial (GG^T)^{-1}}{\partial \vec{\alpha}_o} w_o + y^T \frac{\partial V^T V}{\partial \vec{\alpha}_o} y}{w_o^T w_o}.$$

The final simulation results are shown in Figs. 4– 7. In this example, the computation time for an optimal solution is about 4 h in P4 and Win2000 (typically 4–5 h, depends on the initial conditions and used step sizes). Without specifically pointing out, all our simulations are performed in the same system.

Remark 2: One problem of using the left-invariant metric is that different choices of A will in general lead to different optimal grasps. Since force and moment are two different quantities which can be measured both in a physically meaningful way, Mirtich and Canny considered to optimize both the force and moment transfer function. Some successful applications of this method to optimal grasp planning can be found in [24] and [25]. For two-fingered planar grasps, the antipodal grasp with the largest distance between the two contact points is found to be the optimal, and for three-fingered planar grasps the equilateral grasp with the maximal outer triangle (symmetric grasps if the object is a circle) is the best.

B. Max-Normal-Grasping-Force Problem

For frictional point contacts, the normal component of the finger force x_i is $x_{i,3}$. We define

$$x_n = \sum_{i=1}^{k} x_{i,3} = \xi^T x > 0$$

as the normal grasping force, where $\xi = [0, 0, 1, \ldots, 0, 0, 1]^T$. Note that for a balance grasp $Gx = w_o, x_n$ measures how stable the grasp is. Physically, it represents how much passive forces it can produce to the object to resist external disturbances. Motivated by this, we introduce the following problem.

Problem 2: Max-Normal-Grasping-Force Problem: Find grasp $\vec{\alpha}_o$ such that

$$g_2(\vec{\alpha}_o) = \max_{w_o^T A w_o - 1=0} \min_{Gx=w_o, P(x)\geq 0} \frac{1}{\xi^T x} \tag{16}$$

is minimal.

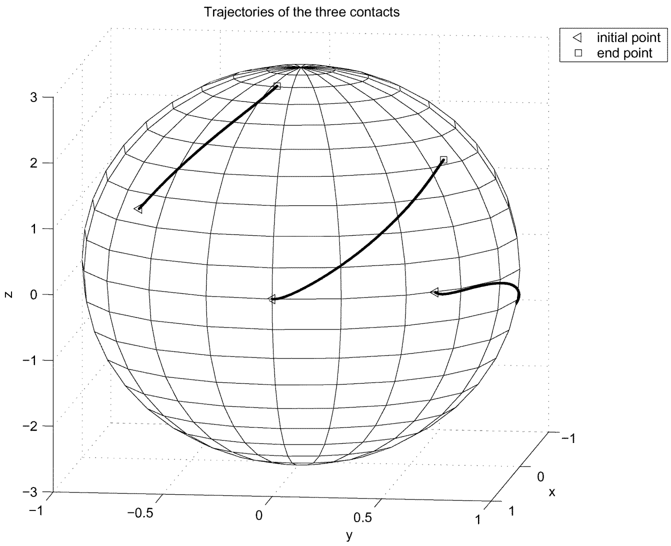

Fig. 5. View 1: Trajectories of the three fingers.

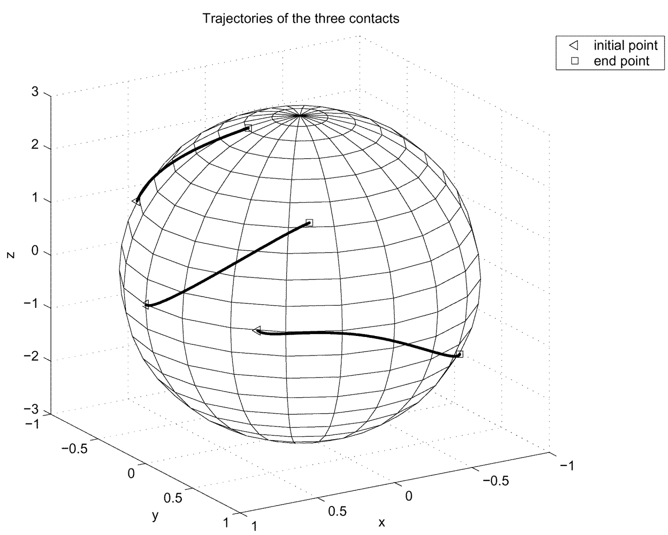

Fig. 6. View 2: Trajectories of the three fingers.

This is a min–max–min problem. For given $\vec{\alpha}_o$ and w_o, the problem in the most internal layer of (16)

$$\max_{Gx=w_o, P(x)\geq 0} \xi^T x$$

can be transformed into a semi-definite problem [37], [38].

Example 2: Optimal Grasp Planning Using g_2: Example 1 Continued: In this example, we adopt g_2 to optimize the grasp for the ellipsoid of Example 1 with the same initial grasp as before. First, we compute $\nabla_{w_o}\xi^T x$ and $\nabla_{\vec{\alpha}_o}\xi^T x$ as follows. Since

$$\xi^T x = \xi^T G^T (GG^T)^{-1} w_o + \xi^T V y$$

we have

$$\frac{\partial \xi^T x}{\partial w_o} = (GG^T)^{-1} G\xi$$

and $\nabla_{w_o}\xi^T x$ is its projection to $w_o^T A w_o = 1$. Similarly

$$\nabla_{\vec{\alpha}_o}\xi^T x = \xi^T \frac{\partial G^T (GG^T)^{-1}}{\partial \vec{\alpha}_o} w_o + \xi^T \frac{\partial V}{\partial \vec{\alpha}_o} y.$$

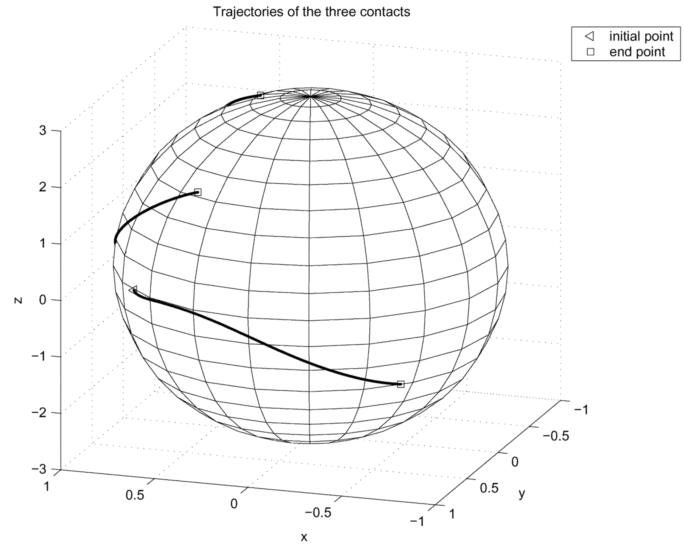

Fig. 7. View 3: Trajectories of the three fingers.

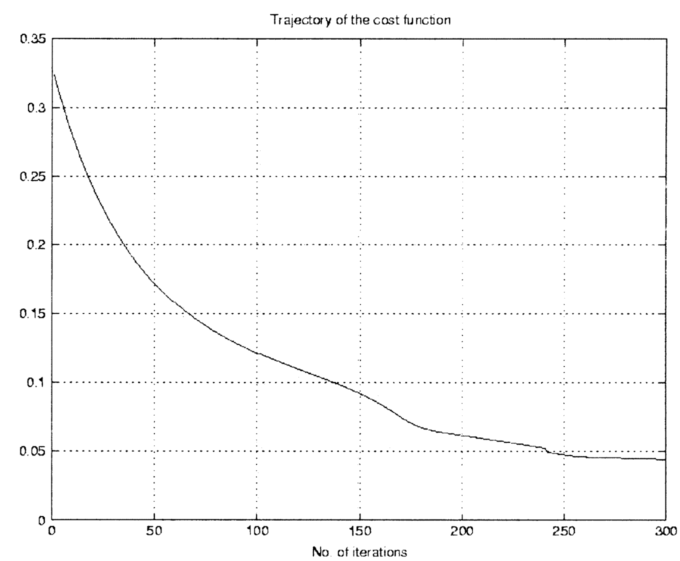

Fig. 8. Trajectory of cost function.

Using Algorithm 1 and 2 of Section IV, we obtain simulation results as shown in Fig. 8–11. In this example, it takes about 2 h to compute an optimal solution.

C. Min-Analytic-Center Problem

From the works on grasping force optimization [10], [11], [9], [28], we see that given $\vec{\alpha}_o$ and w_o, we can assign a unique analytic center to $\Omega_x := \{x \,|\, Gx = w_o, P(x) > 0\}$ as

$$\begin{aligned} \phi(\vec{\alpha}_o, w_o) = \min_x \operatorname{logdet} P(x)^{-1} \\ \text{subject to} \\ Gx = w_o \\ P(x) > 0. \end{aligned}$$

The smaller ϕ is, the farther the optimal x is from the boundary of the friction cone (i.e., more stable). Based on this, we formulate the following problem.

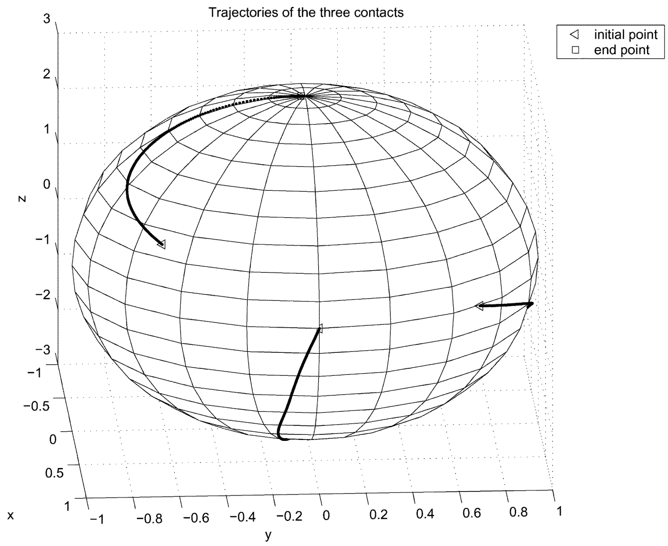

Fig. 9. View 1: Trajectories of the three fingers.

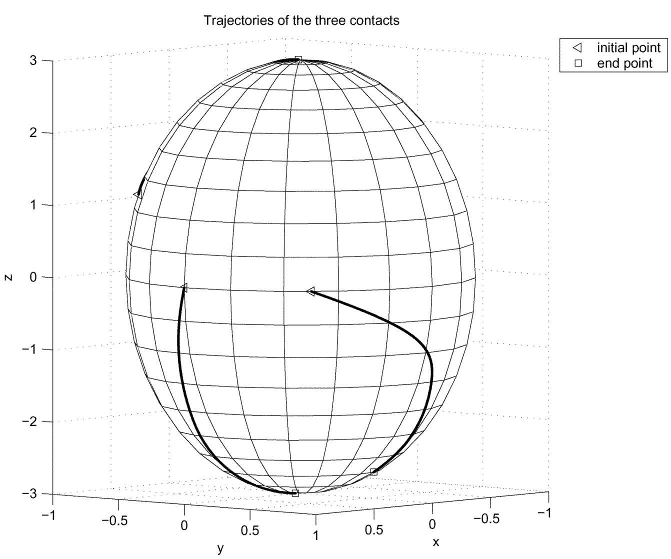

Fig. 10. View 2: Trajectories of the three fingers.

Problem 3: Min-Analytic-Center Problem: Find grasp $\vec{\alpha}_o$, such that

$$g_3(\vec{\alpha}_o) = \max_{w_o^T A w_o = 1} \phi(\vec{\alpha}_o, w_o)$$
$$= \max_{w_o^T A w_o=1} \min_{Gx=w_o, P(x)>0} \text{logdet} P(x)^{-1}$$

is minimal.

Clearly, the min-analytic-center problem is a min–max–min problem. Given $\vec{\alpha}_o$ and $w_o, \phi(\vec{\alpha}_o, w_o)$ can be solved as follows. Eliminating the force balance equation by substituting $x = x_0 + Vy$ into ϕ yields

$$\phi(\vec{\alpha}_o, w_o) = \min_y \text{logdet } \tilde{P}(y)^{-1}$$
$$\text{subject to}$$
$$\tilde{P}(y) = P(x_0 + Vy) = \tilde{A}_0 + \tilde{A}_1 y_1 + \cdots + \tilde{A}_{n-6} y_{n-6} > 0 \tag{17}$$

where $\tilde{A}_0 = \sum_i A_i x_{0,i}, \tilde{A}_i = \sum_j A_j V_{j,i}, x_0 = [x_{0,1}, \ldots, x_{0,n}]^T$, and $V_{j,i}$ is the jith element of V. If the solution set for $\tilde{P}(y) > 0$ is empty, i.e., Problem (17) is infeasible, the system is not force closure at $\vec{\alpha}_o$. For these grasp configurations, we assign a large number to g_3, e.g., 2000. Otherwise, an optimal solution can be obtained. Denote by S_{++}^N the set of positive definite $N \times N$ matrices. Since $\text{logdet}\tilde{P}^{-1} \to \infty$ as $\tilde{P}(y)$ goes to the boundary of S_{++}^N, it is minimal if and only if its gradient is equal to zero, i.e.,

$$\frac{\partial \text{logdet}\tilde{P}(y)^{-1}}{\partial y_j} = -\text{Tr}(\tilde{P}(y)^{-1}\tilde{A}_j) = 0, \qquad j = 1, \ldots, n-6.$$

Let y^* be the solution of the above equalities, which can be shown to be unique and smoothly depend on both $\vec{\alpha}_o$ and w_o [31]. Although it is hard to derive the analytical expression of $y^*(\vec{\alpha}_o, w_o)$, Problem (17) is a standard analytic-center problem and can be solved numerically by interior point algorithms [38] when $\vec{\alpha}_o$ and w_o are given in advance.

Example 3: Planning of Optimal Grasps Using g_3: Example 1 Continued: Consider again the grasp case in Example 1 with the initial configuration: $\alpha_{o1} = (-0.2684, -0.0820)^T, \alpha_{o2} = (-0.2097, 1.4009)^T$, and $\alpha_{o3} = (0.7099, -1.3849)^T$. Algorithm 1 and 2 of Section IV require us to compute $\nabla_{\vec{\alpha}_o} \text{logdet}\tilde{P}(y)^{-1}$ and $\nabla_{w_o} \text{logdet}\tilde{P}(y)^{-1}$. Note that

$$\frac{\partial \text{logdet}\tilde{P}^{-1}}{\partial \tilde{P}} = -\tilde{P}^{-1}.$$

We have

$$\frac{\partial \text{logdet}\tilde{P}(y)^{-1}}{\partial w_o} = -\frac{\partial \text{vec}(\tilde{A}_0)}{\partial w_o}^T \text{vec}(\tilde{P}^{-1})$$
$$= -(GG^T)^{-1} GH \text{vec}(\tilde{P}^{-1})$$

and $\nabla_{w_o} \text{logdet}\tilde{P}(y)^{-1}$ is its projection to $w_o^T A w_o = 1$, where

$$H = \begin{bmatrix} \text{vec}(A_1)^T \\ \vdots \\ \text{vec}(A_n)^T \end{bmatrix}$$

and vec denotes the vector operator. $\nabla_{\vec{\alpha}_o} \text{logdet}\tilde{P}(y)^{-1}$ is given by

$$-\left(w_o^T \frac{\partial (GG^T)^{-1}G}{\partial \vec{\alpha}_o} + y^T \frac{\partial V^T}{\partial \vec{\alpha}_o}\right) H \text{vec}(\tilde{P}^{-1}).$$

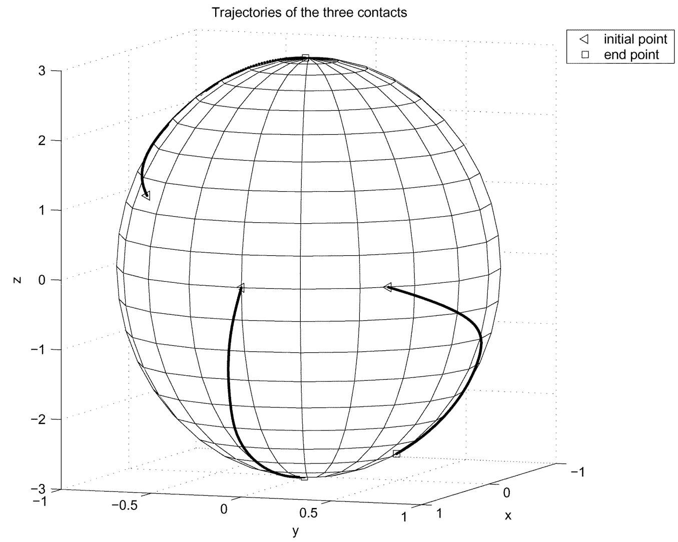

Fig. 11. View 3: Trajectories of the three fingers.

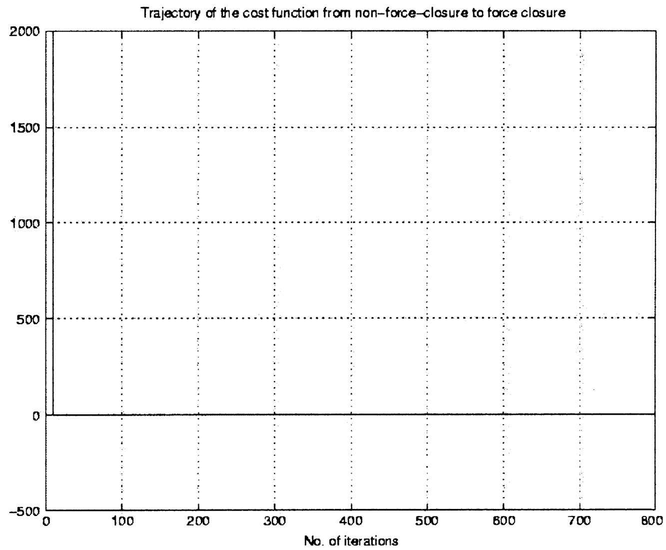

Fig. 12. Trajectory of the cost function: from nonforce closure to force closure.

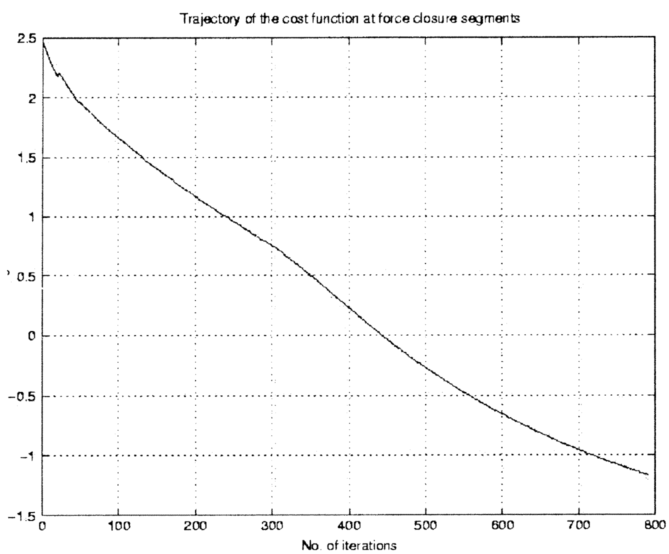

Fig. 13. Trajectory of the cost function: force-closure part.

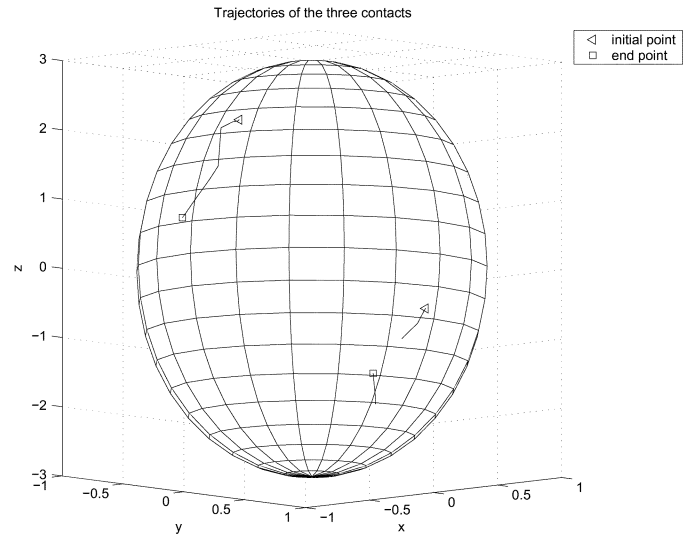

Fig. 14. View 1: Trajectories of finger 2 and 3.

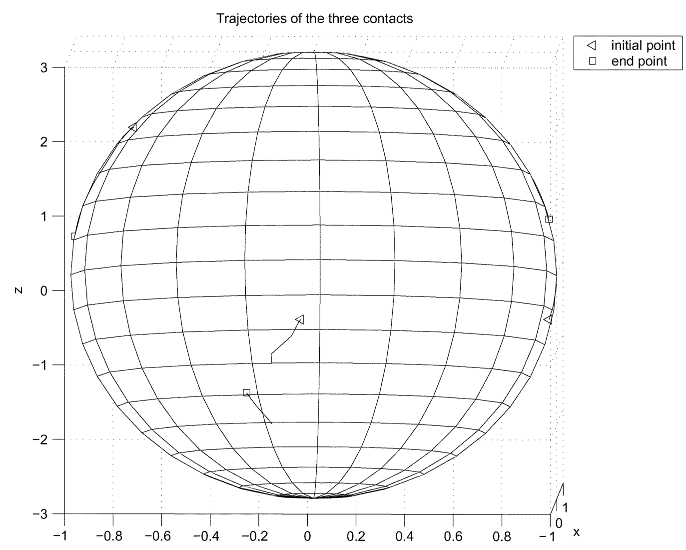

Fig. 15. View 2: Trajectories of the three fingers.

The simulation results are shown in Figs. 12–16. From Fig. 12 and 13, we conclude that using g_3, and Algorithm 1 and 2, the grasp will evolve from nonforce-closure states to force-closure states as g_3 goes from 2000 (nonforce-closure states) to a much smaller value (force-closure states). Second, the nonconvergence of g_3 in Fig. 13 is because the grasps goes into states where we can apply arbitrary large normal finger forces to the object, and $\det\tilde{P}$ will go to ∞ (correspondingly, $\mathrm{logdet}(\tilde{P}^{-1})$ to $-\infty$). In this example, it typically takes 2 hours for 800 iterations. To ensure the convergence of the algorithm, we can add a linear term $\xi^T x$ to the quality function so as to restrict the normal grasping force. Then, the original quality function is changed into

$$g_3(\vec{\alpha}_o) = \max_{w_o^T A w_o = 1} \min_{Gx = w_o, P(x) > 0} \xi^T x + \mathrm{logdet}\, P(x)^{-1}.$$

Compared with the previous grasp synthesis problems, e.g., those in [24] and [25], both the max-normal-grasping-force problem and the min-analytic-center problem are formulated based on the optimal grasping forces. They are closely related to the real-time grasping force optimization problem [9], which can be formulated as a semi-definite programming problem, an analytic center problem, or a max-det (determinant) problem. Thus, the physical meaning of the above two problems can be explained as finding optimal grasp configurations such that the worst case optimal graphing forces is optimal. These two problems are general enough to be applied to objects with smooth surfaces grasped by hands with any number of fingers as long as the geometric model of the surface is known. Powerful algorithms exist for computing an optimal grasping force which will be used in Algorithm 1 and 2, and thereby improve their computation efficiency.

VI. Two Simplified Problems

The max-transfer and Min-analytic-center problems can be simplified via estimation. The derived simple analytic expres-

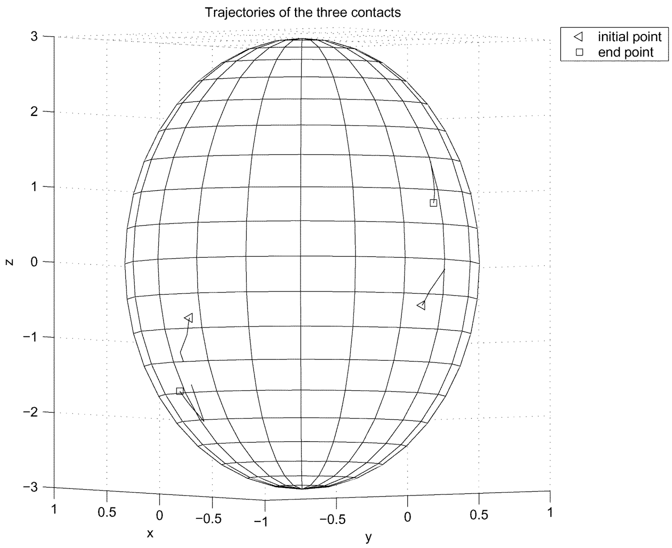

Fig. 16. View 3: Trajectories of finger 1 and 2.

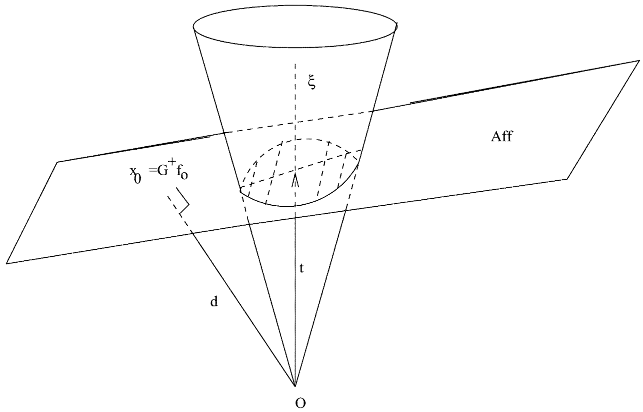

Fig. 17. Relative configuration between Aff and the product of SOC.

sions of their respective grasp quality functions are suitable for real-time grasp planning.

A. Simplifying the Max-Transfer Problem

$\mathcal{FC}$ is a subset with rotational symmetry (assume that $\mu_{t,i} = \mu_i$ for SFCE contacts). Its center of symmetry is a line passing through the vertex O (the origin of $\mathbb{R}^n$, see Fig. 17) with the direction given by

$$\xi = [\underbrace{1}_{1},\ldots,\underbrace{1}_{k}]^T \quad \text{for} \quad \text{FPC}$$
$$\xi = [\underbrace{0,0,1}_{1},\ldots,\underbrace{0,0,1}_{k}]^T \quad \text{for} \quad \text{PCWF}$$
$$\xi = [\underbrace{0,0,1,0}_{1},\ldots,\underbrace{0,0,1,0}_{k}]^T \quad \text{for} \quad \text{SFCE.}$$

The size of $\mathcal{FC}$ is determined by the vector of cone angles $\theta_c = (\theta_1,\ldots,\theta_k)^T \in \mathbb{R}^k$ with $\theta_i \in \mathbb{R}, i = 1,\ldots,k$, being the angle of cone i. $\theta_i = 0$ for FPC contacts and $\theta_i = \tan^{-1}\mu_i$ for PCWF and SFCE models.

Without the friction-cone constraints and adopting the idea used in the max-transfer problem, the maximal distance from the origin O to the affine set $\text{Aff} = \{x \,|\, Gx = w_o\}$ among all unit external wrenches measures the capability of finger forces in resisting the external wrenches

$$\phi(\vec{\alpha}_o) = \max_{w_o^T(GG^T)^{-1}w_o=1} d(O,\text{Aff}).$$

Here, we have implicitly used the 2-norm for finger forces and the metric $A = (GG^T)^{-1}$ for external wrenches. By the projection theorem, $d(O,\text{Aff})$ is calculated as

$$d(O,\text{Aff}) = \|G^T(GG^T)^{-1}w_o\|_2$$

i.e., d is the magnitude of the manipulating force. To take into account the friction cone constraints, we need to modify the distance d into its projection to the center of the friction cone

$$d_T(O,\text{Aff}\cap\mathcal{FC}) = \frac{\|\xi^T G^T(GG^T)^{-1}w_o\|}{(\xi^T\xi)^{\frac{1}{2}}}.$$

and $\phi(\vec{\alpha}_o)$ to

$$\phi_T(\vec{\alpha}_o) = \max_{w_o^T(GG^T)^{-1}w_o=1} d_T(O,\text{Aff}\cap\mathcal{FC}).$$

In fact, at a given grasp configuration $\vec{\alpha}_o, d_T$ is maximal if and only if $(GG^T)^{-1}w_o \parallel (GG^T)^{-1}G\xi$, i.e.,

$$w_o = \pm\frac{1}{\gamma}G\xi$$
$$\gamma = \sqrt{\xi^T G^T(GG^T)^{-1}G\xi}$$

from which, we have

$$\phi_T(\vec{\alpha}_o) = \frac{\sqrt{\xi^T G^T(GG^T)^{-1}G\xi}}{(\xi^T\xi)^{\frac{1}{2}}}.$$

We introduce the following simplified max-transfer problem.

Problem 4: Simplified Max-Transfer Problem: Find grasp configurations $\vec{\alpha}_o$ such that

$$\phi_T(\vec{\alpha}_o)$$

is minimal.

Since ξ is constant, the above problem is equivalent to

$$\min_{\vec{\alpha}_o} \xi^T G^T(GG^T)^{-1}G\xi.$$

B. Simplifying the Min-Analytic-Center Problem

The analytic center x^* for $\min_x \text{logdet}P(x)^{-1}$ at a given grasp configuration $\vec{\alpha}_o$ and under a given external force w_o (satisfying $w_o^T A w_o = 1$) can be physically interpreted as the one which is farthest from the boundary of the friction cone. x^* can be estimated as the intersection point between the center of symmetry ξ and the affine set Aff, see Appendix A for a complete derivation. Let $x^* = t\xi, t > 0 \in \mathbb{R}$, then

$$G\xi t = w_o \tag{18}$$
$$t^2 = \frac{1}{\xi^T G^T A G\xi} \tag{19}$$
$$x^* = \sqrt{\frac{1}{\xi^T G^T A G\xi}}\,\xi. \tag{20}$$

From the Helmke *et al.* expression of $P(x), \det P(x)^{-1}$ is calculated as

$$\det P(x^*)^{-1} = \left.\frac{1}{\Pi_{i=1}^k \left(\mu_i^2 x_{i,3}^2 - x_{i,1}^2 - x_{i,2}^2\right)}\right|_{x^*} = \frac{(\xi^T G^T A G \xi)^k}{\Pi_{i=1}^k \mu_i^2}.$$

Then

$$g_3(\vec{\alpha}_o) = \log \frac{(\xi^T G^T A G \xi)^k}{\Pi_{i=1}^k \mu_i^2}.$$

Since the log function is monotone increasing, we introduce the following simplified min-analytic-center problem:

Problem 5: Simplified Min-Analytic-Center Problem: Find grasp configurations $\vec{\alpha}_o$ such that

$$g_3 = \xi^T G^T A G \xi$$

is minimal.

C. Practical Examples

The objective functions of the simplified max-transfer and the simplified min-analytic-center problems are simple analytic functions. They can be used to efficiently determine the optimal grasps of several classical examples. It turns out that the obtained optimal grasps coincide with those by the traditional heuristic approaches. Moreover, the developed quality measures are general and suitable for real-time grasp optimization.

Example 4: Antipodal Configurations: Optimal two-fingered Grasps: Consider a two-fingered hand grasping a spherical object with radius R, as shown in Fig. 2. The hand makes contacts with the object at $\alpha_{o1} = (u_{o1}, v_{o1})^T$ and $\alpha_{o2} = (u_{o2}, v_{o2})^T$. Assume that both contacts are SFCE, the grasp map is calculated as

$$G = [G_1 \; G_2]$$
$$G_i = \begin{bmatrix} -s_{u_{oi}} c_{v_{oi}} & s_{v_{oi}} & c_{u_{oi}} c_{v_{oi}} & 0 \\ -s_{u_{oi}} s_{v_{oi}} & -c_{v_{oi}} & c_{u_{oi}} s_{v_{oi}} & 0 \\ c_{u_{oi}} & 0 & s_{u_{oi}} & 0 \\ R s_{v_{oi}} & R s_{u_{oi}} c_{v_{oi}} & 0 & c_{u_{oi}} c_{v_{oi}} \\ -R c_{v_{oi}} & R s_{u_{oi}} s_{v_{oi}} & 0 & c_{u_{oi}} s_{v_{oi}} \\ 0 & -R c_{u_{oi}} & 0 & s_{u_{oi}} \end{bmatrix}$$
$$i = 1, 2$$

where $c_{u_{oi}} = \cos u_{oi}, c_{v_{oi}} = \cos v_{oi}, s_{u_{oi}} = \sin u_{oi}$, and $s_{v_{oi}} = \sin v_{oi}$. The finger forces are restricted to the friction cones

$$\mathcal{FC} = \mathcal{FC}_1 \times \mathcal{FC}_2$$

with the center of symmetry $\xi = [0, 0, 1, 0, 0, 0, 1, 0]^T$. Applying the simplified min-analytic-center problem by substituting G and ξ into g_3, we obtain

$$g_3 = \xi^T G^T A G \xi = \left\| \begin{bmatrix} c_{u_{o1}} c_{v_{o1}} + c_{u_{o2}} c_{v_{o2}} \\ c_{u_{o1}} s_{v_{o1}} + c_{u_{o2}} s_{v_{o2}} \\ s_{u_{o1}} + s_{u_{o2}} \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\|^2$$

where A is assumed to be the identity matrix (the same results will be obtained if A is chosen to be other positive definite matrices). It is minimal if and only if

$$[c_{u_{o1}} c_{v_{o1}}, c_{u_{o1}} s_{v_{o1}}, s_{u_{o1}}]^T = -[c_{u_{o2}} c_{v_{o2}}, c_{u_{o2}} s_{v_{o2}}, s_{u_{o2}}]^T.$$

That is, the two contacts are antipodal. It should be noted that antipodal grasps are also the optimal solutions for the simplified max-transfer problem.

In general, if the grasped object has a complex geometry

$$g_3 = \xi G^T A G \xi = \left\| \begin{bmatrix} n(\alpha_{o1}) + n_2(\alpha_{o2}) \\ X(\alpha_{o1}) \times n(\alpha_{o1}) + X(\alpha_{o2}) \times n(\alpha_{o2}) \end{bmatrix} \right\|^2$$

which is equal to zero (minimal) if and only if

$$n(\alpha_{o1}) = -n(\alpha_{o2})$$
$$(X(\alpha_{o1}) - X(\alpha_{o2})) \times n(\alpha_{oi}) = 0, i = 1, 2.$$

Again, the optimal solutions are antipodal grasps. Antipodal grasps are not unique for a given object. To determine the optimal one from the set of antipodal grasps, we need to go back to the max-transfer problem and find the grasp with the maximal distance to be the optimal one [25].

Simply applying the gradient algorithm, g_3 can also be used to plan trajectories of the two contact points from an arbitrary initial configuration to the optimal one. In the current case, we have

$$g_3 = 2\left(1 + s_{u_{o1}} s_{u_{o2}} + c_{u_{o1}} c_{u_{o2}} c_{v_{o1} - v_{o2}}\right)$$

where $c_{v_{o1} - v_{o2}} = \cos(v_{o1} - v_{o2})$. Its Euclidean gradient is calculated as

$$\nabla_{\vec{\alpha}_o} g_3 = \begin{bmatrix} 2\left(c_{u_{o1}} s_{u_{o2}} - s_{u_{o1}} c_{u_{o2}} c_{v_{o1} - v_{o2}}\right) \\ -2 c_{u_{o1}} c_{u_{o2}} s_{v_{o1} - v_{o2}} \\ 2\left(s_{u_{o1}} c_{u_{o2}} - c_{u_{o1}} s_{u_{o2}} c_{v_{o1} - v_{o2}}\right) \\ 2 c_{u_{o1}} c_{u_{o2}} s_{v_{o1} - v_{o2}} \end{bmatrix}.$$

Fig. 18(a) and (b) gives trajectories of g_3 and the distance between the two fingers of the hand, respectively. It clearly shows that the grasp tends to be antipodal (see Fig. 19). The parameters used are $R = 100$ mm, $\alpha_{o1}(0) = (0, 0)$ and $\alpha_{o2}(0) = (0, 2.5)$. The computation time for achieving the final antipodal configuration is 0.047 second, which shows that the computation time has been greatly reduced compared with those in Section V.

Example 5. Symmetric Configurations: Optimal 3-Fingered Grasps of a Spherical Object: Consider a three-fingered hand grasping a spherical object of radius R. PCWF model is assumed for the three contacts with local coordinates $\alpha_{o1} = (u_{o1}, v_{o1})^T, \alpha_{o2} = (u_{o2}, v_{o2})^T$, and $\alpha_{o3} = (u_{o3}, v_{o3})^T$. The grasp map is given by

$$G = [G_1 \; G_2 \; G_3]$$

where

$$G_i = \begin{bmatrix} -s_{u_{oi}} c_{v_{oi}} & s_{v_{oi}} & c_{u_{oi}} c_{v_{oi}} \\ -s_{u_{oi}} s_{v_{oi}} & -c_{v_{oi}} & c_{u_{oi}} s_{v_{oi}} \\ c_{u_{oi}} & 0 & s_{u_{oi}} \\ R s_{v_{oi}} & R s_{u_{oi}} c_{v_{oi}} & 0 \\ -R c_{v_{oi}} & R s_{u_{oi}} s_{v_{oi}} & 0 \\ 0 & -R c_{u_{oi}} & 0 \end{bmatrix}, \quad i = 1, \ldots, 3.$$

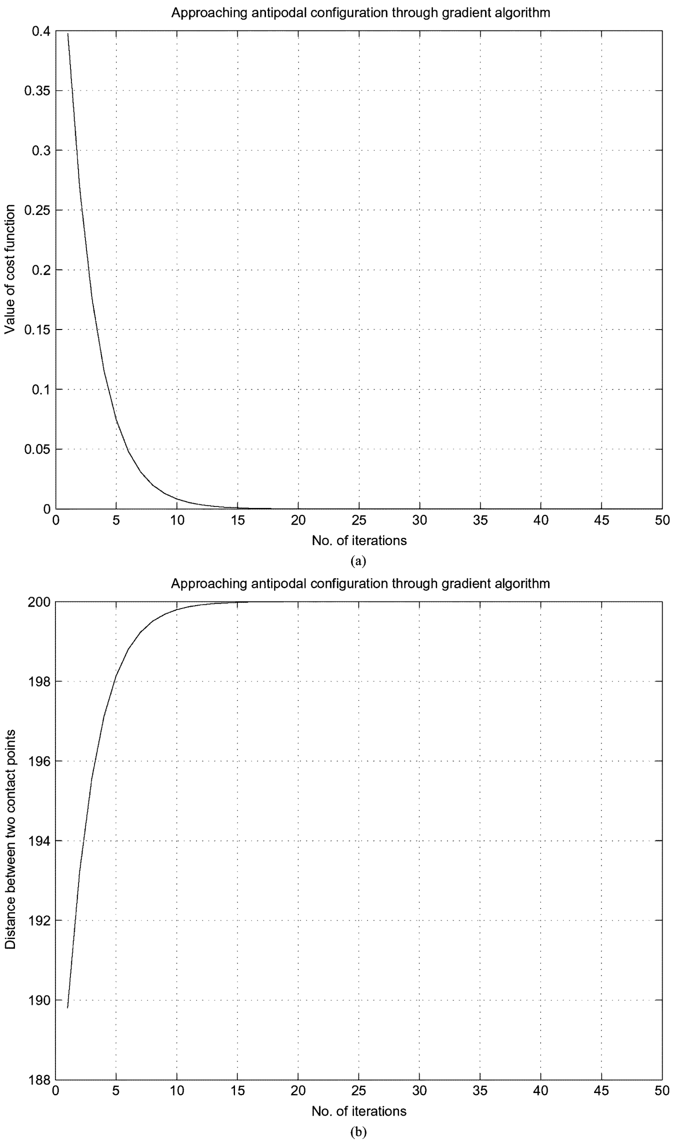

Fig. 18. (a) Trajectory of g_3. (b) Distance between two fingers.

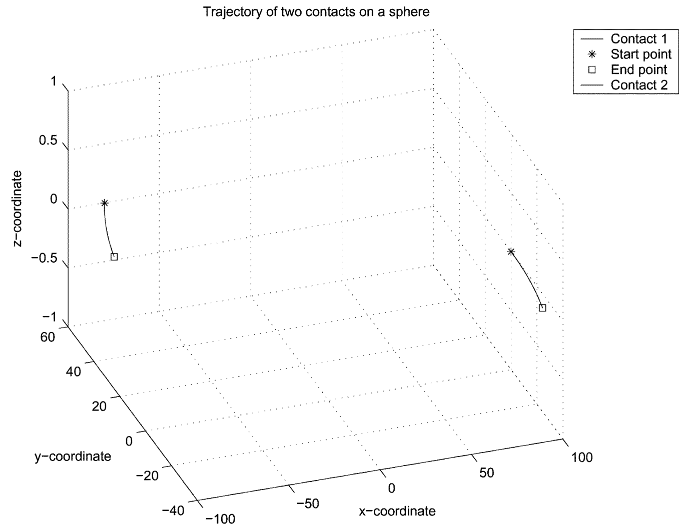

Fig. 19. Trajectories of the two contact points.

The finger forces are restricted to

$$\mathcal{FC} = \mathcal{FC}_1 \times \mathcal{FC}_2 \times \mathcal{FC}_3$$

with the center

$$\xi = [0, 0, 1, 0, 0, 1, 0, 0, 1]^T.$$

Applying the simplified min-analytic-center problem, the cost function is calculated as

$$g_3 = \left\| \begin{bmatrix} c_{u_{o1}} c_{v_{o1}} + c_{u_{o2}} c_{v_{o2}} + c_{u_{o3}} c_{v_{o3}} \\ c_{u_{o1}} s_{v_{o1}} + c_{u_{o2}} s_{v_{o2}} + c_{u_{o3}} s_{v_{o3}} \\ s_{u_{o1}} + s_{u_{o2}} + s_{u_{o3}} \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\|^2.$$

It is minimal if and only if

$$\begin{bmatrix} c_{u_{o1}} c_{v_{o1}} + c_{u_{o2}} c_{v_{o2}} + c_{u_{o3}} c_{v_{o3}} \\ c_{u_{o1}} s_{v_{o1}} + c_{u_{o2}} s_{v_{o2}} + c_{u_{o3}} s_{v_{o3}} \\ s_{u_{o1}} + s_{u_{o2}} + s_{u_{o3}} \end{bmatrix} = 0.$$

This clearly shows that the three fingers should be at three symmetric points of a big circle of the object. The same conclusion can also be reached by applying the simplified max-transfer problem.

In general, when the object is not spherical, the cost function g_3 is given by the equation at the bottom of the page, and it is zero (or minimal) if and only if

$$\sum_{i=1}^{3} n(\alpha_{oi}) = 0$$

$$\sum_{i=1}^{3} X(\alpha_{oi}) \times n(\alpha_{oi}) = 0.$$

The first equality means that the three normal vectors are 120° from each other. The two equalities together mean that the three normal vectors intersect at the same point, and are contained in one plane. This is exactly what Mirtich and Canny [25] termed a symmetric grasp. Again, all such grasps form a nonempty set for a given object. To find the optimal one from this set, we go back to the max-transfer problem and find the grasp with the largest outer triangle.

Second, we use g_3 and its respective gradient algorithm to optimize the grasp from an arbitrary initial configuration. Note that

$$\begin{aligned} g_3 = 3 + 2 \big(& s_{u_{o1}} s_{u_{o2}} + s_{u_{o2}} s_{u_{o3}} + s_{u_{o1}} s_{u_{o3}} \\ & + c_{u_{o1}} c_{u_{o2}} c_{v_{o1} - v_{o2}} + c_{u_{o2}} c_{u_{o3}} c_{v_{o2} - v_{o3}} \\ & + c_{u_{o1}} c_{u_{o3}} c_{v_{o1} - v_{o3}} \big). \end{aligned}$$

$$\left\| \begin{bmatrix} n(\alpha_{o1}) + n_2(\alpha_{o2}) + n_3(\alpha_{o3}) \\ X(\alpha_{o1}) \times n(\alpha_{o1}) + X(\alpha_{o2}) \times n(\alpha_{o2}) + X(\alpha_{o3}) \times n(\alpha_{o3}) \end{bmatrix} \right\|^2$$

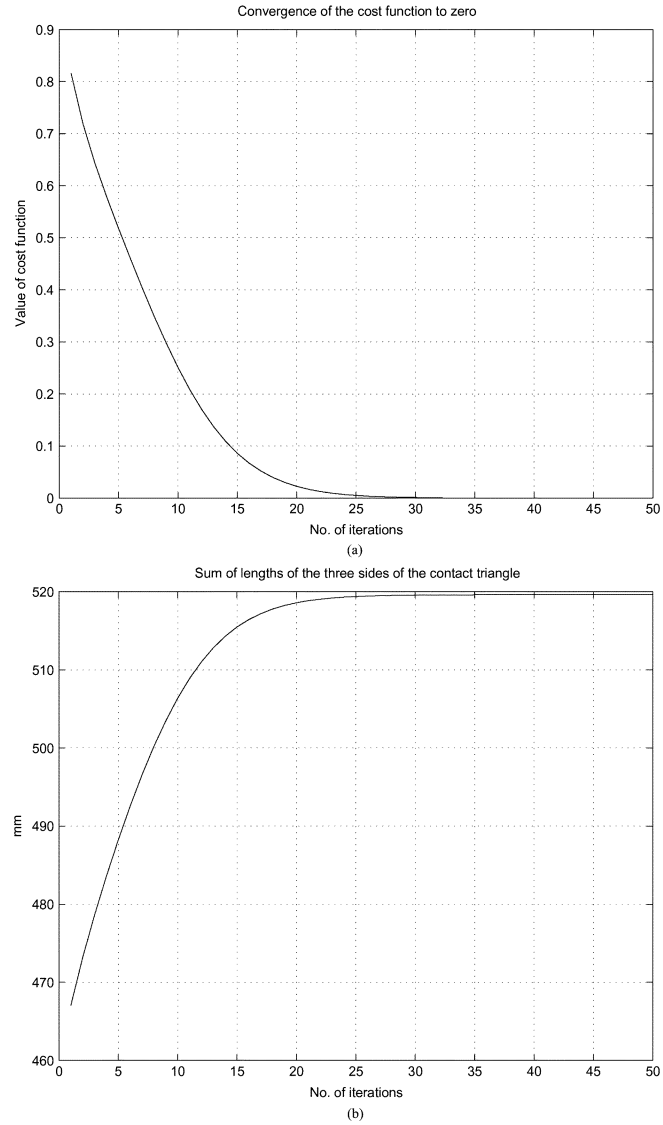

Fig. 20. (a) Trajectory of g_3 (b) The sum of the length of three sides of the grasp triangle.

Its Euclidean gradient ∇g_3 is given by

$$\begin{bmatrix} 2\left[c_{u_{o1}}\left(s_{u_{o2}}+s_{u_{o3}}\right)-s_{u_{o1}}\left(c_{u_{o2}}c_{v_{o1}-v_{o2}}+c_{u_{o3}}c_{v_{o1}-v_{o3}}\right)\right] \\ -2c_{u_{o1}}\left(c_{u_{o2}}s_{v_{o1}-v_{o2}}+c_{u_{o3}}s_{v_{o1}-v_{o3}}\right) \\ 2\left[c_{u_{o2}}\left(s_{u_{o1}}+s_{u_{o3}}\right)-s_{u_{o2}}\left(c_{u_{o1}}c_{v_{o1}-v_{o2}}+c_{u_{o3}}c_{v_{o2}-v_{o3}}\right)\right] \\ -2c_{u_{o2}}\left(c_{u_{o1}}s_{v_{o2}-v_{o1}}+c_{u_{o3}}s_{v_{o2}-v_{o3}}\right) \\ 2\left[c_{u_{o3}}\left(s_{u_{o1}}+s_{u_{o2}}\right)-s_{u_{o3}}\left(c_{u_{o2}}c_{v_{o2}-v_{o3}}+c_{u_{o1}}c_{v_{o1}-v_{o3}}\right)\right] \\ -2c_{u_{o3}}\left(c_{u_{o2}}s_{v_{o3}-v_{o2}}+c_{u_{o1}}s_{v_{o3}-v_{o1}}\right) \end{bmatrix}.$$

The trajectories of g_3 and the sum of the length of the three sides of the triangle formed by the three contact points are shown in Fig. 20(a) and (b), respectively. Fig. 21 gives the trajectories of the three contact points of the hand. Here, we have used the following parameters $R = 100$ mm, $\alpha_{o1}(0) = (0,0), \alpha_{o2}(0) = (0, 2.5)$, and $\alpha_{o3}(0) = (0, 3.3)$. As is expected, the grasp tends to be symmetric as the sum of the length of the three sides of the triangle formed by the three contact points approaches to $3\sqrt{3}R$. It takes $0.047s$ to arrive at the symmetric grasp configuration, showing again the efficiency of the simplified problem.

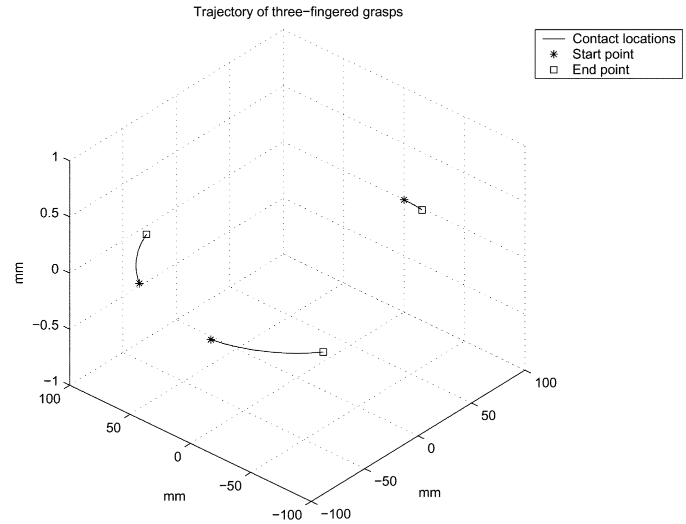

Fig. 21. Trajectory of the grasp configuration.

VII. Application to Coordinated Manipulation With Contact Points Servoing

In this section, we apply real-time grasp planning to coordinated manipulation with contact points servoing. As shown in Fig. 1, given a desired trajectory of the grasped object, $g_{po}^d(t)$, we wish to find the corresponding finger velocity V_{pf_i} that: 1) executes the desired object trajectory, and 2) maintains or optimizes the grasp quality. Readers are referred to [20], [9] for generation of optimal finger forces that balance a given object wrench. The transformation taking O to P can be expressed as

$$g_{po} = g_{pf_i} g_{f_i l_{f_i}} g_{l_{f_i} l_{o_i}} g_{l_{o_i} o}$$

from which we have

$$Ad_{g_{l_{o_i} o}} V_{po} = Ad_{g^{-1}_{f_i l_{o_i}}} V_{pf_i} - Ad_{g_{l_{o_i} l_{f_i}}} V_{l_{o_i} l_{f_i}}$$

and thus

$$V_{pf_i} = Ad_{g_{f_i o}} V_{po} + Ad_{g_{f_i l_{f_i}}} V_{l_{o_i} l_{f_i}}. \tag{21}$$

In (21), the object velocity V_{po} is the input and the fingertip velocity V_{pf_i} is the output to be determined. We wish to specify the contact velocity $V_{l_{o_i} l_{f_i}} \in \mathbb{R}^6$ so that: (i) grasp quality (and thus force-closure condition) is maintained or optimized, and (ii) the fingers impart on the object a desired object wrench. In order to prevent sliding, and maintain finger forces inside the friction cone, we impose the following constraints on the contact velocity:

$$V_{l_{o_i} l_{f_i}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_x^i \\ \omega_y^i \end{bmatrix} := B_{c_i}^c \begin{bmatrix} \omega_x^i \\ \omega_y^i \end{bmatrix}$$

where $(\omega_x^i, \omega_y^i)^T$ are relative rolling velocities to be determined. By inverting Montana's kinematic equations of contact

$$\begin{bmatrix} \omega_x^i \\ \omega_y^i \end{bmatrix} = R_{\psi_i}(K_{oi} + \tilde{K}_{fi}) M_{oi} \dot{\alpha}_{o_i}$$

Fig. 22. The HKUST three-fingered hand manipulating a spherical object.

where $(K_{oi}+\tilde{K}_{fi})$ is the relative curvature form, M_{oi} the metric form of the object [29], and

$$R_{\psi_i} = \begin{bmatrix} \cos\psi_i & -\sin\psi_i \\ -\sin\psi_i & -\cos\psi_i \end{bmatrix}.$$

We specify $\dot{\alpha}_{o_i}$ using the negative gradient of the grasp quality function

$$g_3 : \vec{\alpha}_o \to \mathbb{R}$$

and

$$\dot{\vec{\alpha}}_o = -\lambda \nabla g_3(\vec{\alpha}_o), \qquad \lambda \in (0,1).$$

Let x_i^d be the optimal finger forces needed to balance an object wrench, (see [10], [9] for computation of x_i^d). The net finger velocity required to accomplish all the objectives is given by

$$\begin{aligned} V_{pf_i} &= Ad_{g_{f_io}}(\eta_i)V_{po}^d \\ &\quad + Ad_{g_{f_il_{f_i}}} B_{c_i}^c R_{\psi_i}(K_{oi}+\tilde{K}_{fi})M_{oi}(-\lambda\nabla_i g_3(\vec{\alpha}_o)) \\ &\quad + C_i\left(x_i^d - x_i\right) \end{aligned} \tag{22}$$

where $C_i \in \mathbb{R}^{6\times n_i}$ is a compliance matrix, and x_i the actual finger force. Note that in (22), the first term allows the finger to accommodate the object motion V_{po}^d, the second term servos the contact points to an optimal grasp configuration and the last term enables the fingers to impart a net object wrench with optimal finger forces. Once the hand achieves an optimal grasp configuration, the second correction term disappears with vanishing of the gradient vector field.

Remark 3: During manipulation, the real-time optimization of the grasp quality function is necessary to keep the stability of the system, and thereby prevent the dropping of the object [21]. Here, we adopt the simplified grasp quality function g_3, which makes the real-time execution of manipulation tasks possible as suggested in the simulation example (0.047 s).

Several experiments are conducted with the HKUST three-fingered hand (see Fig. 22). Each finger of the hand consists of a Motorman K-3S robot equipped with force/torque sensor and a 16×16 tactile array fingertip. A VME based multiprocessor control system with three 8-axis DSP motion control boards is provided for joint-level control and two Motorola 68 040 processors are used for object-level motion and grasping force control, along with a VxWorks real-time operating system and a Sun workstation. In the experiments, the object is required to move 100 mm along the twist $(0,0,1,0,0,0)$ in 10 seconds.

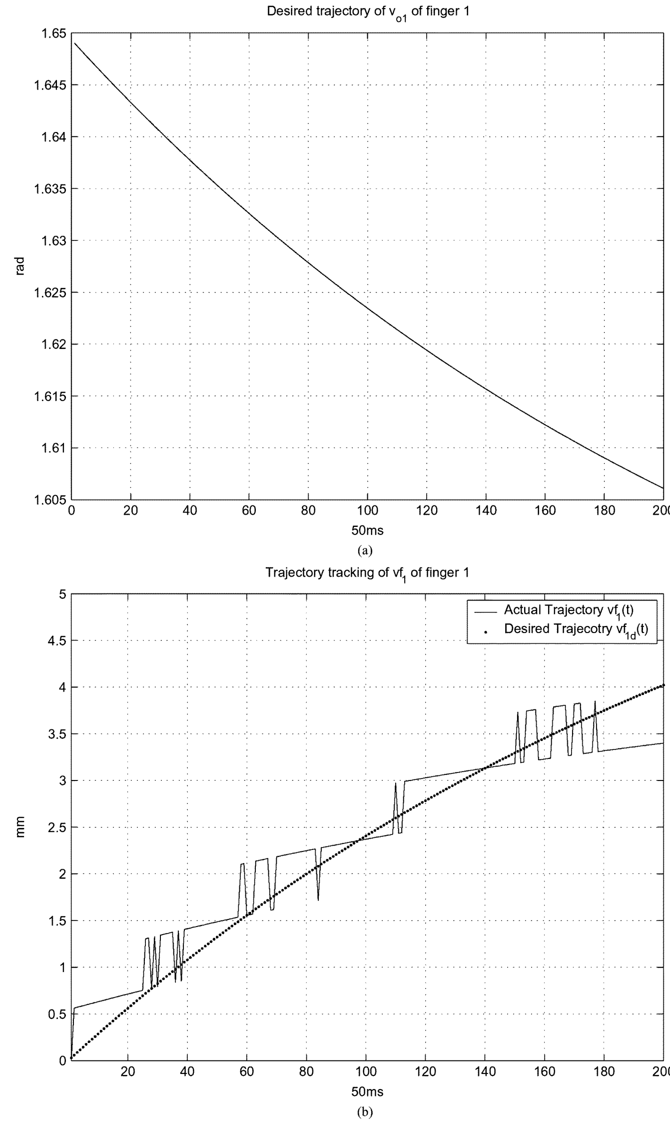

Fig. 23. (a) Finger 1: Desired trajectory of v_{o1}, (b) Finger 1: Trajectory tracking of v_{f1}.

We first manipulate a ball of radius $R = 93$ mm using only two fingers, the initial coordinates of the two contacts are $\alpha_{o1}(0) = (0,((1/2)+(1/40))\pi)^T$ and $\alpha_{o2}(0) = (0,(-(1/2)-(1/40))\pi)^T$. The desired curves of the two contacts, as planned in Example 5, are given in Figs. 23(a) and 24(a), which show that the two fingers tend to be antipodal at the big circle $u_o = 0$. Figs. 23(b) and 24(b) show the trajectory tracking results of $v_{\mathrm{fi}}(u_{\mathrm{fi}} = 0), i = 1,2$. The desired curves $v_{\mathrm{fi}}^d(t)$ are obtained from $v_{oi}^d(t)$ by inverting Montana's contact kinematics equations [30], [29].

In the second experiment, we manipulate a ball of radius $R = 122$ mm using three fingers. Other parameters are $\alpha_{o1}(0) = (0,((5/6)+(1/100))\pi)^T, \alpha_{o2}(0) = (0,(-(1/2)-$

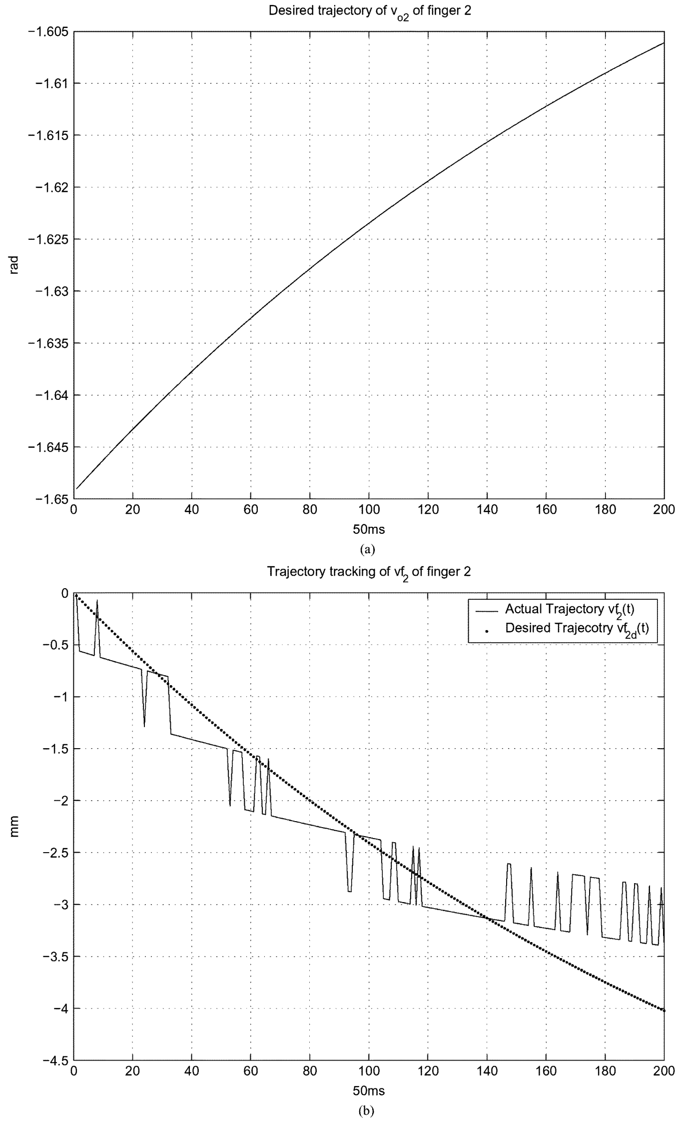

Fig. 24. (a) Finger 2: Desired trajectory of v_{o2}, (b) Finger 2: Trajectory tracking of v_{f2}.

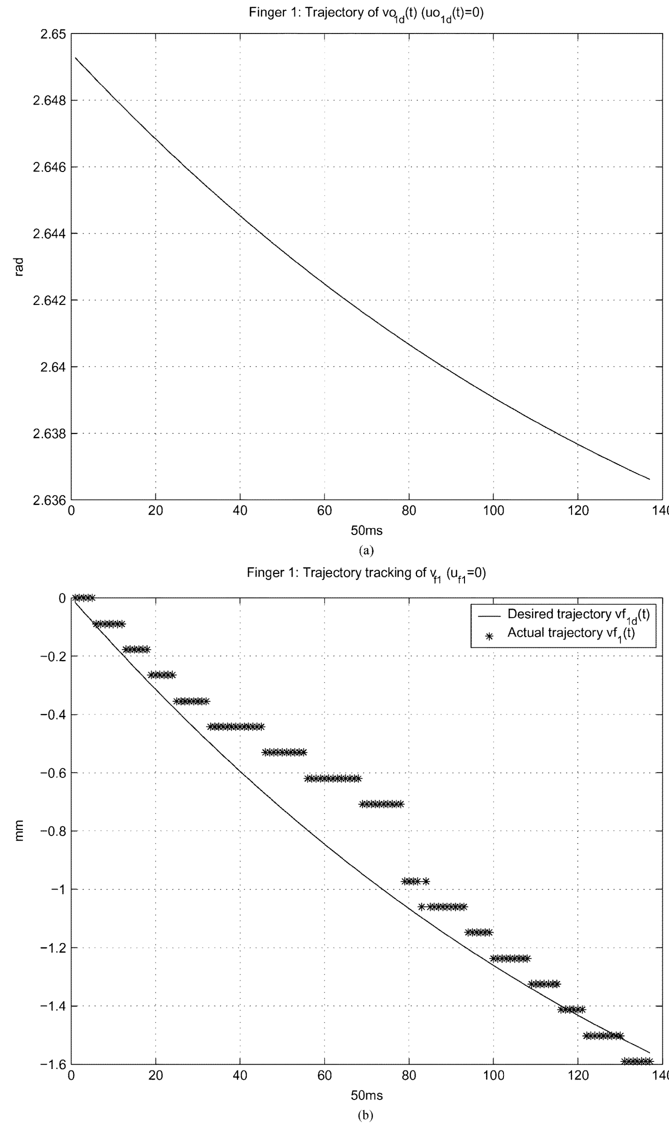

Fig. 25. (a) Finger 1: Desired trajectory of v_{o1}, (b) Finger 1: Trajectory tracking of v_{f1}.

$(1/40))\pi)^T$, and $\alpha_{o3}(0) = (0, ((1/6) + (1/40))\pi)^T$. The desired curves of the three contacts, as planned in Example 6, are shown in Figs. 25(a) and 26(a), from which we can see that the three fingers tend to the three symmetric points of the great circle $u_o = 0$. The trajectory tracking results of $v_{\rm fi}\ (u_{\rm fi} = 0), i = 1, \ldots, 3$ are shown in Figs. 25(b), 26(b), and 27(b), respectively.

VIII. Conclusion

This paper presented a general formulation of the optimal grasp synthesis problem as optimization problem of three grasp quality functions. We discussed physical significances and gave algorithms for computing the gradient solutions of these quality functions. We also provided simplified versions of two problems when real-time grasp planning solutions are needed. we showed in particular that optimal solutions of the simplified problems coincide with the familiar optimal grasps obtained using heuristic approaches. We applied real-time grasping optimization to coordinated manipulation with contact points servoing. Simulation and experimental studies were conducted to illustrate validity of the proposed methods.

In future works, we wish to extend the methods and algorithms to objects with edges and vertices and obtain accelerated results when objects have special geometries, e.g., polyhedral objects.

Appendix A
Approximate Solution to the Analytic-Center Problem

The analytic-center problem

$$\min_{Gx=w_o, P(x)>0} \text{logdet } P(x)^{-1}$$

is equivalent to

$$\max_{Gx=w_o, P(x)>0} \text{logdet } P(x). \tag{23}$$

Based on the structure of the matrix $P(x)$ (5), we have

$$\det P(x) = \Pi_{i=1}^k \left(\mu_i^2 x_{i,3}^2 - x_{i,1}^2 - x_{i,2}^2\right) \le \Pi_{i=1}^k \mu_i^2 x_{i,3}^2.$$

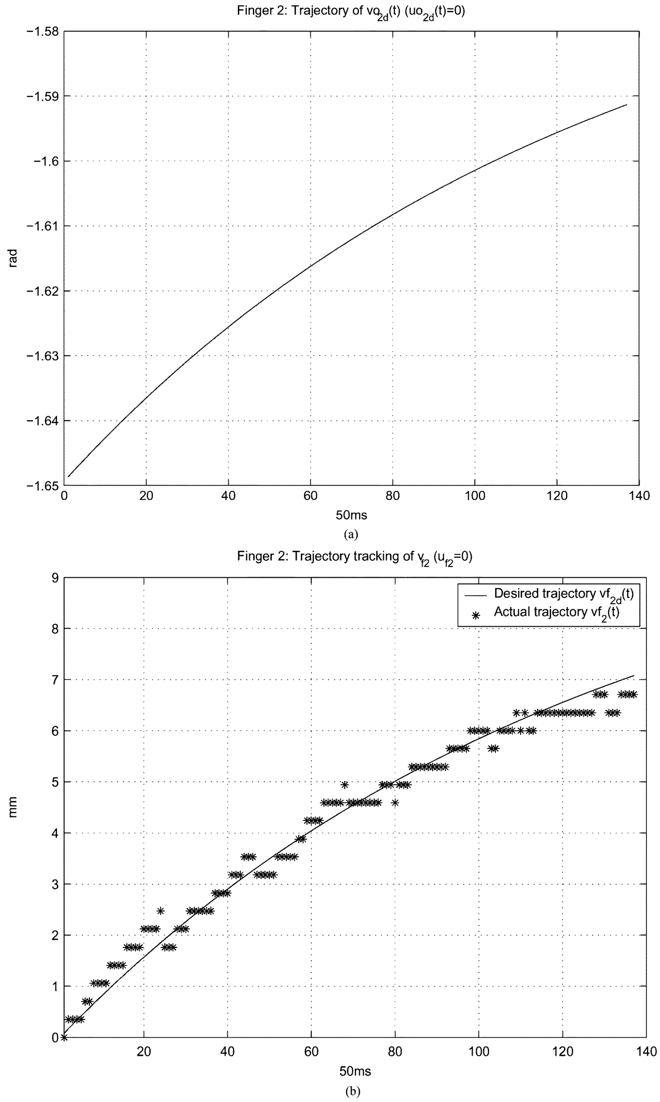

Fig. 26. (a) Finger 2: Desired trajectory of v_{o2}, (b) Finger 2: Trajectory tracking of v_{f2}.

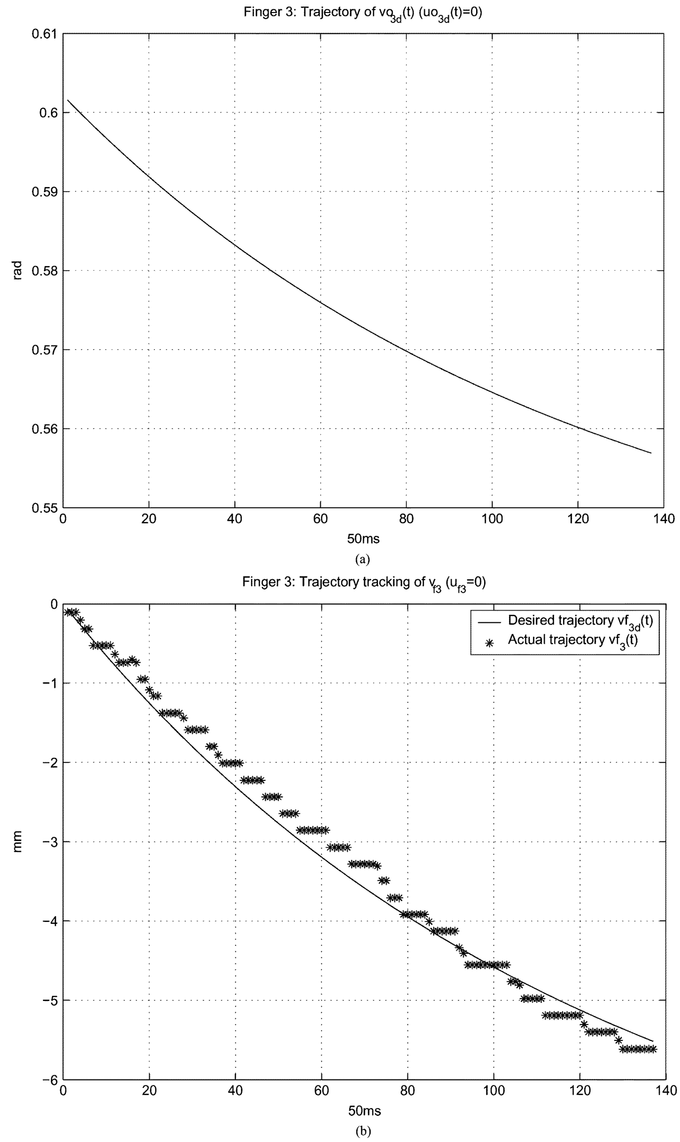

Fig. 27. (a) Finger 3: Desired trajectory of v_{o3}, (b) Finger 3: Trajectory tracking of v_{f3}.

Here, without loss of generality we assume that the friction model is PCWF. Note also that all finger forces x with $x_{i,1} = x_{i,2} = 0$ and $x_{i,3} > 0, i = 1, \ldots, k$, will automatically satisfy $P(x) > 0$. Thus a sufficient condition for $x^\star$ to be a solution of (23) is $x^\star = t\xi, t > 0$, and $Gx^\star = w_o$ with

$$\xi = [\underbrace{0, 0, 1}_{1}, \ldots, \underbrace{0, 0, 1}_{k}]^T.$$

If $x = t\xi$ and $Gx = w_o$ can not be simultaneously satisfied, we seek a t as in (18), (19), and (20), such that $\|G\xi t\| = \|w_o\|$ is satisfied, which, of course, only provides an approximation of the solution to (23). We could also adopt the following optimal approximation:

$$\min_t \|G\xi t - w_o\|.$$

However, the resultant optimal solution is still a function of w_o. This will not simplify too much of the cost function.

REFERENCES

[1] F. Reuleaux, *Theoretische Kinematik. Translated as Kinematics of Machinery*. New York: Dover, 1875.

[2] J. K. Salisbury and B. Roth, "Kinematic and force analysis of articulated mechanical hands," *ASME J. Mech. Transmissions Automat. Des.*, vol. 105, no. 1, pp. 35–41, 1983.

[3] B. Mishra, J. T. Schwartz, and M. Sharir, "On the existence and synthesis of multifinger positive grips," *Algorithmica*, vol. 2, no. 4, pp. 541–558, 1987.

[4] I. M. Chen and J. W. Burdick, "Finding antipodal point grasps on irregularly shaped objects," *IEEE Trans. Robot. Automat.*, vol. 9, pp. 507–512, Aug. 1993.

[5] V. Nguyen, "Constructing force-closure grasps," *Int. J. Robot. Res.*, vol. 7, no. 3, pp. 2–17, 1988.

[6] J. C. Trinkle, "On the stability and instantaneous velocity of grasped frictionless objects," *IEEE Trans. Robot. Automat.*, vol. 8, pp. 560–572, Oct. 1992.

[7] A. Bicchi, "On the closure properties of robotic grasping," *Int. J. Robot. Res.*, vol. 14, no. 4, pp. 319–334, 1995.

[8] Y. H. Liu, "Qualitative test and force optimization of 3-D frictional form-closure grasps using linear programming," *IEEE Trans. Robot. Automat.*, vol. 15, pp. 163–173, Feb. 1999.

[9] L. Han, J. C. Trinkle, and Z. X. Li, "Grasp analysis as linear matrix inequality problems," *IEEE Trans. Robot. Automat.*, vol. 16, pp. 663–674, Dec. 2000.
[10] M. Buss, H. Hashimoto, and J. Moore, "Dextrous hand grasping force optimization," *IEEE Trans. Robot. Automat.*, vol. 12, pp. 406–418, June 1996.
[11] M. Buss, L. Faybusovieh, and J. Moore, "Dikin-type algorithms for dextrous grasping force optimization," *Int. J. Robot. Res.*, vol. 17, no. 8, pp. 831–839, 1998.
[12] Z. Ji and B. Roth, "Contact force in grasping and kinematic constraints," in *Proc. 7th IFTOMM World Congr.*, 1987, pp. 1219–1222.
[13] J. Ponce and B. Faverjon, "On computing three-finger force-closure grasps of polygonal objects," *IEEE Trans. Robot. Automat.*, vol. 11, pp. 868–881, Dec. 1995.
[14] J. Ponce *et al.*, "On computing four-finger equilibrium and force-closure grasps of polyhedral objects," *Int. J. Robot. Res.*, vol. 16, no. l, pp. 11–35, 1997.
[15] D. Ding, Y. H. Liu, M. Y. Wang, and S. G. Wang, "Automatic selection of fixturing surfaces and fixturing points of polyhedral workpieces," *IEEE Trans. Robot. Automat.*, vol. 17, pp. 833–841, Dec. 2001.
[16] Y. H. Liu, "Computing n-finger form-closure grasps on polygonal objects," *Int. J. Robot. Res.*, vol. 18, no. 2, pp. 149–158, 2000.
[17] M. Cutkosky, *Robotic Grasping and Fine Manipulation*. Boston, MA: Kluwer, 1985.
[18] Z. Li and S. Sastry, "Task oriented optimal grasping by multifingered robotic hands," in *Proc. IEEE Int. Conf. Robotics Automation*, 1987, pp. 389–394.
[19] J. W. Hong, G. Lafferriere, B. Mishra, and X. L. Tang, "Fine manipulation with multifinger hands," in *Proc. IEEE Int. Conf. Robotics Automation*, 1990, pp. 1568–1573.
[20] Z. X. Li, Z. Qin, S. Jiang, and L. Han, "Coordinated motion generation and real-time grasping force control for multifingered manipulation," in *Proc. IEEE Int. Conf. Robotics Automation*, 1998, pp. 3631–3638.
[21] S. L. Jiang, K. K. Choi, and Z. X. Li, "Coordinated motion generation for multifingered manipulation using tactile feedback," in *Proc. IEEE Int. Conf. Robotics Automation*, 1999, pp. 3032–3037.
[22] L. Han *et al.*, "Dextrous manipulation with rolling contacts," in *Proc. IEEE Int. Conf. Robotics Automation*, 1997, pp. 992–997.
[23] D. Kirkpatrick, B. Mishra, and C. K. Yap, "Quantitative Steinitz's theorem with applications to multifingered grasping," *Discr. Comput. Geom.*, vol. 7, no. 3, pp. 295–318, 1992.
[24] C. Ferrari and J. Canny, "Planning optimal grasps," in *Proc. IEEE Int. Conf. Robotics Automation*, 1992, pp. 2290–2295.
[25] B. Mirtich and J. Canny, "Easily computable optimum grasps in 2-D and 3-D," in *Proc. IEEE Int. Conf. Robotics Automation*, 1994, pp. 739–747.
[26] X. Y. Zhu, H. Ding, and J. Wang, "Grasp analysis and synthesis based on a new quantitative measure," *IEEE Trans. Robot. Automat.*, vol. 19, pp. 942–953, Dec. 2003.
[27] X. Y. Zhu and J. Wang, "Synthesis of force-closure grasps on 3-D objects based on the Q-distance," *IEEE Trans. Robot. Automat.*, vol. 19, pp. 669–679, Aug. 2003.
[28] G. F. Liu and Z. X. Li, "Real-time grasping force optimization for multifingered manipulation: Theory and experiments," *IEEE/ASME Trans. Mechatron.*, vol. 9, pp. 65–77, Mar. 2004.
[29] R. Murray, Z. X. Li, and S. Sastry, *A Mathematical Introduction to Robotic Manipulation*. Boca Raton, FL: CRC, 1994.
[30] D. J. Montana, "The kinematics of contact and grasp," *Int. J. Robot. Res.*, vol. 7, no. 3, pp. 17–25, 1988.
[31] U. Helmke, K. Hueper, and J. B. Moore, "Quadratically convergent algorithms for optimal dextrous hand grasping," *IEEE Trans. Robot. Automat.*, vol. 18, pp. 138–146, Apr. 2002.
[32] M. D. Do Carmo, *Differential Geometry of Curves and Surfaces*. Englewood Cliffs, NJ: Prentice-Hall, 1976.
[33] T. Omata, "Fingertip position planning for a multifingered hand," in *Proc. IEEE Int. Workshop Intelligent Robots Systems*, 1990, pp. 859–864.
[34] B. Faverjon and J. Ponce, "On computing two-finger force-closure grasps of curved 2-D objects," in *Proc. IEEE Int. Conf. Robotics Automation*, 1991, pp. 424–429.
[35] V. F. Demyanov and V. N. Malozemov, *Introduction to Minmax*. New York: Dover, 1990.
[36] T. Yoshikawa and K. Nagai, "Manipulating and grasping forces in manipulation by multifingered robot hands," *IEEE Trans. Robot. Automat.*, vol. 7, pp. 67–77, Feb. 1991.
[37] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge Univ. Press, 2004.
[38] L. Vandenberghe, S. Boyd, and S.-P. Wu, "Determinant maximization with linear matrix inequality constraints," *SIAM J. Matrix Anal. Applicat.*, vol. 19, no. 2, pp. 499–533, 1998.

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