A Geometric Theory for Synthesis and Analysis of Sub-6 DoF Parallel Manipulators

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I. Introduction

Parallel mechanism design is in general a difficult problem. It consists of interactively solving two tightly coupled problems: *(i) mechanism synthesis* and *(ii) dimensional optimization.* Over the last two decades or so, we have witnessed the enormous progresses toward the second problem, namely for a given mechanism the kinematic (forward, inverse and instantaneous)([1],[2],[3]) and singularity ([4], [5], [6], [7]) analysis, the determination and computation of properties such as workspace and stiffness ([8],[9],[10]), and the different approaches for formulating and optimizing the various performance indices ([11],[12],[13]); we have also seen numerous architectures or topological configurations being proposed in the literature, but solution to the first problem remains to be ad hoc, it is more of an art than an engineering science. Human experience and intuitions instead of mathematically rigorous and justifiable procedures are relied on when a new design problem is called for. The lack of a rigorous and yet precise and complete mathematical approach to the synthesis problem has not only prevented globally optimal solutions in many design cases, but also rendered the mission of automating design solutions, an ultimate objective of design, extremely difficult, if not impossible.

Majority of the parallel mechanism architectures being proposed in the literature have six degrees-of-freedom (DoF). Most industrial applications, however, rarely need it, even those for which some of the 6-DoF platforms are designed for. For example, only 3-DoF is needed for any orientation device, 4-DoF is sufficient for most pick-and-place applications, and 5-DoF is adequate for every conceivable machine tool application. Having a mechanism with more DoF than necessary is not only wasteful in terms of the hardware resources but also increases the programming and maintenance cost. The Delta manipulator ([14], [15]) is perhaps the most famous and successful example of a sub-6 DoF parallel mechanism design. Other examples include the H4 robot by Pierrot [16], [17], the orientation device by Gosselin [18], [19], the haptic devices by T. Salcudean [13], the Tsai's manipulators [1]. More examples are found in Gao [20], Huang [21], and Merlet [22].

Synthesis design of sub-6 DoF parallel mechanisms is in general more difficult than that of their 6-DoF counterpart, for the reasons being that, first of all and most importantly, the set of desired end-effector motions, except for a few special cases, does not admit a so-called *canonical* or (Lie) subgroup description, a description which is global and free of local coordinates; secondly, the subchains that make up a sub-6 DoF mechanism can have in general much richer and more diversified structures than that of their counterpart; and lastly but not the least, much fewer design cases are available in an area where experiences count the most.

A rigorous mathematical approach with a strong geometric flavor thus appears to be the only alternative for addressing the immense difficulties encountered in synthesis design of parallel mechanisms, especially sub 6-DoF parallel mechanisms, and for achieving the ultimate design objectives.

Herve ([23], [24]) is perhaps the first to initiate such a program. Using group theory, he studied synthesis of several subgroups of the Euclidean group $SE(3)$ with 1-dimensional lower pairs. He also considered generations of the product of the planar subgroup and the rotational subgroup and explained how some of the subchains employed by existing parallel manipulator designs such as the Delta arise. Related effort can also be found in the works J. Angeles [25], and Z. Huang and Herve [26].

Inspired by Herve's work, the aim of this paper is to develop a rigorous and yet precise and complete geometric theory for the synthesis and analysis of sub-6 DoF parallel manipulators. The single most important and yet natural mathematical tool we will rely on is the powerful Lie group theory, one that has already found widespread use in kinematic analysis of serial manipulators as well as other branches of robotics and control ([27], [28]). Starting with a review of the basic properties of the Euclidean group $SE(3)$, especially its Lie algebra and the exponential map that connects the Lie algebra with the Lie group, we recall a classic result that classifies, up to a conjugacy class, all subgroups of $SE(3)$. With the low dimensional subgroups providing models for lower pairs or so called primitive generators, the high dimensional subgroups are used as models for some desired set of end-effector motions. We show that, aside from these few special *canonical* cases, the set of desired end-effector motions can also be modelled as regular submanifolds of $SE(3)$, and give two important classes of regular submanifolds that can be expressed globally, and independent of local coordinates. One class is given by

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the product of two Lie subgroups and another the product of a submanifold of the translational subgroup $T(3)$ with that of the rotational subgroup $SO(3)$. Starting from a given list of primitive generators, we give a rigorous definition of the synthesis problem for a serial manipulator subchain, and develop a general procedure for solving the synthesis problem when the set of desired end-effector motions is a Lie subgroup or a regular submanifold. In doing so, several important new concepts are introduced, which substantially broaden our understanding of the synthesis problem. Finally, for the parallel manipulator synthesis problem, we introduce a general proposition that dictates how the subchains should be chosen so as to generate the desired set of end-effector motions for the resulting manipulator. We again present a general procedure for solving the parallel manipulator synthesis problem for the case of a Lie subgroup and a regular submanifold.

The paper is organized as follows: In Section II, we develop the mathematical preliminaries needed for the paper; In Section III, we study the synthesis problem for serial manipulator subchains; and in Section IV, we study the parallel manipulator synthesis problem. Finally, in Section V we conclude the paper with a few comments for future works.

II. Mathematical Tool Boxes

A. The Special Euclidean Group $SE(3)$

We assume the reader is familiar with the basic notions of differentiable manifolds and Lie groups (See e.g., [29] and [30]). Examples of the former include $\mathbb{R}^n$ and S^n, the unit sphere in $\mathbb{R}^{n+1}$, and of the latter include $\mathbb{R}^n$, $SO(n)$, the Special Orthogonal group of $\mathbb{R}^n$, and $GL(n,\mathbb{R})$, the set of $n \times n$ nonsingular matrices.

Our primary interest lies in the Special Euclidean group

$$SE(3) = \left\{ \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \mid p \in \mathbb{R}^3, R \in SO(3) \right\}$$

It is well known that by attaching a Cartesian coordinate frame to a rigid body, a configuration of the body relative to a *reference configuration* can be identified with an element g of $SE(3)$, and a trajectory of the body can be identified with a curve $g(t), t \in (-\epsilon, \epsilon)$ of $SE(3)$. Thus, rigid body motions can be studied by investigating the geometric properties of $SE(3)$.

Apparently, $SE(3)$ is a closed linear subgroup of $GL(4,\mathbb{R})$ as it is defined by polynomial equations. Thus, by Theorem 0.15 [30], $SE(3)$ with its relative topology becomes a Lie group. The manifold structure of $SE(3)$ is diffeomorphic to that of $\mathbb{R}^3 \times SO(3)$, i.e,

$$\Psi : \mathbb{R}^3 \times SO(3) \to SE(3) : (p, R) \mapsto \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

However, $SE(3)$ with its group operation given by $(p_1, R_1) \cdot (p_2, R_2) = (R_1 p_2 + p_1, R_1 R_2)$ is not a product in a group theoretic sense.

Associated with a Lie group G is its Lie algebra $\mathcal{G}$, defined as the tangent space to G at the identity e, i.e, $\mathcal{G} \triangleq T_e G$. The Lie algebra of $SE(3)$, denoted $se(3)$, has the form

$$se(3) = \left\{ \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \mid \omega, v \in \mathbb{R}^3 \right\}$$

where for $\omega \in \mathbb{R}^3$

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3)$$

and $so(3)$ is the Lie algebra of the rotation group $SO(3)$. Clearly, $se(3)$ can be identified with $\mathbb{R}^6$ via the map

$$\wedge : \mathbb{R}^6 \to se(3) : \xi = (v^T, \omega^T)^T \mapsto \hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

An element $\hat{\xi}$ of $se(3)$ is called a twist, with twist coordinates $\xi = \begin{pmatrix} v \\ \omega \end{pmatrix} \in \mathbb{R}^6$. The pitch ρ of ξ is defined as

$$\rho = \begin{cases} \frac{\omega^T v}{||\omega||^2}, & \text{if } \omega \neq 0 \\ \infty, & \text{if } \omega = 0 \end{cases}$$

The exponential map

$$exp : se(3) \to SE(3) : \hat{\xi} \mapsto e^{\hat{\xi}} \tag{1}$$

defines a local diffeomorphism taking the zero vector to the identity element of $SE(3)$. See [27] for an explict formula of (1). Physically, $e^{\hat{\xi}}$ corresponds to a screw motion along the axis of ξ([27]). Given a basis $(\hat{v}_1, \cdots, \hat{v}_6)$ of $se(3)$, there are two ways to coordinatize $SE(3)$ around the identity. Let

$$g = e^{\zeta_1 \hat{v}_1 + \cdots + \zeta_6 \hat{v}_6}$$

then $\zeta = (\zeta_1, \cdots, \zeta_6)$ is called the *canonical coordinates of the first kind*. On the other hand, let

$$g = e^{\eta_1 \hat{v}_1} \cdots e^{\eta_6 \hat{v}_6}$$

then $\eta = (\eta_1, \cdots, \eta_6)$ is the *canonical coordinates of the second kind*. Conversion from ζ to η coordinates can be done via the Baker-Campbell-Hausdorff formula

$$e^{\hat{\xi}_1} e^{\hat{\xi}_2} = e^{\hat{\xi}_1 + \hat{\xi}_2 + \frac{1}{2}[\hat{\xi}_1, \hat{\xi}_2] + \cdots}$$

where for $\xi_1 = (v_1, \omega_1)$, $\xi_2 = (v_2, \omega_2)$, the Lie bracket operator

$$\begin{array}{rcl} se(3) \times se(3) & \to & se(3) \\ (\hat{\xi}_1, \hat{\xi}_2) & \mapsto & [\hat{\xi}_1, \hat{\xi}_2] = \hat{\xi}_1 \hat{\xi}_2 - \hat{\xi}_2 \hat{\xi}_1 \\ & & = (\omega_1 \times v_2 - \omega_2 \times v_1, \omega_1 \times \omega_2)^\wedge \end{array}$$

satisfies skew-symmetry and the Jacobi identity.

Denote by L_g and R_g, the left and the right translation map, respectively, and $I_g = L_g \circ R_{g^{-1}}$ the conjugation map. Let $g(t) \in SE(3), t \in (-\epsilon, \epsilon)$, be a trajectory of the rigid body, then

$$\begin{aligned} \hat{V}^b &= L_{g(t)^{-1}*} \cdot \dot{g}(t) = \begin{bmatrix} R^T \dot{R} & R^T \dot{p} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \end{aligned}$$

and $\hat{V}^s = R_{g^{-1}*} \cdot \dot{g}$ give, respectively, the body and the spatial velocities of the rigid body. These two velocities are related by

$$V^s = Ad_g V^b$$

where for $g = (p, R)$, the Adjoint map

$$Ad_g = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}$$

satisfies $(Ad_g)^{-1} = Ad_{g^{-1}}$, $Ad_{g_1 \cdot g_2} = Ad_{g_1} \cdot Ad_{g_2}$ and $Ad_e = I_{6\times 6}$, i.e, $Ad : SE(3) \to GL(6, \mathbb{R})$ is a group homomorphism.

Motion representation with respective to a coordinate frame displaced by g_0 is obtained from the original description via the conjugacy map

$$I_{g_0} : SE(3) \to SE(3) : h \mapsto g_0 h g_0^{-1} \quad (2)$$

Differentiating (2) with respective to h yields relations between velocities represented in different coordinate frames

$$V_2 = Ad_{g_0} \cdot V_1$$

Note that the pitch of a twist is invariant under change of coordinate frames, i.e, $\rho(\xi) = \rho(Ad_g \xi)$. One can also verify that

$$I_{g_0}(e^{\hat{\xi}}) = e^{(Ad_{g_0}\xi)^\wedge}$$

which means that screw motion about a changed axis $Ad_{g_0}\xi$ is the same as a screw motion viewed from a coordinate frame displaced by g_0.

B. Lie Subgroups of $SE(3)$

Constraint motions between two rigid bodies are often modelled by the level set $Q \triangleq C^{-1}(0)$ of a certain function:

$$C : SE(3) \to \mathbb{R}^m$$

The most general structure one can expect Q to have is that of a regular submanifold of $SE(3)$. However, when Q is also closed under the group operation, it is said to be a *Lie subgroup* of $SE(3)$. Examples of 1-dimensional Lie subgroups are that generated by the exponential map :

$$H = \{e^{\hat{\xi}\theta} \mid \theta \in \mathbb{R}\}$$

with $\hat{\xi} \in se(3)$. Other examples of higher dimensional Lie subgroups are the spatial translational subgroup

$$T(3) = \Psi(\mathbb{R}^3 \times \{0\}) = \left\{ \begin{bmatrix} I & p \\ 0 & 1 \end{bmatrix} \mid p \in \mathbb{R}^3 \right\}$$

and the rotational subgroup, also denoted as

$$SO(3) = \left\{ \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \mid R \in SO(3) \right\}$$

Let G be a Lie group. A subgroup H of G is a normal subgroup if $gHg^{-1} = H$ for all $g \in G$. The Lie algebra $\mathfrak{h}$ of H is an ideal of $\mathcal{G}$, i.e, $[\mathfrak{h}, \mathcal{G}] \subset \mathfrak{h}$. Apparently, $T(3)$ is a normal subgroup of $SE(3)$. Each subgroup H of G defines an equivalence relation on G. Two elements g_1 and g_2 of G are said to be equivalent, $g_1 \sim g_2$, if $g_1 = hg_2$ for some $h \in H$. The equivalent classes are called cosets, and the space of cosets for such a relation is called the *quotient space* of G by H, denoted G/H. G/H has the structure of a differentiable manifold, and is often referred to as a homogeneous space. When H is a normal subgroup, G/H becomes a Lie group. For example, with $G = SE(3)$ and $H = T(3)$, G/H can be identified with $SO(3)$.

Let H be a subgroup of $SE(3)$, then, the conjugate subgroup of H, denoted

$$I_g(H) = \{ghg^{-1} \mid h \in H\}$$

is also a subgroup. Two subgroups H_1 and H_2 are said to be equivalent if $\exists g \in SE(3)$ such that $H_1 = I_g(H_2)$. Clearly, if H_1 and H_2 are equivalent, then their Lie subalgebras are related by $\mathfrak{h}_1 = Ad_g(\mathfrak{h}_2)$.

Example *1:* Let $H = \{e^{\hat{\xi}\theta} \mid \theta \in [0, 2\pi]\}$ be a one-parameter subgroup of screw motions with a fixed $\hat{\xi} \in se(3)$. Then,

$$I_g(e^{\hat{\xi}\theta}) = e^{(Ad_{g_0}\xi)^\wedge \theta}$$

corresponds to screw motions about a displaced twist axis $(Ad_{g_0}\xi)^\wedge \in se(3)$. Since the pitch of a twist is Ad_g-invariant, we see that screw motions of the same pitch are equivalent. Let

$$T(z) = \left\{ \begin{bmatrix} I & \alpha z \\ 0 & 1 \end{bmatrix}, \alpha \in \mathbb{R} \right\}$$

then

$$I_{g_0}(T(z)) = \left\{ \begin{bmatrix} I & \alpha R_0 z \\ 0 & 1 \end{bmatrix}, \alpha \in \mathbb{R} \right\} \triangleq T(\mathbf{v}), \ \mathbf{v} = R_0 z$$

represents translations along direction $\mathbf{v}$. For

$$\mathcal{R}(0, z) = \left\{ \begin{bmatrix} e^{\hat{z}\theta} & 0 \\ 0 & 1 \end{bmatrix}, \theta \in [0, 2\pi] \right\},$$

$$\begin{aligned} I_{g_0}(\mathcal{R}(0, z)) &= \left\{ \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})p_0 \\ 0 & 1 \end{bmatrix}, \theta \in [0, 2\pi] \right\} \\ &\triangleq \mathcal{R}(p_0, \omega), \ \omega = R_0 z \end{aligned}$$

represents rotations about an axis with direction $\omega = R_0 z$, and a point p_0 on the axis.

As we will see, Lie subgroups of $SE(3)$ play an important role in kinematic studies of both serial and parallel manipulators. An interesting problem is to classify, up to a conjugacy class, all Lie subgroups of $SE(3)$. J.M. Selig [28] gave a classification directly at the group level, and Herve ([23], [24]) enumerated all subgroups of $SE(3)$. Using the

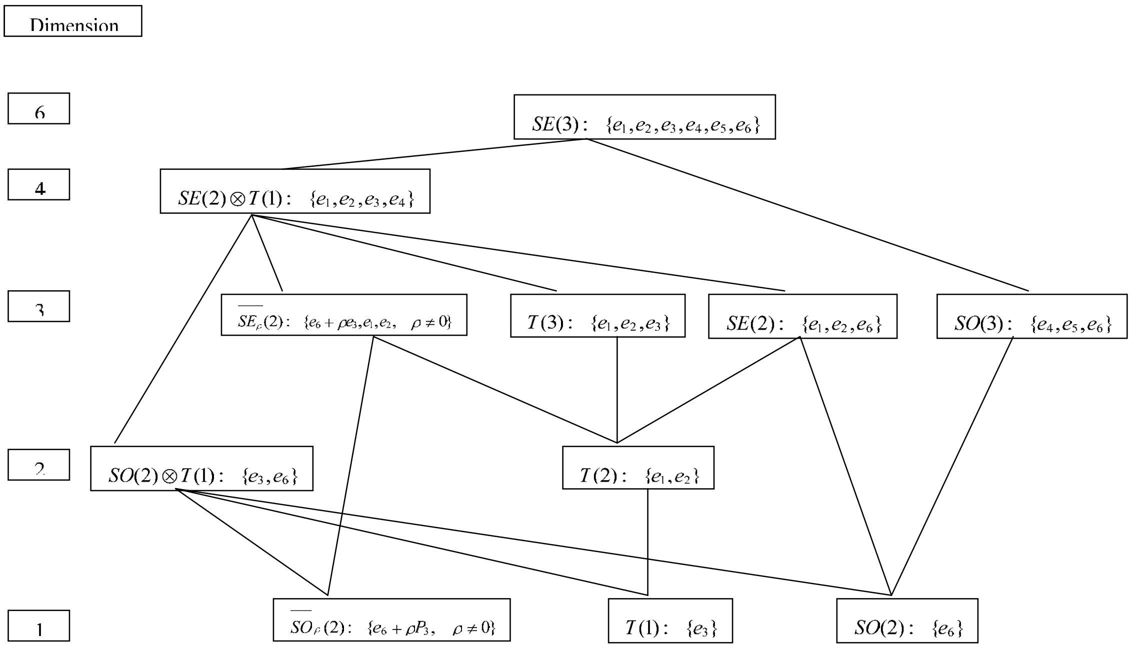

Fig. 1. A classification of Lie subalgebras and subgroups of $SE(3)$

Subgroup	Nominal Configuration	Conjugation Subgroup
$SO(2)$	$\mathcal{R}(0,z) = \left\{\begin{bmatrix} e^{\hat{z}\theta} & 0 \\ 0 & 1\end{bmatrix}, \theta\in[0,2\pi]\right\}$	$\mathcal{R}(p,\omega) = I_g(\mathcal{R}(0,z)) = \left\{\begin{bmatrix} e^{\hat{\omega}\theta} & (I-e^{\hat{\omega}\theta})p \\ 0 & 1\end{bmatrix}, \theta\in[0,2\pi]\right\}, \ \omega = Rz$
$T(1)$	$T(z) = \left\{\begin{bmatrix} I & \alpha z \\ 0 & 1\end{bmatrix}, \alpha\in\mathbb{R}\right\}$	$T(\mathbf{v}) = I_g(T(z)) = \left\{\begin{bmatrix} I & \alpha \mathbf{v} \\ 0 & 1\end{bmatrix}, \alpha\in\mathbb{R}\right\}, \ \mathbf{v} = Rz$
$\overline{SO}_\rho(2)$	$H_\rho(0,z) = \left\{\begin{bmatrix} e^{\hat{\omega}\theta} & \rho\omega\theta \\ 0 & 1\end{bmatrix}, \theta\in[0,2\pi]\right\}$	$H_\rho(p,\omega) = I_g(H_\rho(0,z)) = \left\{\begin{bmatrix} e^{\hat{\omega}\theta} & (I-e^{\hat{\omega}\theta})p+\rho\omega\theta \\ 0 & 1\end{bmatrix}, \theta\in[0,2\pi]\right\}, \ \omega = Rz$
$SO(2)\otimes T(1)$	$C(0,z) = \left\{\begin{bmatrix} e^{\hat{z}\theta} & \alpha z \\ 0 & 1\end{bmatrix}, \theta\in[0,2\pi], \alpha\in\mathbb{R}\right\}$	$C(p,\omega) = I_g(C(0,z)) = \left\{\begin{bmatrix} e^{\hat{\omega}\theta} & (I-e^{\hat{\omega}\theta})p+\alpha\omega \\ 0 & 1\end{bmatrix}, \theta\in[0,2\pi], \alpha\in\mathbb{R}\right\}, \ \omega = Rz$
$T(2)$	$T_2(z^n) = \left\{\begin{bmatrix} I & \alpha_1 x+\alpha_2 y \\ 0 & 1\end{bmatrix}, \alpha_1,\alpha_2\in\mathbb{R}\right\}$	$T_2(\omega^n) = I_g(T_2(z^n)) = \left\{\begin{bmatrix} I & \alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2 \\ 0 & 1\end{bmatrix}, \alpha_1,\alpha_2\in\mathbb{R}\right\}, \ \begin{matrix}\omega = Rz \\ \mathbf{v}_1,\mathbf{v}_2\perp\omega\end{matrix}$
$T(3)$	$T(3) = \left\{\begin{bmatrix} I & q \\ 0 & 1\end{bmatrix}, q\in\mathbb{R}^3\right\}$	$T(3) = I_g(T(3)) = \left\{\begin{bmatrix} I & q \\ 0 & 1\end{bmatrix}, q\in\mathbb{R}^3\right\}$
$SE(2)$	$PL(z^n) = \left\{\begin{bmatrix} e^{\hat{z}\theta} & \alpha_1 x+\alpha_2 y \\ 0 & 1\end{bmatrix}, \begin{matrix}\theta\in[0,2\pi], \\ \alpha_1,\alpha_2\in\mathbb{R}\end{matrix}\right\}$	$PL(\omega^n) = I_g(PL(z^n)) = \left\{\begin{bmatrix} e^{\hat{\omega}\theta} & \alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2 \\ 0 & 1\end{bmatrix}, \begin{matrix}\theta\in[0,2\pi], \\ \alpha_1,\alpha_2\in\mathbb{R}\end{matrix}\right\}, \ \begin{matrix}\omega = Rz \\ \mathbf{v}_1,\mathbf{v}_2\perp\omega\end{matrix}$
$SO(3)$	$S(0) = \left\{\begin{bmatrix} R' & 0 \\ 0 & 1\end{bmatrix}, R'\in SO(3)\right\}$	$S(p) = I_g(S(0)) = \left\{\begin{bmatrix} R' & (I-R')p \\ 0 & 1\end{bmatrix}, R'\in SO(3)\right\}$
$SE(2)\otimes T(1)$	$X(z) = \left\{\begin{bmatrix} e^{\hat{z}\theta} & q \\ 0 & 1\end{bmatrix}, \theta\in[0,2\pi], q\in\mathbb{R}^3\right\}$	$X(\omega) = I_g(X(z)) = \left\{\begin{bmatrix} e^{\hat{\omega}\theta} & q \\ 0 & 1\end{bmatrix}, \theta\in[0,2\pi], q\in\mathbb{R}^3\right\}, \ \omega = Rz$

TABLE I

Subgroups of $SE(3)$ in a nominal and a general configuration relation between Lie algebras and Lie groups through the exponential map, we follow [31] to provide here a complete classification of all Lie subgroups of $SE(3)$.

Let $\{e_i\}_{i=1}^6$ be the canonical basis of $\mathbb{R}^6$, with $e_i \in \mathbb{R}^6$ having an 1 in the i^{th} entry and 0 otherwise. Thus, $\{\hat{e}_i\}_{i=1}^6$ provides a basis of $se(3)$. Note that $\hat{e}_i$, $i = 1, \cdots, 3$ represents instantaneous translations about the three principal axes of the reference frame, and $\hat{e}_j$, $j = 4, \cdots, 6$ instantaneous rotations about these axes. Recall that the subalgebra $t(3) = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$, i.e., the span of $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$, forms an ideal of $se(3)$. This implies that the quotient $se(3)/t(3)$ is also a Lie algebra. To see this, let $\hat{\xi}, \hat{\eta} \in se(3)/t(3)$, and define the Lie bracket operation $[\hat{\xi}, \hat{\eta}]$ using that of $se(3)$. If $\hat{\xi}_1 = \hat{\xi} + \hat{t}_1$, $\hat{\eta}_1 = \hat{\eta} + \hat{s}_1$, with $\hat{t}_1, \hat{s}_1 \in t(3)$, then

$$[\hat{\xi}_1, \hat{\eta}_1] = [\hat{\xi} + \hat{t}_1, \hat{\eta} + \hat{s}_1] = [\hat{\xi}, \hat{\eta}] + t(3).$$

Thus, the Lie bracket operation on $se(3)/t(3)$ is well defined. It is not difficult to see that $se(3)/t(3)$ can be identified with $so(3) = \{\hat{e}_4, \hat{e}_5, \hat{e}_6\}$. Consider now the projection map

$$\pi : se(3) \rightarrow se(3)/t(3) \cong so(3). \tag{3}$$

It is a Lie algebra homomorphism. Thus, the preimages under π of all subalgebras of $so(3)$ reveal, up to a conjugacy class, all subalgebras of $se(3)$, and the exponentials of which provide a complete classification, up to a conjugacy class, of all connected Lie subgroups of $SE(3)$.

The subalgebras of $so(3)$ are easy to find and are given by $0 = \{0\}$, $so(2) = \{\hat{e}_6\}$, and $so(3) = \{\hat{e}_4, \hat{e}_5, \hat{e}_6\}$. $\pi^{-1}(0)$ consists of $\{0\}$, $t(1) = \{\hat{e}_3\}$, $t(2) = \{\hat{e}_1, \hat{e}_2\}$, and $t(3) = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$. On the other hand, $\pi^{-1}(so(2))$ consists of, up to a conjugacy class, the following elements:

$$\begin{array}{rcl} so(2) = \{\hat{e}_6\} & , & \overline{so}(2) = \{\hat{e}_6 + \rho\hat{e}_3 \mid \rho \neq 0\} \\ so(2) \otimes t(1) = \{\hat{e}_6, \hat{e}_3\} & , & se(2) = \{\hat{e}_1, \hat{e}_2, \hat{e}_6\} \\ se(2) \otimes t(1) & = & \{\hat{e}_6, \hat{e}_1, \hat{e}_2, \hat{e}_3\}. \end{array}$$

The preimage of $so(3)$ consists of simply $so(3)$ itself and the whole $se(3)$. Fig. 1 summarizes the various Lie subgroups of $SE(3)$ obtained by taking the exponentials of the above subalgebras. Note that the three 1-dimensional Lie subgroups, namely, $SO(2)$, $T(1)$ and $\overline{SO}_\rho(2)$ that represent, screw motions of zero, infinity, and non-zero finite pitches, provide a model for the relative displacements of two rigid bodies connected by a revolute, prismatic and helical joint of pitch ρ, respectively. These joints are collectively referred to as lower pairs by Reuleaux. Other higher dimensional lower pairs include the cylindrical subgroup $SO(2) \otimes T(1)$, the planar subgroup $SE(2)$ and the rotation subgroup $SO(3)$(for spherical joints). For practical reasons, we define the set of primitive generators to include all these lower pairs except the planar Euclidean subgroup.

Subgroups of dimensions equal or higher than 3 provide models for the configuration spaces of some common manipulators, e.g, $T(3)$, $SO(3)$, and $SE(2)\otimes T(1)$. The latter, which consists of three translations and rotation about a fixed axis, is also referred to as the Schoenflies group, denoted $X(\omega)$ with ω being the axis of rotation. The popular SCARA robot and the H_4 parallel manipulator both have $X(\omega)$ as their configuration spaces.

The descriptions of these subgroups in their nominal configurations, the corresponding conjugation subgroups and some of their associated names to be used in the sequal are shown in Table I.

Another problem to consider is the following : Given two Lie subgroups G_1 and G_2, determine if they are actually equal. This is in general difficult to do at the group level. The problem can be significantly simplified by checking, for

$$\mathcal{G}_i \triangleq T_e G_i = L_{g^{-1}*} T_g G_i, \quad \forall g \in G_i, \ i = 1, 2,$$

if

$$\mathcal{G}_1 = \mathcal{G}_2$$

as two subspaces of $se(3)$, and both sets contain the identity element.

Example *2:* As an illustration, we show that the intersection of the planar subgroup $PL(\omega^n)$ [1] about a plane with normal ω, and the set of rotations $S(N)$ about a point N is equal to the rotational subgroup about the axis ω and N being a point on the axis :

$$PL(\omega^n) \cap S(N) = \mathcal{R}(N, \omega) \tag{4}$$

A general representation of $PL(\omega^n)$ and $S(N)$, as seen from Table I, is given by

$$PL(\omega^n) = \left\{ \begin{bmatrix} e^{\hat{\omega}\theta} & \alpha_1 v_1 + \alpha_2 v_2 \\ 0 & 1 \end{bmatrix}, \theta \in [0, 2\pi], \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$

where $v_1, v_2 \perp \omega$, and

$$S(N) = \left\{ \begin{bmatrix} R & (I - R)N \\ 0 & 1 \end{bmatrix}, R \in SO(3) \right\}$$

Their Lie algebras are given by

$$pl(\omega^n) \quad = \quad \{\xi_1, \xi_2, \xi_3\}$$

with $\xi_1 = (v_1^T, \mathbf{0})^T$, $\xi_2 = (v_2^T, \mathbf{0})^T$ and $\xi_3 = (\mathbf{0}, \omega^T)^T$, and

$$s(N) \quad = \quad \{\eta_1, \eta_2, \eta_3\}$$

with $\eta_1 = ((N \times (1,0,0)^T)^T, (1,0,0)^T)^T$, $\eta_2 = ((N \times (0,1,0)^T)^T, (0,1,0)^T)^T$, and $\eta_3 = ((N \times (0,0,1)^T)^T, (0,0,1)^T)^T$. Since the Lie algebra of $PL(\omega^n) \cap S(N)$ is $pl(\omega^n) \cap s(N)$, and from the equations above, it is obvious that $pl(\omega^n) \cap s(N) = \{((N \times \omega)^T, \omega^T)^T\} = r(N, \omega)$, where $r(N, \omega)$ denotes the Lie algebra of $\mathcal{R}(N, \omega)$. Therefore, $PL(\omega^n) \cap S(N) = \mathcal{R}(N, \omega)$.

C. Submanifolds of $SE(3)$

In general, for the constraint space $Q \triangleq C^{-1}(0)$ to be a Lie subgroup is a rather strong requirement. Recall from Fig.1 that there is only one Lie subgroup of dimension 4, and no Lie subgroup of dimension 5 at all. In order to

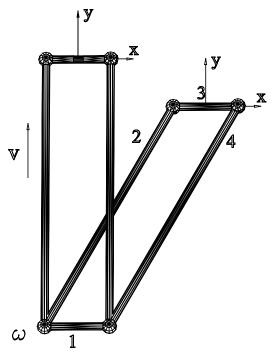

Fig. 2. Motion of the Parallelogram

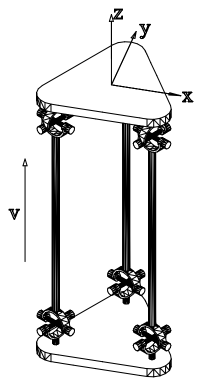

Fig. 3. U^*-joint

broaden our scope, we need to look at submanifolds, especially regular submanifolds of $SE(3)$.

The first example of a submanifold that comes into mind is the configuration space of a plane-hinged parallelogram that is widely utilized in the Delta manipulator design(See Fig.2). Let ω be the unit vector normal to the parallelogram, and $\mathbf{v}$ the vector of link 2 or 4 at the home position, then the set of displacements generated by the parallelogram has the form

$$P_a(\omega, \mathbf{v}) = \left\{ \begin{bmatrix} I & (e^{\hat{\omega}\theta} - I)\mathbf{v} \\ 0 & 1 \end{bmatrix} \mid \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \right\}$$

It's interesting to note that $P_a(\omega, \mathbf{v})$ is actually isomorphic to an open subset of S^1 in $\mathbb{R}^3$, and thus a 1-dimensional submanifold of $SE(3)$. A parallelogram produces linear motion along an open subset of the unit circle S^1.

A U-joint consists of two consecutive $\mathcal{R}$ joints with orthogonally intersecting axes. The configuration space of a U-joint with its axes initially aligned with the x and y axes is given by

$$\begin{aligned} U(0,x,y) &= \mathcal{R}(0,x)\cdot\mathcal{R}(0,y) \\ &= \left\{ \begin{bmatrix} e^{\hat{x}\alpha}e^{\hat{y}\beta} & 0 \\ 0 & 1 \end{bmatrix}, \alpha, \beta \in [0, 2\pi] \right\} \end{aligned}$$

which is a 2-dimensional submanifold of $SO(3)$ and thus $SE(3)$.

[1]The superscript n indicates that ω is a normal vector.

A generalization of the plane-hinged parallelogram to 2-dimension is the so-called U^*-joint introduced by [25] and [20]. The U^*-joint consists of three identical subchains symmetrically arranged in parallel as shown in Fig.3. Each subchain in turn consists of two U-joints, where the second and the third $\mathcal{R}$ joints and the first and the fourth $\mathcal{R}$ joints are, respectively, parallel to each other. The configuration space of the U^*-joint has the form

$$\begin{aligned} U^*(\omega_1, \omega_2, \mathbf{v}) &= \{ \begin{bmatrix} I & (e^{\hat{\omega}_1\alpha}e^{\hat{\omega}_2\beta} - I)\mathbf{v} \\ 0 & 1 \end{bmatrix}, \\ & \alpha \in (0, 2\pi), \beta \in (-\pi/2, \pi/2)\} \end{aligned}$$

where ω_1, ω_2 are the direction vectors of the first two axes of the subchain, and $\mathbf{v}$ is the direction vector of the link that connects the two U-joints of the subchain, all at the home position. For simplicity, we assume that $\omega_1 \perp \omega_2 \perp \mathbf{v}$. Note that $U^*(\omega_1, , \omega_2, \mathbf{v})$ is isomorphic to an open subset of S^2 in $\mathbb{R}^3$, and thus a 2-dimensional submanifold of $SE(3)$. The U^*-joint generates 2 degree-of-freedom (DoF) linear motions along the unit sphere S^2.

For practical reasons, we will include in the list of primitive generators the plane-hinged parallelogram and the U^*-joint.

Unlike Lie subgroups, there are in general no canonical expressions for submanifolds of $SE(3)$. However, there are two special families of regular submanifolds that are of particular importance in our study. These are summarized in the following Propositions.

Proposition *1:* Let H_1 and H_2 be two closed Lie subgroups of $SE(3)$, of dimension n_1 and n_2, respectively. Let $H = H_1 \cap H_2$, and $n = dim(H)$. Then, the product $H_1 \cdot H_2$ is a regular submanifold of $SE(3)$ of dimension $n_1 + n_2 - n$.
Proof: See appendix A.

As a regular submanifold, $H_1 \cdot H_2$ possesses preferred coordinate systems. For this, let $(\hat{\eta}_1, \cdots, \hat{\eta}_n)$ be a basis of $\mathfrak{h}$, and complete the basis so that $(\hat{\eta}_1, \cdots, \hat{\eta}_n, \hat{\eta}_{n+1}, \cdots, \hat{\eta}_{n_1})$ forms a basis of $\mathfrak{h}_1$, and $(\hat{\eta}_1, \cdots, \hat{\eta}_n, \hat{\eta}_{n_1+1}, \cdots, \hat{\eta}_{n_1+n_2-n})$ a basis of $\mathfrak{h}_2$. Then

$$(\theta_1, \cdots, \theta_{n_1+n_2-n}) \mapsto e^{\hat{\eta}_1\theta_1} \cdots e^{\hat{\eta}_{n_1+n_2-n}\theta_{n_1+n_2-n}} \in H_1 \cdot H_2$$

defines the required coordinate system.

If H_1 and H_2 commute, then $H_1 \cdot H_2$ is a Lie subgroup of $SE(3)$. For example, the cylindrical subgroup $SO(2)\otimes T(1)$ is the product of two commuting Lie subgroups

$$\left\{ \begin{bmatrix} e^{\hat{z}\alpha} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & \hat{z}\beta \\ 0 & 1 \end{bmatrix} \mid \alpha \in [0, 2\pi), \beta \in \mathbb{R} \right\}$$

and the Schoenflies subgroup is also the product of two commuting subgroups

$$X(z) = \left\{ \begin{bmatrix} e^{\hat{z}\alpha_1} & \alpha_2 x + \alpha_3 y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & \alpha_4 \hat{z} \\ 0 & 1 \end{bmatrix} \mid \begin{matrix} \alpha_1 \in [0, 2\pi), \\ \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R} \end{matrix} \right\}$$

Using Proposition 1, we see that the configuration space of the U-joint is the product of two 1-dimensional Lie subgroups and thus a 2-dimensional submanifold of $SO(3)$ and

DoF	Submanifolds	Conjugation	Description	Applications
5	$C_5^3 = T(2) \cdot SO(3) = T2(z^n) \cdot SO(3)$	$C_5^1(\omega^n, N) = T2(\omega^n) \cdot S(N)$	2-DoF translations parallel to the XY plane, and 3-DoF rotations	
5	$C_5^2 = T(3) \cdot U(0, x, y)$	$C_5^2(\omega_1, \omega_2) = T(3) \cdot U(q, \omega_1, \omega_2),\ \omega_1 \perp \omega_2$	3-DoF translations and 2-DoF rotations about x and y axis	5-axis machine center
4	$C_4^2 = T(2) \cdot U(0, z, x)$	$C_4^2(q, \omega_1^n, \omega_1, \omega_2) = T2(\omega_1^n) \cdot U(q, \omega_1, \omega_2), \omega_1 \perp \omega_2$	2-DoF translations parallel to the XY plane, and 2-DoF rotations about z and x-axis	Haptic devices
4	$C_4^3 = T(1) \cdot SO(3) = T(z) \cdot SO(3)$	$C_4^3(\omega, N) = T(\omega) \cdot S(N)$	1-DoF translations along z axis, and 3-DoF rotations	Flight simulator
3	$C_3^2 = T(1) \cdot U(0, z, x)$	$C_3^2(q, \omega_1, \omega_2) = T(\omega_1) \cdot U(q, \omega_1, \omega_2),\ \omega_1 \perp \omega_2$	1-DoF translations along z axis and 2-DoF rotations about z and x-axis	
3	$C_3^1 = T(2) \cdot R(0, x) = T2(z^n) \cdot \mathcal{R}(0, x)$	$C_3^4(q, \omega_1^n, \omega_2) = T2(\omega_1^n) \cdot \mathcal{R}(q, \omega_2),\ \omega_1 \perp \omega_2$	2-DoF translations parallel to the XY plane and 1-DoF rotations about x- axis	Hana manipulator

TABLE II
SUBMANIFOLDS OF $SE(3)$

hence $SE(3)$. On the other hand, the product $PL(\omega^n) \cdot S(N)$, as seen from (4), is a regular submanifold of dimension 5.

In general, the product of a regular submanifold with another or a Lie subgroup or the product of three or more subgroups may not always be a regular submanifold because of possible singularities.

Proposition *2:* Let M_1 and M_2 be, respectively, a regular submanifold of $T(3)$ and $SO(3)$, of dimension n_1 and n_2. Then, $M_1 \cdot M_2$ is a regular submanifold of $SE(3)$, of dimension $n_1 + n_2$.

Proof: Since $SE(3)$ is diffeomorphic to $T(3) \times SO(3)$, an element $m_1 \cdot m_2$ of $M_1 \cdot M_2$ corresponds directly to $(m_1, m_2) \in T(3) \times SO(3)$. Thus, $M_1 \cdot M_2$ corresponds with $M_1 \times M_2$, a $(n_1 + n_2)$-dimensional regular submanifold of $T(3) \times SO(3)$. Hence, $M_1 \cdot M_2$ is a $(n_1 + n_2)$-dimensional regular submanifold of $SE(3)$ by the equivalent manifold structure of $SE(3)$ and $T(3) \times SO(3)$.

Since the conjugation of a product is equal to the product of conjugations, i.e,

$$gM_1 \cdot M_2 g^{-1} = (gM_1 g^{-1}) \cdot (gM_2 g^{-1}),$$

it is possible, based on the above proposition, to derive many interesting submanifolds with dimensions ranging from 3 to 5. Table II shows the "canonical forms" of such a list. The submanifold C_5^2 [2] that consists of 3 translations and 2 rotations about two perpendicular axes, provides a model for the configuration space of a 5-axes machine tool ([32]). C_4^3, that consists of one translation along the z-axis and three rotations provides a model of a flight simulator([33]).

[2]The subscript denotes the total number of DoF and the superscript the rotational DoF.

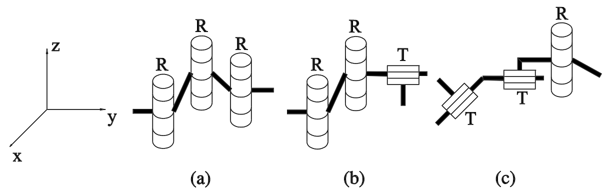

Fig. 4. Equivalent Generators of $SE(2)(PL(z^n))$

Finally, to conclude this section, we remark that the quotient space G/H for G a Lie subgroup of $SE(3)$ and H a subgroup of G is a differentiable manifold (of dimension $(dim(G) - dim(H))$), but not necessarily a submanifold of $SE(3)$. In certain cases, G/H can be identified with a submanifold of the latter, e.g., $SO(3)/R(0, z) \sim U(0, x, y)$.

III. KINEMATIC SYNTHESIS OF SERIAL SUBCHAINS

Using the mathematical tools developed in the previous section, we present in this section a formal theory for kinematic synthesis of serial manipulator subchains. This in turn will be applied in the following section to the synthesis of parallel manipulators.

We will be mainly concerned with manipulators (parallel or serial) having less than six degrees-of-freedom (or sub-6 DoF). For this reason, we restrict our attention to the synthesis of sub-6 DoF serial manipulator subchains.

The synthesis problem starts with a specification of the desired set Q of end-effector motions for the manipulators. We assume that Q is either a Lie subgroup as specified in Table I, or a regular submanifold of $SE(3)$, as specified in Table II, and $n = dim(Q)$. With a pre-specified list of primitive generators or allowable joints (e.g., those given in Table III), the problem amounts to finding all possible (serial) arrangements of the primitive generators so that the re-

DoF	Description/Notation	Remark
1	$T(\mathbf{v})$ (T)	Prismatic joint
	$\mathcal{R}(p,\omega)$ $(\mathcal{R})$	Revolute joint
	$H_\rho(p,\omega)$ (H)	Helical joint
	$P_a(\omega,\mathbf{v})$ (P_a)	Parallelogram
2	$C(p,\omega)$ (C)	Cylindrical joint
	$U^*(\omega_1,\omega_2,\mathbf{v})$ (U^*)	U^*-joint
3	$S(N)$ (S)	Spherical joint

TABLE III
Primitive generators

sulting manipulators have the desired end-effector motions. To illustrate this, we consider synthesis of a $SE(2)$ manipulator using two primitive generators $T(\mathbf{v})$ and $\mathcal{R}(p,\omega)$. We let

$$M = G_1 \cdot G_2 \cdot G_3 \qquad (5)$$

be a manipulator obtained by cascading in series three primitive generators $G_i, i = 1, \cdots, 3$, since $dim(SE(2)) = 3$. Associated with (5) is its forward kinematic map

$$F : \Theta \to SE(3) : \theta \mapsto F(\theta) \qquad (6)$$

Without loss of generality, we assume that $F(0) = e$. The product in (6) may or may not be a submanifold. This is, however, not important. What is important is *if M contains an open neighborhood of e in $SE(2)$*. In this case, we say that M is a *mechanical generator* of $SE(2)$.

By an abuse of notation, we also denote by $\mathcal{G}_i$ the twist representation of the primitive generator G_i, which corresponds to a basis of its Lie algebra when G_i is a Lie subgroup, or to $((\omega \times v)^T, 0)$ when G_i is $P_a(\omega)$ and to $((\omega_1 \times v)^T, 0)$ and $(\omega_2 \times v)^T, 0)$ when G_i is $U^*(\omega_1, \omega_2, v)$. Differentiating (5) at the identity or (6) at $\theta = 0$, we see that M is a mechanical generator of $SE(2)$ if and only if the span of $\mathcal{G}_i$, $i = 1, \cdots, 3$, equals to $se(2)$, i.e,

$$span(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = se(2) \qquad (7)$$

Note that the order where the primitive generators appear in the serial arrangement is immaterial. Fig.4 shows all possible configurations of the $SE(2)$ or $PL(\omega^n)$ generators.

Generalizing from this example, we let $M = G_1 \cdot G_2 \cdots G_l$, and define $DoF(M)$ to be that of its joint space Θ.

Definition *1:* M is a ***mechanical generator*** of Q, if $DoF(M) = n$ and M contains an open neighborhood of e in Q. Similarly, we say that the (serial) cascading arrangement of k manipulators $M_1, \cdots, M_k$, denoted $M = M_1 \cdot M_2 \cdot \cdots \cdot M_k$, is a mechanical generator of Q if $DoF(M) = n$, and M contains an open neighborhood of e in Q.

Note that this definition excludes redundant manipulators from consideration, since $DoF(M) = n$ ensures the resulting manipulator to have the minimum number primitive joints. However, a slight modification of this condition could handle the latter case.

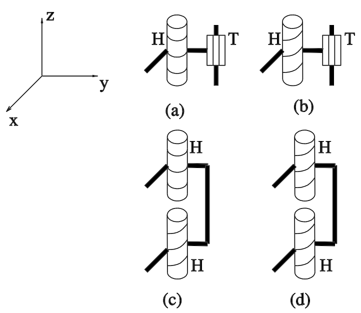

Fig. 5. Equivalent Generators of $C(0, z)$

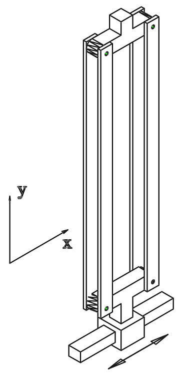

Fig. 6. A Generator of $T(2)$

Problem *1:* **Kinematic Synthesis Problem**
Given $Q \subset SE(3)$, and a list of primitive generators, find all mechanical generators of Q.

We separate the problem into the case of Q being a Lie subgroup, and a regular submanifold.

A. Subgroup Motion Generators

Let $\overline{M}$ be the ordered set of twists of its (primitive) generators, and $span(\overline{M})$ the subspace spanned by the elements of $\overline{M}$.

Proposition *3:* M is a mechanical generator of a Lie subgroup Q if and only if $DoF(M) = n$ and $span(\overline{M}) = T_eQ$.

The synthesis of some of the Lie subgroups given in Table I are discussed in the following.

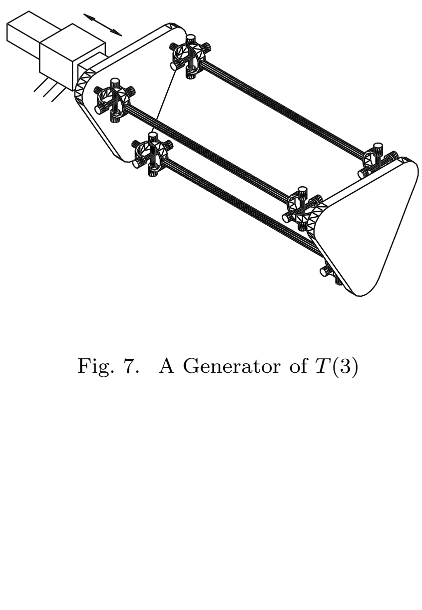

Fig. 7. A Generator of $T(3)$

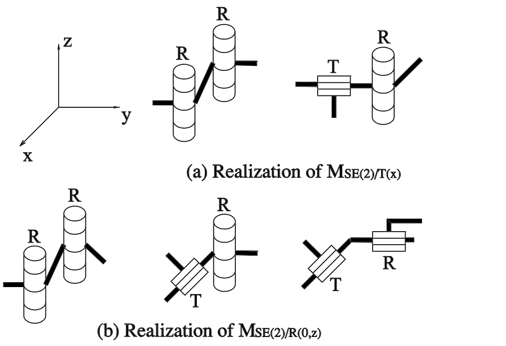

Fig. 8. Realizations of Quotient Manipulaters of $SE(2)$

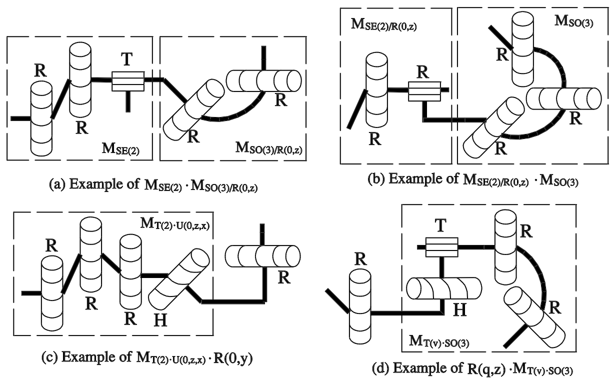

Fig. 9. Realizations of Product Submanfold $T(2)\cdot SO(3)$

1. Synthesis of $C(0,z)$ or $C(p,\omega)$ subchain

The Lie algebra of $C(0,z)$ is the subspace spanned by $(\mathbf{z}^T,0)^T$ and $(0,\mathbf{z}^T)^T$. Thus, by Proposition 3, all the serial mechanisms in Fig. 5 generate $C(0,z)$. Also, the order of the prismatic, revolute and helical joints in Fig.5 can be arranged freely.

If the direction of the revolute or helical joints in Fig.5 is replaced by ω, with p being a point on the axis, then $C(p,\omega)$ will be generated. Again, the order of the constituting pairs can be changed freely.

2. Synthesis of $T(2)$ and $T(3)$ subchains

It is easy to see that cascading two prismatic pairs, or a prismatic pair with a parallelogram, as in Fig.6, generates $T(2)$. Similarly, cascading three prismatic pairs, or a prismatic pair with a U^*-pair, as in Fig.7, generates $T(3)$. The order in which the constituting pairs appear is immaterial.

3. Synthesis of $X(z)$ or $X(\omega)$ subchain

Since

$$X(z)=T(3)\cdot\mathcal{R}(p,z)=PL(z^n)\cdot T(1)=T_2(z^n)\cdot C(p,z),$$

the most direct way to synthesis a $X(z)$ subchain is by cascading in series a $T(3)$ chain with a revolute joint $\mathcal{R}(p,z)$, a $PL(z^n)$ chain with a prismatic joint $T(1)$, or a $T_2(z^n)$ chain with a $C(p,z)$ chain. Herve and Sparacino [23] gave a complete list of combinations of 1-DoF lower pairs that generate $X(z)$, as shown in Table IV, where the order of the primitive joints can be freely exchanged.

Observe also that any of the prismatic pair T in Table IV for $X(z)$ can be replaced by the parallelogram P_a without affecting the result. For example, replacing the T by P_a in the combination $\mathcal{R}\cdot\mathcal{R}\cdot T\cdot\mathcal{R}$ gives rise to the subchain of the Delta manipulator.

Replacing the direction of the revolute and the helical joints from z to ω in Table IV generates the $X(\omega)$ chain.

We summarize the various subgroup motion generators in Table IV.

B. Submanifold Motion Generators

First, we consider motion generators for a submanifold Q given by $Q=N_1\cdot N_2$, with N_1 a submanifold of $T(3)$ and N_2 a submanifold of $SO(3)$.

Proposition *4: Given a desired submanifold $Q=N_1\cdot N_2$, with $N_1\subset T(3)$ and $N_2\subset SO(3)$. Suppose that M_1 generates N_1 and M_2 generates N_2, then $M=M_1\cdot M_2$ generates $Q=N_1\cdot N_2$.*

Proof. Since M_1 contains an open neighborhood U of e, M_2 an open neighborhood V of e, and $U\times V$ is an open neighborhood of (e,e) of $N_1\times N_2$, $U\cdot V$ is an open neighborhood of e of $N_1\cdot N_2$ because of the equivalent manifold structures of $N_1\times N_2$ with $N_1\cdot N_2$. Thus M contains an open neighborhood of e in $N_1\cdot N_2$. Since $DoF(M)=DoF(M_1)+DoF(M_2)=dim(N_1)+dim(N_2)=dim(Q)$, we conclude that M generates Q.

Next, we consider motion generators for a submanifold Q of the form $Q=H_1\cdot H_2$, with H_1 and H_2 Lie subgroups of $SE(3)$. Let G be a Lie subgroup of $SE(3)$ and H a Lie subgroup of G. Denote by M_H the mechanical generator of H, and $M_{G/H}$ the *quotient manipulator* such that $M_{G/H}\cdot M_H$ generates G, i.e, if $span(\overline{\mathcal{M}}_H)=\mathfrak{h}$, then

$$span(\overline{\mathcal{M}}_{G/H})\oplus span(\overline{M}_H)=\mathcal{G}.$$

Some examples of quotient manipulators of $SE(2)$ are shown in Fig.8.

Proposition *5: Given a desired regular submanifold of the form $Q=H_1\cdot H_2$, with H_1 and H_2 Lie subgroups of $SE(3)$. Let $H=H_1\cap H_2$, and assume that M_1 generates H_1 and M_2 generates H_2. Then, (i) if $H=\{e\}$, $M=M_1\cdot M_2$ generates Q; (ii) if $H>\{e\}$, $M=M_{H_1/H}\cdot M_2$ and $N=M_1\cdot M_{H_2/H}$ generate Q.*

Proof. We only show case (ii) as (i) follows as a special case. Assume that $dim(H_1)=m_1$, $dim(H_2)=m_2$ and $dim(H)=n$. Also assume that $\overline{\mathcal{M}}_1=(\zeta_1,\ldots,\zeta_{m_1})$, $\overline{\mathcal{M}}_2=(\eta_1,\ldots,\eta_{m_2})$, $\overline{\mathcal{M}}_{H_1/H}=(\delta_1,\cdots,\delta_{m_1-n})$, $\overline{M}_{H_2/H}=(\gamma_1,\cdots,\gamma_{m_2-n})$. Thus, the twist representation of $M=M_{H_1/H}\cdot M_{H_2}$ is given by

$$\overline{M}=(\delta_1,\cdots,\delta_{m_1-n},\eta_1,\ldots,\eta_{m_2})$$

and the forward kinematic map of the associated manipulator has the form

$$\begin{aligned} f_1 \quad &: \mathbb{R}^{m_1+m_2-n}\to SE(3)\\ &: (\theta_1,\cdots,\theta_{m_1+m_2-n})\mapsto \quad e^{\delta_1\theta_1}\cdots e^{\delta_{m_1-n}\theta_{m_1-n}}\\ &\qquad\qquad \cdot e^{\eta_1\theta_{m_1-n+1}}\cdots e^{\eta_{m_2}\theta_{m_1-n+m_2}}\end{aligned}$$

DoF	Subgroup	Representative generators
4	$X(z)$	$\mathcal{R}(q_1,z)\cdot\mathcal{R}(q_2,z)\cdot T(\mathbf{u})\cdot T(\mathbf{v})$, $\mathcal{R}(q_1,z)\cdot H_\rho(q_2,z)\cdot T(\mathbf{u})\cdot T(\mathbf{v})$, $H_{\rho_1}(q_1,z)\cdot H_{\rho_2}(q_2,z)\cdot T(\mathbf{u})\cdot T(\mathbf{v})$, $H_{\rho_1}(q_1,z)\cdot H_{\rho_2}(q_2,z)\cdot H_{\rho_3}(q_3,z)\cdot T(\mathbf{u})$, $\mathcal{R}(q_1,z)\cdot\mathcal{R}(q_2,z)\cdot\mathcal{R}(q_3,z)\cdot T(\mathbf{u})$, $\mathcal{R}(q_1,z)\cdot\mathcal{R}(q_2,z)\cdot H_\rho(q_3,z)\cdot T(\mathbf{u})$, $\mathcal{R}(q_1,z)\cdot H_{\rho_1}(q_2,z)\cdot H_{\rho_2}(q_3,z)\cdot T(\mathbf{u})$, $H_{\rho_1}(q_1,z)\cdot H_{\rho_2}(q_2,z)\cdot H_{\rho_3}(q_3,z)\cdot H_{\rho_4}(q_4,z)$, $H_{\rho_1}(q_1,z)\cdot H_{\rho_2}(q_2,z)\cdot H_{\rho_3}(q_3,z)\cdot\mathcal{R}(q_4,z)$, $H_{\rho_1}(q_1,z)\cdot H_{\rho_2}(q_2,z)\cdot\mathcal{R}(q_3,z)\cdot\mathcal{R}(q_4,z)$, $H_\rho(q_1,z)\cdot\mathcal{R}(q_2,z)\cdot\mathcal{R}(q_3,z)\cdot\mathcal{R}(q_4,z)$
3	$T(3)$	$T(\mathbf{u})\cdot T(\mathbf{v})\cdot T(\mathbf{w})$, $T(\mathbf{u})\cdot P_a(z,\mathbf{v})\cdot T(\mathbf{w})$, $T(\mathbf{u})\cdot U^*(\omega_1,\omega_2,\mathbf{v})$
3	$SE(2)$	$T(\mathbf{u})\cdot T(\mathbf{v})\cdot\mathcal{R}(q,z)$, $T(\mathbf{u})\cdot\mathcal{R}(q_1,z)\cdot\mathcal{R}(q_2,z)$, $\mathcal{R}(q_1,z)\cdot\mathcal{R}(q_2,z)\cdot\mathcal{R}(q_3,z)$
3	$SO(3)$	$\mathcal{R}(0,\mathbf{u})\cdot\mathcal{R}(0,\mathbf{v})\cdot\mathcal{R}(0,\mathbf{w})$
2	$C(0,z)$	$T(z)\cdot\mathcal{R}(0,z)$, $T(z)\cdot H_\rho(0,z)$, $H_\rho(0,z)\cdot\mathcal{R}(0,z)$, $H_{\rho_1}(0,z)\cdot H_{\rho_2}(0,z)$
2	$T(2)$	$T(\mathbf{u})\cdot T(\mathbf{v})$, $T(\mathbf{u})\cdot P_a(z,\mathbf{v})$

TABLE IV

Equivalent generators for Lie subgroups of $SE(3)$

Since $e^{\delta_i\theta_i}$ and $e^{\eta_j\theta_{m_1-n+j}} \in H_2$, by the product closure, the image of f_1 lies in Q. As Q is a regular submanifold of $SE(3)$, by [29], f_1 defines a smooth map into Q

$$\hat{f}_1 \quad : \mathbb{R}^{m_1+m_2-n} \to Q$$
$$: (\theta_1,\cdots,\theta_{m_1+m_2-n}) \mapsto \quad e^{\delta_1\theta_1}\cdots e^{\delta_{m_1-n}\theta_{m_1-n}} \cdot e^{\eta_1\theta_{m_1-n+1}}\cdots e^{\eta_{m_2}\theta_{m_1-n+m_2}}$$

Since the twists in M are linearly independent, we have that $span(\overline{M}) = T_eQ$ and $\hat{f}_{1*0}$ is an isomorphism. Thus by the inverse function theorem, $\hat{f}_1$ is a local diffeomorphism at 0. Hence, its image contains an open neighborhood of e in Q. Furthermore, as $DoF(\mathcal{M}) = m_1+m_2-n = dim(Q)$, we see that M generates Q.

A similar argument shows that N also generates Q.

Propositions 5 and 4 provide us with a set of tools needed for finding the mechanical generators of these special submanifolds. These tools can be applied separately or in combination when a submanifold is expressed in its *canonical form* or its conjugation form. Note that the quotient manipulator is in general not unique, and the order of its primitive joints can be changed freely, too. Together with the abundance of subgroup motion generators, they combine to exhibit the diversity of the problem.

We now illustrate the application of these tools by finding some representative generators for some of the submanifolds given in Table II.

1. Synthesis of $T(2)\cdot U(0,z,x)$ *subchain*

$M_1 = T(\mathbf{u})\cdot T(\mathbf{v})$, where $\mathbf{u},\mathbf{v}$ are linearly independent and parallel to the $x-y$ plane, generates $T(2)$, and $M_2 = R(0,z)\cdot R(0,x)$ generates $U(0,z,x)$. According to Proposition 4, $M = M_1\cdot M_2$ generates $T(2)\cdot U(0,z,x)$. The twist representation for M is $\overline{M} = (\overline{T}(\mathbf{u}),\overline{T}(\mathbf{v}),\overline{\mathcal{R}}(0,z),\overline{\mathcal{R}}(0,x))$. Since $span(\overline{T}(\mathbf{u}),\overline{T}(\mathbf{v}),\overline{\mathcal{R}}(0,z)) = se(2)$, the Lie algebra of $SE(2)$, so these three twists form a generator of $SE(2)$ and can be equivalently replaced by any set of generators of $se(2)$ as shown in Fig.4. For instance, $M = R(q_1,z)\cdot R(q_2,z)\cdot R(q_3,z)\cdot R(0,x)$ gives another generator of $T(2)\cdot U(0,z,x)$. Furthermore, since $R(0,x) = M_{C(0,x)/T(x)}$ as $span(\overline{\mathcal{R}}(0,x))\oplus span(\overline{T}(x))$ equals to the Lie algebra of $C(0,x)$, we see that $M = M_{SE(2)}\cdot M_{C(0,x)/T(x)}$ gives another mechanical generator of $T(2)\cdot U(0,z,x)$. By Proposition 5, $N = M_{SE(2)/T(x)}\cdot M_{C(0,x)}$ is also an equivalent generator of the submanifold. While $M_{C(0,x)}$ can be replaced by any equivalent $C(0,x)$ generators listed in Fig.5, there are also more than one candidates for $M_{C(0,x)/T(x)}$ and $M_{SE(2)/T(x)}$. For example, $H_\rho(0,x)$ can replace $M_{C(0,x)/T(x)}$ in M instead of $R(0,x)$. We summarize all the new submanifold generators discussed here in Table VI and quotient manipulators in Table V, where realizations for $M_{SE(2)}$ and $M_{C(0,x)}$ can be found in the previous section.

Quotient Manipulator	Reprensentative generators
$M_{C(0,x)/T(x)}$	$\mathcal{R}(0,x)$, $H_\rho(0,x)$
$M_{SE(2)/T(x)}$	$T(\mathbf{u})\cdot\mathcal{R}(q,z)$, $\mathcal{R}(q_1,z)\cdot\mathcal{R}(q_2,z)$
$M_{C(0,z)/R(0,z)}$	$T(z)$, $H_\rho(0,z)$
$M_{SO(3)/R(0,z)}$	$R(0,\mathbf{u})\cdot R(0,\mathbf{v})$
$M_{SE(2)/\mathcal{R}(0,z)}$	$T(\mathbf{u})\cdot T(\mathbf{v})$, $\mathcal{R}(q,z)\cdot T(\mathbf{v})$, $\mathcal{R}(q_1,z)\cdot\mathcal{R}(q_2,z)$
$M_{X(x)/T(3)}$	$\mathcal{R}(q,x)$, $H(q,x,\rho)$
$M_{X(x)/T_2(y^n)}$	$\mathcal{R}(q_1,x)\cdot\mathcal{R}(q_2,x)$, $\mathcal{R}(q_1,x)\cdot H_\rho(q_2,x)$, $H_{\rho_1}(q_1,x)\cdot H_{\rho_2}(q_2,x)$
$M_{PL(y^n)/T_2(y^n)}$	$\mathcal{R}(q,y)$
$M_{X(x)/T(y)}$	$\mathcal{R}(q_1,x)\cdot\mathcal{R}(q_2,x)\cdot T(\mathbf{v})$, $\mathcal{R}(q_1,x)\cdot H_\rho(q_2,x)\cdot T(\mathbf{v})$, $H_{\rho_1}(q_1,x)\cdot H_{\rho_2}(q_2,x)\cdot T(\mathbf{v})$, $H_{\rho_1}(q_1,x)\cdot H_{\rho_2}(q_2,x)\cdot H_{\rho_3}(q_3,x)$, $\mathcal{R}(q_1,x)\cdot\mathcal{R}(q_2,x)\cdot\mathcal{R}(q_3,x)$, $\mathcal{R}(q_1,x)\cdot\mathcal{R}(q_2,x)\cdot H_\rho(q_3,x)$, $\mathcal{R}(q_1,x)\cdot H_{\rho_1}(q_2,x)\cdot H_{\rho_2}(q_3,x)$,

TABLE V

Representative realizations for some quotient manipulators

Remark *1:* M and N are generators for both $T(2)\cdot U(0,z,x)$ and $SE(2)\cdot C(0,x)$. This does not mean that $T(2)\cdot U(0,z,x)$ and $SE(2)\cdot C(0,x)$ are necessarily equal

Submanifold	Reprensentative generators
$M_{T(2)\cdot U(0,z,x)}$ $= M_{SE(2)\cdot C(0,x)}$	$M_{SE(2)} \cdot M_{C(0,x)/T(x)}$, $M_{SE(2)/T(x)} \cdot M_{C(0,x)}$
$M_{T(1)\cdot SO(3)}$ $= M_{C(0,z)\cdot SO(3)}$	$M_{C(0,z)/R(0,z)} \cdot M_{SO(3)}$, $M_{C(0,z)} \cdot M_{SO(3)/R(0,z)}$
$M_{T(2)\cdot SO(3)}$ $= M_{SE(2)\cdot SO(3)}$	$M_{SE(2)} \cdot M_{SO(3)/\mathcal{R}(0,z)}$, $M_{SE(2)/\mathcal{R}(0,z)} \cdot M_{SO(3)}$, $M_{T(2)\cdot U(0,z,x)} \cdot \mathcal{R}(0,y)$, $T(\mathbf{u}) \cdot M_{T(\mathbf{v})\cdot SO(3)}$, $\mathcal{R}(q,z) \cdot M_{T(\mathbf{v})\cdot SO(3)}$
$M_{T(3)\cdot U(0,x,y)}$ $= M_{X(x)\cdot X(y)}$ $= M_{PL(x^n)\cdot PL(y^n)}$ $= M_{X(x)\cdot PL(y^n)}$ $= M_{X(x)\cdot C(q,y)}$	$M_{T(3)} \cdot M_{U(0,x,y)}$, $M_{X(x)} \cdot M_{X(y)/T(3)}$, $M_{X(x)/T(3)} \cdot M_{X(y)}$, $M_{PL(x^n)/T(z)} \cdot M_{PL(y^n)}$, $M_{PL(x^n)} \cdot M_{PL(y^n)/T(z)}$, $M_{X(x)/T_2(y^n)} \cdot M_{PL(y^n)}$, $M_{X(x)} \cdot M_{PL(y^n)/T_2(y^n)}$, $M_{X(x)/T(y)} \cdot M_{C(q,y)}$, $M_{X(x)} \cdot M_{C(q,y)/T(y)}$
$M_{S(N)\cdot SO(3)}$	$M_{S(N)} \cdot M_{SO(3)/\mathcal{R}(0,t)}$, $M_{S(N)/\mathcal{R}(0,t)} \cdot M_{SO(3)}$

TABLE VI
Equivalent generators for some product submanifolds

globally.[3] It only means that they are equal in a small neighborhood of e.

2. Synthesis of $T(1) \cdot SO(3)$ subchain

$M_1 = T(z)$ generates $T(1)$ and $M_2 = R(0,\mathbf{u}) \cdot R(0,\mathbf{v}) \cdot R(0,\mathbf{w})$, where $\mathbf{u},\mathbf{v},\mathbf{w}$ are linearly independent, generates $SO(3)$. By Proposition 4, $M = M_1 \cdot M_2$ generates $T(1) \cdot SO(3)$. Furthermore, it is easy to verify that

$$\begin{aligned} M &= M_{C(0,z)/R(0,z)} \cdot M_{SO(3)} \\ &= M_{C(0,z)} \cdot M_{SO(3)/R(0,z)} \end{aligned}$$

and the realization of the various quotients are given in Table V.

3. Synthesis of $T(2) \cdot SO(3)$ subchain

With $M_1 = T(\mathbf{u}) \cdot T(\mathbf{v})$, where $\mathbf{u},\mathbf{v}$ are linearly independent and parallel to the $x-y$ plane, and $M_2 = R(0,z) \cdot R(0,x) \cdot R(0,y)$, $M = M_1 \cdot M_2$ provides a realization of $T(2) \cdot SO(3)$. Additional realizations of the submanifold are given by

$$\begin{aligned} M &= M_{SE(2)} \cdot M_{SO(3)/R(0,z)} \\ &= M_{SE(2)/R(0,z)} \cdot M_{SO(3)} \\ &= M_{T(2)\cdot U(0,z,x)} \cdot R(0,y) \\ &= T(u) \cdot M_{T(v)\cdot SO(3)} \\ &= R(q,z) \cdot M_{T(v)\cdot SO(3)} \end{aligned}$$

where realizations for the various factors contained in the above equations can be either found from the previous discussions or obtained by referring to Table IV and Table V.

[3]Although in this special case, through algebraic computation, $T(2) \cdot U(0,z,x) = SE(2) \cdot C(0,x)$, but it does not necessarily hold in general.

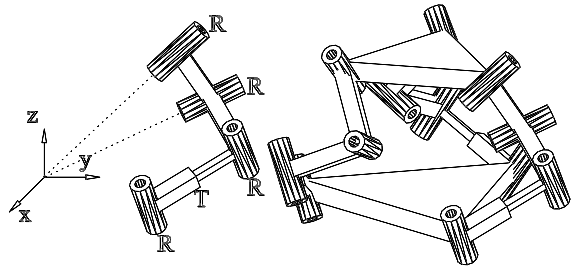

Fig. 10. Example of $SO(3)$ PM Motion Generator with $H_j = PL(\omega_j^n)$

Here we have covered Herve's work [24] into our framework. Some examples are shown in Fig.9.

4. Synthesis of $T(3) \cdot U(0,x,y)$ subchain

With $M_1 = T(x) \cdot T(y) \cdot T(z)$ and $M_2 = R(0,x) \cdot R(0,y)$, $M = M_1 \cdot M_2$ generates $T(3) \cdot U(0,x,y)$, with its twist representation given by

$$\overline{M} = (\overline{T}(x), \overline{T}(y), \overline{T}(z), \overline{\mathcal{R}}(0,x), \overline{\mathcal{R}}(0,y))$$

Since $span(\overline{T}(x), \overline{T}(y), \overline{T}(y), \overline{\mathcal{R}}(0,x))$ agrees with the Lie algebra of $X(x)$, and $R(0,y) = X(y)/T(3)$, we have the following equivalent realizations of M,

$$\begin{aligned} M &= M_{T(3)} \cdot M_{U(0,x,y)} \\ &= M_{X(x)} \cdot M_{X(y)/T(3)} \\ &= M_{X(x)/T(3)} \cdot M_{X(y)} \end{aligned}$$

Reordering the twist elements of $\overline{M}$ and applying Proposition 5 yield additional realizations of the submanifold as follows,

$$\begin{aligned} M &= M_{PL(x^n)/T(z)} \cdot M_{PL(y^n)} \\ &= M_{PL(x^n)} \cdot M_{PL(y^n)/T(z)} \\ &= M_{X(x)/T_2(y^n)} \cdot M_{PL(y^n)} \\ &= M_{X(x)} \cdot M_{PL(y^n)/T_2(y^n)} \\ &= M_{X(x)/T(y)} \cdot M_{C(q,y)} \\ &= M_{X(x)} \cdot M_{C(q,y)/T(y)} \end{aligned}$$

Again, realization of the various quotients are given in Table V, and the subgroup generators can be found in Table IV, with slight modifications when necessary.

IV. Synthesis of Parallel Manipulators

Synthesis of a parallel manipulator (PM) also starts with a set Q of desired end-effector motions, which we assume will be in the form of either a Lie subgroup, as in Table I, or a regular submanifold of $SE(3)$, as in Table II. We also assume that the manipulator under consideration will be arranged to have a fully parallel configuration, with the k-subchains $M_1, \cdots M_k$ all sharing a common base and a common end-effector. Denote by $C_{M_i}, i = 1, \cdots k$ the set of end-effector motions that M_i implements (at least locally), and by $M = M_1 \| \cdots \| M_k$ the parallel arrangement of the

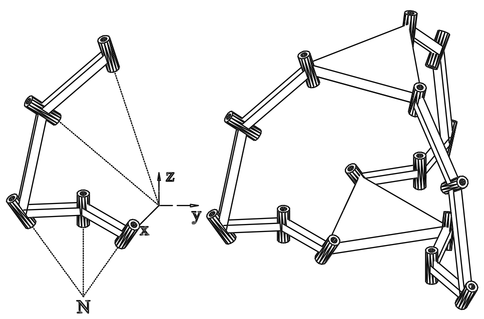

Fig. 11. Example of $SO(3)$ PM Motion Generator with $H_j = S(N_j)$

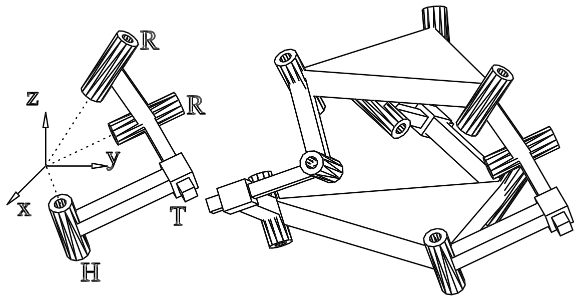

Fig. 12. Example of $SO(3)$ PM Motion Generator with $H_j = C(0, \omega_j)$

$M_i's$. Conceptually, the set of end-effector motions of the resulting PM is given by

$$C_M = C_{M_1} \cap C_{M_2} \cap \cdots \cap C_{M_k} \tag{8}$$

and the set of permissible velocities at each $g \in C_M$ by

$$R_{g^{-1}*}T_g C_M = R_{g^{-1}*}T_g C_{M_1} \cap \cdots \cap R_{g^{-1}*}T_g C_{M_k}. \tag{9}$$

In general, C_M is a very complex subset of SE(3), due to the complex singularities of the PM. If, however, C_M agrees with Q in a neighborhood of e, then we say that M is a *parallel motion (PM) generator* of Q. More precisely, we have

Definition *2:* Given a desired set Q of end-effector motions, with $n = dim(Q)$, a PM $M = M_1 \| \cdots \| M_k$ is a PM generator of Q if there exists an open neighborhood U of e in SE(3) such that $C_M \cap U = Q \cap U$.

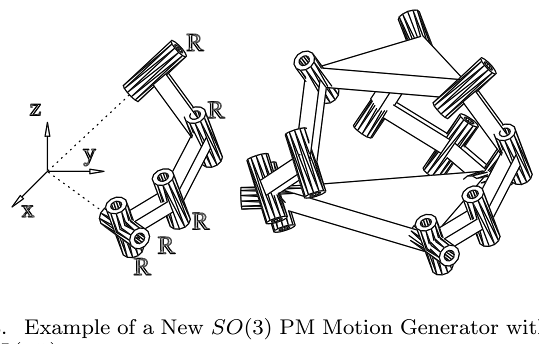

Fig. 13. Example of a New $SO(3)$ PM Motion Generator with $H_j = PL(\omega_j^n)$

In order to place only one actuator at the base of each subchain, we will assume that $k = n$. We will also assume, without loss of generality, that all the subchains are identical and thus the $C'_{M_i}s$ differ from each other by only a conjugation.

Because of the complexity of the algebraic set C_M (as well as any attempt to compute it), the PM synthesis problem is in general difficult to solve. This, however, can be substantially simplified with the following proposition, which is a special case of a more general theorem proved in Appendix B.

Proposition *6:* Given $Q \subset SE(3), k = dim(Q)$. Assume that $M = M_1 \| \cdots \| M_k$, and all of the $C_{M_j}, j = 1, \cdots k$ contains a connected open subset Q_U of Q around e,

$$Q_U \subseteq C_{M_j}, j = 1, \cdots, k \tag{10}$$

and consequently $Q_U \subseteq C_M$. If the following condition

$$T_e Q = span(\overline{M}_1) \cap \cdots \cap span(\overline{M}_k) \tag{11}$$

or the dual condition

$$(T_e^* Q)^\perp = (T_e^* C_{M_1})^\perp + \cdots + (T_e^* C_{M_k})^\perp \tag{12}$$

holds, where

$$(T_e^* Q)^\perp = \{f \in \mathbb{R}^6 \| \langle f, \xi \rangle = 0, \forall \xi \in T_e Q\}$$

denotes the set of constraint forces for $T_e Q$. Then, there exists a connected open neighborhood U_e of e in SE(3) such that $U_e \cap Q = U_e \cap C_M$. Furthermore, if for every $g \in Q_U$,

$$R_{g^{-1}*}T_g(\mathbf{C}) = R_{g^{-1}*}T_g(C_{M_1}) \cap \cdots \cap R_{g^{-1}*}T_g(C_{M_k}). \tag{13}$$

(or its dual condition holds), then there exists a connected open subset W of $SE(3)$ such that $Q_U = W \cap C_M$.

Remark *2:* The importance of Proposition 6 lies in the fact that if we choose $C_{M_j}, j = 1, \cdots k$ so that $Q \subseteq C_{M_j}$ and hence $Q \subseteq C_M$, and either Condition (11) or (12) holds, then $M = M_1 \| \cdots \| M_k$ is a PM generator of Q. Furthermore, if we want to know to what extent M covers Q, i.e., to find the maximal Q_U such that $Q_U = W \cap C_M$, with W an open subset of SE(3), we have to check Condition (13) or its dual for every $g \in Q_U$.

We will refer to (11) and its dual (12) the *velocity matching condition*(VMC) and the *force matching condition*(FMC), respectively. Computational savings may result by using the FMC when the dimension of the problem is relatively high($\geq$ 4), otherwise the using of the VMC is preferred. (13) will be referred to as the global VMC (GVMC).

Using Proposition 6, we now study the PM synthesis problem for Q being, respectively, a Lie subgroup and a regular submanifold of SE(3).

A. Subgroup PM Generators

We will consider PM generators for the Lie subgroups $SO(3)$, $T(3)$, $SE(2)$, and $X(z)$ only, as these are the most common cases in practice, e.g., the Delta, H4([16], [17]) and the Tsai's manipulators([1]). We will ignore the trivial case where each subchain generates identical motions as the end-effector of the resulting parallel manipulator.

1. Synthesis of $SO(3)$ PM Generators

By Proposition 6, we seek 3 subchains $M_{C_j}, j = 1, \cdots, 3$, such that $SO(3) \subseteq C_j, j = 1, \cdots, 3$. From Section II and Proposition 1, a solution to this problem is suggested by expressing

$$\mathbf{C}_j = H_j \cdot SO(3) \quad j = 1, \cdots, 3$$

where H_j is a subgroup of SE(3) to be determined by the VMC or FMC.

First, observe that H_j should not have trivial intersection with $SO(3)$ because, if $H_j \cap SO(3) = \{e\}, j = 1, \cdots, 3$, then the mechanical joints that generate H_j in the subchain will be dead-locked when the loop or closure constraints are imposed. Thus, removing these dead joints will not affect the final motions of the resulting PM. Taking this into considerations, we are left with the following subgroups (and their conjugate subgroups) for H_j:$SE(2), S(N)$ and $C(0, z)$. Thus, a general form of C_j is

$$C_j = \begin{cases} PL(w_j^n) \cdot SO(3) \\ S(N_j) \cdot SO(3) \\ C(0, w_j) \cdot SO(3) \end{cases} \quad (14)$$

C_j is, respectively, a 5- and 4-dimensional submanifold of SE(3) containing SO(3). Realizations of the various products in (14) have been either discussed or obtained using the tools from the previous section, see also Table VI. The constraint force for a subchain generating $PL(w^n) \cdot SO(3)$ at e is given by $(w^T, 0)$, and that for $S(N) \cdot SO(3)$ by $(t^T, 0)$, where t is the unit direction vector from the origin to N. Thus, if

$$\mathbf{C}_j = PL(\omega_j^n) \cdot SO(3) \quad j = 1, \cdots, 3$$

and with

$$(T_e^* C_j)^\perp = \{(w_j^T, 0)\}, \quad j = 1, \cdots 3$$

the FMC shows that if the $\omega_j' s, j = 1, \cdots, 3$ are linearly independent, then $M = M_{C_1} \| M_{C_2} \| M_{C_3}$ will be a PM generator for $SO(3)$.

Similarly, if

$$\mathbf{C}_j = S(N_j) \cdot SO(3) \quad j = 1, \cdots, 3$$

then

$$(T_e^* C_j)^\perp = \{(t_j^T, 0)\}, \quad j = 1, \cdots 3$$

and$M = M_{C_1} \| M_{C_2} \| M_{C_3}$ is a PM generator for $SO(3)$ if the t_j's are linearly independent. Fig.10 and 11 give examples of these two types of PM generators discussed above.

The third type of PM generators for SO(3) is obtained when

$$\mathbf{C}_j = C(0, \omega_j) \cdot SO(3) \quad j = 1, \cdots, 3$$

Note that

$$span(\overline{M}_{\mathbf{C}_j}) = \{\xi_{j,1}, \cdots, \xi_{j,4}\}$$

where

$$\begin{aligned} \xi_{j,1} &= (\omega_j^T, 0)^T, \quad \xi_{j,2} = (\mathbf{0}, (1,0,0))^T \\ \xi_{j,3} &= (\mathbf{0}, (0,1,0))^T, \quad \xi_{j,4} = (\mathbf{0}, (0,0,1))^T \end{aligned}$$

Hence, in order to satisfy the VMC, at least two of $\omega_j s, j = 1, \cdots, 3$ need to be linearly independent. Examples of this types of SO(3) PM generators are shown in Fig.12.

We now turn into a few comments regarding realizations of $M = M_{C_1} || M_{C_2} || M_{C_3}$ with

$$C_j = H_j \cdot SO(3)$$

First, since $dim(C_j) > dim(SO(3)) = 3$, certain motions of C_j are to be constrained when the loop constraints are imposed, resulting in possibly dead-locking certain joints of M_{C_j} if C_j is not properly realized. More precisely, if C_j is realized as

$$M_{C_j} = M_{H_j/(H_j \cap SO(3))} \cdot M_{SO(3)}$$

then clearly, motions generated by $M_{H_j/(H_j \cap SO(3))}$ will be constrained and hence $M_{H_j/(H_j \cap SO(3))}$ will be dead-locked. If the subchain contains a generator of H_j, then M_{H_j} can only undergo motions in $H_j \cap SO(3)$, i.e., it is equivalent to a mechanical generator of $H_j \cap SO(3)$ with the loop constraints in place. In this case, if M_{H_j} contains some joints that form a $H_j \cap SO(3)$ generator, then the remaining joints will become dead-locked. Consider, for example,

$$C_j = PL(\omega_j^n) \cdot SO(3), j = 1, \cdots, 3$$

Two possible realizations of C_j are given by, respectively, $M_{PL(\omega_j^n)/R(0,\omega_j)} \cdot M_{SO(3)}$ and $M_{PL(\omega_j^n)} \cdot M_{SO(3)/R(0,\omega_j)}$. In the former case, motions generated by $M_{PL(\omega_j^n)/R(0,\omega_j)}$ will be fully constrained with loop constraints imposed. Hence, this realization is undesirable and should be avoided. In the latter case, with the motions other than $R(0, \omega_j)$ in $M_{PL(\omega_j^n)}$ being constrained, $M_{PL(\omega_j^n)}$ is equivalent to $M_{R(0,\omega_j)}$. As long as the realization of $M_{PL(\omega_j^n)}$ does not contain a $R(0, \omega_j)$ joint, no joints of $M_{PL(\omega_j^n)}$ will be dead-locked.

Finally, suppose that M_{C_j} is composed by n primitive joints

$$M_{C_j} = \quad P_{j1} \cdots P_{jm} \cdot P_{jm+1} \cdots P_{jn}, \quad j = 1, \cdots, 3$$

where $P_{j1} \cdots P_{jm}$ is a realization of M_{H_j} (containing no joints of mechanical generator of $H_j \cap SO(3)$), and $P_{jm+1} \cdots P_{jn}$ realization of $M_{SO(3)/(H_j \cap SO(3))}$. With loop constraints imposed, $P_{j1}\cdots P_{jm}$ is equivalent to $M_{H_j\cap SO(3)}$, and M_{C_j} is equivalent to a mechanical generator of $SO(3)$. Thus, changing the order of the joints of $M_{SO(3)/(H_j\cap SO(3))}$ produces another optional structure of subchain which contains an open subset of $SO(3)$ around e :

$$N_j = \quad P_{jm+1}\cdot P_{j1}\cdots P_{jm}\cdots\cdot P_{jn}, \quad j=1,\cdots,3$$

The FMC is also trivially satisfied. Hence, $N = N_1||N_2||N_3$ is a variational $SO(3)$ PM generator for M. For example, suppose M is a $SO(3)$ PM generator with subchains

$$M_{C_j} = \mathcal{R}(q_{j_1},\omega_j)\cdot\mathcal{R}(q_{j_2},\omega_j)\cdot\mathcal{R}(q_{j_3},\omega_j)\cdot\mathcal{R}(\mathbf{0},\mathbf{u})\cdot\mathcal{R}(\mathbf{0},\mathbf{v}) \quad j=1,\cdots,3,\quad q_{ji}\neq\mathbf{0}$$

where $\mathcal{R}(q_{j_1},\omega_j)\cdot\mathcal{R}(q_{j_2},\omega_j)\cdot\mathcal{R}(q_{j_3},\omega_j)$ is a realization of $M_{PL(\omega_j^n)}$, and $\mathcal{R}(\mathbf{0},\mathbf{u})\cdot\mathcal{R}(\mathbf{0},\mathbf{v})$ that of $M_{SO(3)/\mathcal{R}(0,\omega_j)}$. Then $N = N_1||N_2||N_3$ with

$$N_j = \mathcal{R}(\mathbf{0},\mathbf{u})\cdot\mathcal{R}(q_{j_1},\omega_j)\cdot\mathcal{R}(q_{j_2},\omega_j)\cdot\mathcal{R}(q_{j_3},\omega_j)\cdot\mathcal{R}(\mathbf{0},\mathbf{v}) \quad j=1,\cdots,3,\quad q_{ji}\neq\mathbf{0}$$

is also a $SO(3)$ PM generator, as in Fig.13 .

2. Synthesis of $SE(2)$ PM Generators

Following the same procedure as in the $SO(3)$ case, we need to find subchains $M_{\mathbf{C}_j}$, $j=1,\cdots 3$, such that

$$\mathbf{C}_j = SE(2)\cdot H_j \quad j=1,\cdots,3$$

where H_j is either $S(N_j)$, $C(0,\omega_j)$, or any of their conjugate subgroups. Realizations for each of the product terms in the above form have already been discussed previously.

If

$$\mathbf{C}_j = SE(2)\cdot S(N_j) \qquad j=1,\cdots,3$$

then the constraint force space of each subchain at e is spanned by $f_j = (z^T,(N_j\times z)^T)$. Applying the FMC to f_j, we see that $M = M_{C_1}\|M_{C_2}\|M_{C_3}$ is a SE(2) PM generator if the N_j's are different from each other.

If

$$\mathbf{C}_j = SE(2)\cdot C(0,\omega_j) \qquad j=1,\cdots,3$$

then

$$span(\overline{M}_{\mathbf{C}_j}) = \{\xi_{j,1},\cdots,\xi_{j,4}\}$$

with

$$\begin{aligned}\xi_{j,1} &= ((1,0,0),\mathbf{0})^T, \quad \xi_{j,2} = ((0,1,0),\mathbf{0})^T\\ \xi_{j,3} &= (\mathbf{0},(0,0,1))^T \quad \xi_{j,4} = (\mathbf{0},\omega_j^T)^T\end{aligned}$$

and applying the VMC, we see that M is a PM generator of SE(2) if at least two of the ω_j's, $j=1,\cdots,3$ are linearly independent.

3. Synthesis of $X(z)$ PM Generators

Let

$$\mathbf{C}_j = X(z)\cdot H_j \quad j=1,\cdots,4$$

where H_j is any conjugate subgroups of $X(y), PL(y^n)$ and $C(q,y)$. The subchains that generate $X(z)\cdot X(\omega)$, $X(z)\cdot PL(\omega^n)$, and $X(z)\cdot C(q,\omega)$ are all equivalent to $M_{T(3)\cdot U(0,z,\omega)}$, the generators for which have been studied in the previous section. It is interesting to note that $M_{SE(2)}\cdot M_{PL(\omega_j^n)}$, an equivalent generator of $M_{T(3)\cdot U(0,z,\omega)}$, provides another option of subchains of $X(z)$ PM generators.

With

$$(T_e^*C_j)^\perp = \{(\mathbf{0},\mathbf{u}_j)\}$$

where $\mathbf{u}_j$ is the normal of the plane determined by z and ω_j, and applying the FMC, we see that $M = M_{C_1}\|\cdots\|M_{C_4}$ is a $X(z)$ PM generator if at least two of the $w_j's$ are linearly independent.

Note that in order to avoid joint dead-locking, we need to restrict our realizations of M_{C_j} to: $M_{X(z)/T(3)}\cdot M_{X(\omega_j)}$, $M_{X(z)/T2(\omega_j^n)}\cdot M_{PL(\omega_j^n)}$ and $M_{X(z)/T(\omega_j)}\cdot M_{C(q,\omega_j)}$.

Reordering the joints of $M_{X(z)/T(3)}, M_{X(z)/T2(\omega_j^N)}, M_{X(z)/T(\omega_j)}$ and $M_{SE(2)/T(\mathbf{u}_j)}$ will give, amazingly, wide variations on the structures of the subchains of the $X(z)$ PM generators.

4. Synthesis of $T(3)$ PM Generators

Since $T(3)$ is contained in $X(z)$, we can have

$$C_j = X(\omega_j), j=1,\cdots 3$$

Applying the VMC, we see that $M = M_{C_1}\|M_{C_2}\|M_{C_3}$ is a $T(3)$ PM generator if at least two of the $w_j's$ are linearly independent. The Delta manipulator is a good example of this type of PM generators.

We can also use for the subchain the following products: $X(\omega_1)\cdot X(\omega_2)$, $X(\omega_1)\cdot PL(\omega_2^n)$, $X(\omega_1)\cdot C(q,\omega_2)$, or $PL(\omega_1^n)\cdot PL(\omega_2^n)$, all of which are in fact equivalent to $M_{T(3)\cdot U(0,\omega_1,\omega_2)}$. Since

$$(T_e^*C_j)^\perp = \{(\mathbf{0},\mathbf{u}_j)\} \quad j=1,\cdots,3$$

where $\mathbf{u}_j$ is the normal of the plane determined by ω_{j1} and ω_{j2}, it is easy to see that $\mathbf{u}_j, j=1,\cdots,3$ must be linearly independent in order for M to be a PM generator of $X(\omega_1)$.

By reordering the twist elements, one can derive the Tsai's manipulators from here.

B. Submanifold PM Generators

Proposition 6 and the synthesis procedure for subgroup PM generators naturally extend to regular submanifolds. For example, Huang's manipulator([21]) with desired motion $T(2)\cdot SO(3)$ is constructed by connecting five subchains, each producing identical motion as the end effector of the manipulator. It is also straightforward to explain PM generation for other regular submanifolds. Instead of continuing our study along this line, however, we will pursue a different application of the proposed theory in computing the the global motions of the parallelogram and the U^*-joint. This is in general a problem of considerable difficulties using any local theory.

1. Synthesis of PM Generators for Circular Translations

Let the set of desired motions be

$$Q = \left\{ \begin{bmatrix} I & (e^{\hat{\omega}\theta} - I)\mathbf{v} \\ 0 & 1 \end{bmatrix} \mid \theta \in [0, 2\pi] \right\}$$

where ω is the axis of rotation for the circle, and $\mathbf{v} \perp \omega$ a vector from the center of the circle to the origin of the spatial coordinate frame. Q is thus S^1 embedded in $SE(3)$, and contains e. We now show that the parallelogram is a mechanical generator of Q. Let $M = M_1 \cdot M_2$, where for $i = 1, 2$,

$$M_i = \mathcal{R}(q_{i1}, \omega) \cdot \mathcal{R}(q_{i2}, \omega) = \mathcal{R}(q_{i1}, \omega) \cdot \mathcal{R}(q_{i1} + \mathbf{v}, \omega)$$

and $q_{21} = q_{11} + \mathbf{u}$, see Fig 2. Notice that

$$\begin{aligned} &\begin{bmatrix} e^{\hat{\omega}\theta_{i1}} & (I - e^{\hat{\omega}\theta_{i1}})q_{i1} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{\hat{\omega}\theta_{i2}} & (I - e^{\hat{\omega}\theta_{i2}})q_{i2} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{\hat{\omega}(\theta_{i1}+\theta_{i2})} & (I - e^{\hat{\omega}(\theta_{i1}+\theta_{i2})})q_{i1} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} I & (e^{-\hat{\omega}\theta_{i2}} - I)\mathbf{v} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} I & (e^{\hat{\omega}\theta_{i1}} - I)\mathbf{v} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{\hat{\omega}(\theta_{i1}+\theta_{i2})} & (I - e^{\hat{\omega}(\theta_{i1}+\theta_{i2})})q_{i2} \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Hence,

$$\begin{aligned} C_{M_1} &= \mathcal{R}(q_{11}, \omega) \cdot Q = Q \cdot \mathcal{R}(q_{12}, \omega) && (15) \\ C_{M_2} &= \mathcal{R}(q_{21}, \omega) \cdot Q = Q \cdot \mathcal{R}(q_{22}, \omega) && (16) \end{aligned}$$

from which one derives that,

$$Q \subseteq C_{M_1} \cap C_{M_2}.$$

If at the home configuration e, $\mathbf{v}$ and $\mathbf{u}$ do not coincide, then the spatial velocities of the subchains are given by

$$\begin{aligned} span(\overline{M_1}) &= \{(q_{11} \times \omega, \omega)^T, ((\hat{\omega}\mathbf{v})^T, 0)^T\} \\ span(\overline{M_2}) &= \{(q_{21} \times \omega, \omega)^T, ((\hat{\omega}\mathbf{v})^T, 0)^T\} \end{aligned}$$

with the intersection

$$\{((\hat{\omega}\mathbf{v})^T, 0)^T\} \tag{17}$$

as long as $q_{11} \neq q_{21}$. (17) is precisely the tangent space of Q at e. By the first part of Proposition 6, M generates Q.

Suppose the angle from $\mathbf{u}$ to $\mathbf{v}$ is $\phi = ATAN2(\omega^T(\mathbf{u} \times \mathbf{v}), \mathbf{u}^T\mathbf{v}) > 0$, and define an open subset Q_U of Q as

$$Q_U = \{g \in Q \mid \theta \in (-\phi, \pi - \phi)\}$$

Then for all $g \in Q_U$, the spatial velocity spaces of the two subchains are

$$\begin{aligned} R_{g^{-1}*}T_g C_{M_1} &= \{(-\omega \times q_{11}, \omega)^T, ((e^{\hat{\omega}\theta}\hat{\omega}\mathbf{v})^T, 0)^T\} \\ R_{g^{-1}*}T_g C_{M_2} &= \{(-\omega \times q_{21}, \omega)^T, ((e^{\hat{\omega}\theta}\hat{\omega}\mathbf{v})^T, 0)^T\} \end{aligned}$$

where $\theta = \theta_{j1} = -\theta_{j2}, j = 1, 2$, and their intersection is

$$\{((e^{\hat{\omega}\theta}\hat{\omega}\mathbf{v})^T, 0)^T\} \tag{18}$$

which is precisely the velocity space of Q at g. However, one can check that this equality will not hold when $\theta = -\phi$ or $\pi - \phi$. Therefore, by the second part of Proposition 6, Q_U is the maximal open subset of Q that M generates.

2. Synthesis of PM Generators for Spherical Translations

Let

$$\begin{aligned} Q &= \{ \begin{bmatrix} I & (e^{\hat{\omega}_1\alpha}e^{\hat{\omega}_2\beta} - I)\mathbf{v} \\ 0 & 1 \end{bmatrix}, \\ &\qquad \alpha \in [0, 2\pi], \beta \in [-\pi/2, \pi/2]\} \end{aligned}$$

where ω_1 and ω_2 are perpendicular unit vectors representing the rotation axes of the sphere, and $\mathbf{v} \perp \omega_2$ is a vector from the center of the sphere to the origin of the spatial coordinate frame. Q is actually S^2 embedded in $SE(3)$. We can show that the U-pair based prism (Fig.3) which consists of three UU subchains is a mechanical generator of Q. Suppose $M = M_1 \cdot \cdots \cdot M_3$, where for $i = 1, \cdots 3$,

$$M_i = \mathcal{R}(q_{i1}, \omega_1) \cdot \mathcal{R}(q_{i1}, \omega_2) \cdot \mathcal{R}(q_{i2}, \omega_2) \cdot \mathcal{R}(q_{i2}, \omega_1)$$

and $q_{i2} = q_{i1} + \mathbf{v}$. Notice that

$$\begin{aligned} g &= \begin{bmatrix} e^{\hat{\omega}_1\alpha_{i1}}e^{\hat{\omega}_2\beta_{i1}} & (I - e^{\hat{\omega}_1\alpha_{i1}}e^{\hat{\omega}_2\beta_{i1}})q_{i1} \\ 0 & 1 \end{bmatrix} \cdot \\ &\quad \begin{bmatrix} e^{\hat{\omega}_2\beta_{i2}}e^{\hat{\omega}_1\alpha_{i2}} & (I - e^{\hat{\omega}_2\beta_{i2}}e^{\hat{\omega}_1\alpha_{i2}})q_{i2} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} I & (e^{\hat{\omega}_1\alpha_{i1}}e^{\hat{\omega}_2\beta_{i1}} - I)\mathbf{v} \\ 0 & 1 \end{bmatrix} \in Q, \quad i = 1, \cdots, 3 \end{aligned}$$

if $\alpha_{i1} = -\alpha_{i2}, \beta_{i1} = -\beta_{i2}$. Thus

$$Q \subseteq C_{M_i}, \quad i = 1, \cdots, 3$$

Moreover, if at the home configuration e, $\mathbf{v}$ and ω_1 are not aligned, then the spatial velocity spaces of the three subchains at e are

$$\begin{aligned} &span(\overline{M_i}) = \\ &\{ \begin{bmatrix} q_{i1} \times \omega_1 \\ \omega_1 \end{bmatrix}, \begin{bmatrix} q_{i1} \times \omega_2 \\ \omega_2 \end{bmatrix}, \begin{bmatrix} q_{i2} \times \omega_2 \\ \omega_2 \end{bmatrix}, \begin{bmatrix} q_{i2} \times \omega_1 \\ \omega_1 \end{bmatrix} \} \\ &= \{ \begin{bmatrix} q_{i1} \times \omega_1 \\ \omega_1 \end{bmatrix}, \begin{bmatrix} q_{i1} \times \omega_2 \\ \omega_2 \end{bmatrix}, \begin{bmatrix} \hat{\omega}_1\mathbf{v} \\ 0 \end{bmatrix}, \begin{bmatrix} \hat{\omega}_2\mathbf{v} \\ 0 \end{bmatrix} \}, \end{aligned}$$

The intersection of these three subspaces is

$$\{((\hat{\omega}_1\mathbf{v})^T, 0)^T, ((\hat{\omega}_2\mathbf{v})^T, 0)^T\} \tag{19}$$

Since $\omega_2 \perp \mathbf{v}$, as long as ω_1 is not aligned with $\mathbf{v}$ in the home configuration, (19) is precisely the tangent space of Q at e, and hence M is a mechanical generator of Q.

In a similar manner, assume that the angle from ω_1 to $\mathbf{v}$ is $\phi = ATAN2(\omega_2^T(\omega_1 \times \mathbf{v}), {\omega_1}^T\mathbf{v}) > 0$, and define an open subset Q_U of Q as

$$Q_U = \{g \in Q \mid \alpha \in (-\pi, \pi), \beta \in (-\phi, \pi - \phi)\}$$

Then for all $g \in Q_U$, the spatial velocity spaces of the three subchains are

$$\begin{aligned} &R_{g^{-1}*}T_g(C_{M_i} = \\ &\{ \begin{bmatrix} q_{i1} \times \omega_1 \\ \omega_1 \end{bmatrix}, \begin{bmatrix} q_{i1} \times e^{\hat{\omega}_1\alpha}\omega_2 \\ e^{\hat{\omega}_1\alpha}\omega_2 \end{bmatrix}, \\ &\begin{bmatrix} q'_{i2} \times e^{\hat{\omega}_1\alpha}\omega_2 \\ e^{\hat{\omega}_1\alpha}\omega_2 \end{bmatrix}, \begin{bmatrix} q'_{i2} \times \omega_1 \\ \omega_1 \end{bmatrix} \} = \\ &\{ \begin{bmatrix} q_{i1} \times \omega_1 \\ \omega_1 \end{bmatrix}, \begin{bmatrix} q_{i1} \times e^{\hat{\omega}_1\alpha}\omega_2 \\ e^{\hat{\omega}_1\alpha}\omega_2 \end{bmatrix}, \\ &\begin{bmatrix} \hat{\omega}_1 e^{\hat{\omega}_1\alpha}e^{\hat{\omega}_2\beta}\mathbf{v} \\ 0 \end{bmatrix}, \begin{bmatrix} e^{\hat{\omega}_1\alpha}e^{\hat{\omega}_2\beta}\hat{\omega}_2\mathbf{v} \\ 0 \end{bmatrix} \}, \end{aligned}$$

where

$$\begin{aligned} q'_{i2} &= q_{i1} + e^{\hat{\omega}_1 \alpha_{i1}} e^{\hat{\omega}_2 \beta_{i1}} \mathbf{v} \\ \alpha &= \alpha_{i1} = -\alpha_{i2} \qquad\qquad i = 1, \cdots, 3 \\ \beta &= \beta_{i1} = -\beta_{i2} \end{aligned}$$

The intersection of these three subspaces is

$$\{((\hat{\omega}_1 e^{\hat{\omega}_1 \alpha} e^{\hat{\omega}_2 \beta} \mathbf{v})^T, 0)^T, ((e^{\hat{\omega}_1 \alpha} e^{\hat{\omega}_2 \beta} \hat{\omega}_2 \mathbf{v})^T, 0)^T\}$$

which is just the spatial velocity space of Q at g. However, one can check that this equality will not hold when $\beta = -\phi$ or $\pi - \phi$. Therefore, Q_U is the maximal open subset of Q that M generates.

V. Conclusion

In this paper, we presented a geometric theory for the synthesis and analysis of sub-6 DoF manipulators, both serial and parallel. Aside from showing the previously known results that Lie subgroups can be used to model lower pairs and the desired set of end-effector motions, we also showed that there are at least two classes of regular submanifolds which admit global and coordinates free expressions and can also be used for similar purposes. This substantially broadens the spectrum of manipulators available to design applications. So we gave a formal definition of the serial synthesis problem and developed general procedures as well as technical tools for systematic solutions of the synthesis problem. In the case of the parallel synthesis problem, we presented a general result for guiding the selection of the subchains so that the parallel connection of which generates the desired end-effector motions of the manipulator.

As for future work along this direction, we propose: (1) develop a parametric model of the synthesis problem and introduce performance measures so that the performance of each realization can be computed and bench-marked; (2) develop a tolerance (or error) model of the primitive generators and study the error characteristics of the associated realizations; and (3) introduce a stiffness model of the primitive generators, and exam the resulting stiffness of the various realizations.

Appendix A : Proof of Proposition 1

Proposition 1 : Let H_1 and H_2 be two closed Lie subgroups of $SE(3)$, of dimension n_1 and n_2, respectively. Let $H = H_1 \cap H_2$, and $n = dim(H)$. Then, the product $H_1 \cdot H_2$ is a regular submanifold of $SE(3)$ of dimension $n_1 + n_2 - n$.

Proof: First of all, we endow $H_1 \cdot H_2$ with the subspace topology of $SE(3)$. Second, let $(\hat{\eta}_1, \cdots, \hat{\eta}_n)$ be a basis of $\mathfrak{h}$, and complete the basis so that $(\hat{\eta}_1, \cdots, \hat{\eta}_n, \hat{\eta}_{n+1}, \cdots, \hat{\eta}_{n_1})$ forms a basis of $\mathfrak{h}_1$, $(\hat{\eta}_1, \cdots, \hat{\eta}_n, \hat{\eta}_{n_1+1}, \cdots, \hat{\eta}_{n_1+n_2-n})$ a basis of $\mathfrak{h}_2$, and $\eta_1, \cdots, \eta_{n_1+n_2-n}, \cdots, \eta_6$ a basis of $se(3)$. Let U, V defined as

$$\begin{aligned} U &= \{e^{\hat{\eta}_1 \theta_1} \cdots e^{\hat{\eta}_{n_1+n_2-n} \theta_{n_1+n_2-n}} \in H_1 \cdot H_2 \;\mid\; |\theta_i| < \epsilon\} \\ V &= \{e^{\hat{\eta}_1 \theta_1} \cdots e^{\hat{\eta}_6 \theta_6} \in SE(3) \;\mid\; |\theta_i| < \epsilon\} \end{aligned}$$

respectively. Clearly, U is a slice of V : $U = \{x \in V \mid \theta_{n_1+n_2-n+1} = 0, \cdots, \theta_6 = 0\}$. For sufficiently small ϵ, V is an open coordinate neighborhood of e in $SE(3)$, with local coordinates $(\theta_1, \cdots, \theta_6)$, and $U = V \cap (H_1 \cdot H_2)$. Hence, U is an open coordinate neighorhood of e in $H_1 \cdot H_2$, with preferred local coordinates $(\theta_1, \cdots, \theta_{n_1+n_2-n})$. Using left and right translation, any point of $H_1 \cdot H_2$, $h_1 \cdot h_2$, has such a preferred local coordinate neighborhood $h_1 \cdot U \cdot h_2$. Thus $H_1 \cdot H_2$ has the n-submanifold property, together with its subspace topology, making itself a regular submanifold of $SE(3)$.

Appendix B : A More General Thoerem Than Proposition 6

Theorem *1:* Suppose $C_1, \cdots, C_k$ are k connected regular submanifolds of G(G is a manifold), they all contain a connected regular submanifold $\mathbf{C}$, i.e,

$$\mathbf{C} \subseteq C_j, \quad j = 1, ..., k \tag{20}$$

and consequently $\mathbf{C} \subseteq C_1 \cap \cdots \cap C_k$. Moreover, if for some $x \in \mathbf{C}$, the following equation holds :

$$\Delta_x \mathbf{C} = \Delta_x(C_1) \cap \Delta_x(C_2) \cap \cdots \cap \Delta_x(C_k) \tag{21}$$

where $\Delta_x \mathbf{C}$ denotes the tangent space of $\mathbf{C}$ at x, and $\Delta_x C_j$ the tangent space of C_j at x, $j = 1, \cdots, k$. Then there exists a connected neigborhood U_x of x, where U_x is an open subset of G, such that $U_x \cap \mathbf{C} = U_x \cap (C_1 \cap \cdots \cap C_k)$. If for every $x \in \mathbf{C}$, Equation 21 holds, then there exist a connected open subset of G, W, such that $\mathbf{C} = W \cap (C_1 \cap \cdots \cap C_k)$.

Proof : We prove the case for $k = 2$, similar proof is for $k > 2$. Suppose $dim(G) = m, x \in \mathbf{C}$, and we choose a coordinate neigborhood of x of G, (U, ϕ), such that $x_1, \cdots, x_m$ are the local coordinates in this neighborhood. Suppose in this neighborhood, C_1 is locally cut out by s functions,

$$C_1 \cap U = \phi{-1}\{(x_1, \cdots, x_m) | h_1(x_1, \cdots, x_m) = 0, \cdots, h_s(x_1, \cdots, x_m) = 0\}$$

where s is the codimention of C_1 in G; C_2 is locally cut out by l functions, where l is codimension of C_2 in G

$$C_2 \cap U = \phi{-1}\{(x_1, \cdots, x_m) | g_1(x_1, \cdots, x_m) = 0, \cdots, g_l(x_1, \cdots, x_m) = 0\}$$

Then in local coordinates

$$(C_1 \cap C_2) \cap U = \phi{-1}\{(x_1, \cdots, x_m) | h_1 = 0, \cdots, h_s = 0, g_1 = 0, \cdots, g_l = 0\}$$

Since we know $\Delta_x \mathbf{C} = \Delta_x(C_1) \cap \Delta_x(C_2)$ at $x \in \mathbf{C} \subseteq C_1 \cap C_2$, thus

$$rank \begin{pmatrix} dh_1 \\ \vdots \\ dh_s \\ dg_1 \\ \vdots \\ dg_l \end{pmatrix}_{\phi(x)} = n$$

where n is the codimension of $\mathbf{C}$ in G. Without loss of generality, we assume that the first n rows of the above matrix are linearly independent with each other at $\phi(x) \in$

$\phi(\mathbf{C} \cap U)$. Thus

$$rank \begin{pmatrix} dh_1 \\ \vdots \\ dh_s \\ dg_1 \\ \vdots \\ dg_t \end{pmatrix}_{\phi(x)} = n$$

where $s + t = n$. Then by the submersion theorem, there is an open neighborhood of G around x, $U^{'} \subseteq U$ such that

$$\overline{\mathbf{C}} = \phi^{-1}\{(x_1, \cdots, x_m) \in \phi(U^{'}) | h_1 = 0, \cdots, h_s = 0, g_1 = 0, \cdots, g_t = 0\}$$

is an $m - n$ dimmensional regular submanifold of G containing x. It's also obvious that $(C_1 \cap C_2) \cap U^{'} \subseteq \overline{\mathbf{C}}$. Since we know $\mathbf{C} \subseteq (C_1 \cap C_2)$, we also get $\mathbf{C} \cap U^{'} \subseteq (C_1 \cap C_2) \cap U^{'} \subseteq \overline{\mathbf{C}}$. Since both $\mathbf{C} \cap U^{'}$ and $\overline{\mathbf{C}}$ are $m - n$ dimmensional submanifold of G, and the former is contained in the latter. Thus we know that there exists an open neighborhood of x of G, $V \subseteq U^{'}$ such that $\mathbf{C} \cap V = (C_1 \cap C_2) \cap V = \overline{\mathbf{C}} \cap V$. V is just the U_x as we wanted. If for every $x \in \mathbf{C}$, Equation 21 holds, then every x will have a U_x. Thus the union of all these U_x's will be the W as we wanted, i.e, $W = \cup_{x \in \mathbf{C}} U_x$.

Replacing G in Theorem 1 into $SE(3)$ and C_j the desired end-effector motions of the subchains, we can infer from this theorem Proposition 6.

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