A Geometric Theory for Analysis and Synthesis of Sub-6 DoF Parallel Manipulators

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***Abstract*— Mechanism synthesis is mostly dependent on the designer's experience and intuition and is difficult to automate. This paper aims to develop a rigorous and precise geometric theory for analysis and synthesis of sub-6 DoF (or lower mobility) parallel manipulators. Using Lie subgroups and submanifolds of the special Euclidean group $SE(3)$, we first develop a unified framework for modelling commonly used primitive joints and task spaces. We provide a mathematically rigorous definition of the notion of motion type using conjugacy classes. Then, we introduce a new structure for subchains of parallel manipulators using the product of two subgroups of $SE(3)$ and discuss its realization in terms of the primitive joints. We propose the notion of quotient manipulators that substantially enriches the topologies of serial manipulators. Finally, we present a general procedure for specifying the subchain structures given the desired motion type of a parallel manipulator. The parallel mechanism synthesis problem is thus solved using the realization techniques developed for serial manipulators. Generality of the theory is demonstrated by systematically generating a large class of feasible topologies for (parallel or serial) mechanisms with a desired motion type of either a Lie subgroup or a submanifold.**

***Index Terms*— Lie subgroups, regular submanifolds, motion type, quotient manipulators, kinematic synthesis.**

I. Introduction

Parallel mechanism design involves interactively solving two tightly coupled problems: *(i) mechanism synthesis* and *(ii) dimensional optimization.* Over the last two decades or so, we have witnessed the enormous progresses toward the second problem, namely for a given mechanism the kinematic and singularity analysis ([15], [32], [35], [37], [38], [49]), the determination and computation of properties such as workspace and stiffness ([11], [12], [22], [31]), and the different approaches for formulating and optimizing the various performance indices ([14],[41],[48]); we have also seen numerous architectures or topological configurations being proposed in the literature, but solutions to the synthesis problem remains ad hoc or unsystematic. Human experience and intuitions instead of mathematically rigorous and justifiable procedures are relied on when a new design problem is called for. Ambiguity in the design objective and lack of a rigorous treatment in existing approaches to the synthesis problem has not only prevented globally optimal solutions in many design cases, but also rendered the mission of automating design solutions, an ultimate objective of design, extremely difficult, if not impossible.

Majority of the parallel mechanism architectures proposed in the literature have six degree-of-freedom (DoF) ([27], [39]). Most industrial applications, however, rarely need it. For example, only 3-DoF is needed for an orientation device, 4-DoF is sufficient for most pick-and-place applications, and 5-DoF is adequate for every conceivable machine tool application. Having a mechanism with more DoF than necessary is not only uneconomical but also increases the programming and maintenance complexity. The Delta manipulator ([5], [6]) is perhaps the most famous and successful example of a sub-6 DoF (or lower mobility) parallel mechanism design. Other examples include the H4 robot by Pierrot ([42], [43]), the orientation device by Gosselin ([13], [16]), the haptic devices by T. Salcudean [48], Tsai's manipulators [49], and that in Gao [10], Huang [23], and Merlet [39].

Constraint-synthesis method or its variations derived from screw (or reciprocal screw) theory has long been the main tool for parallel mechanism synthesis ([8], [24], [30], [29], [49], [8]). The emergence of many novel sub-6 DoF parallel manipulators could be attributed to this approach. However, it is rather difficult to provide a rigorous and yet concise proof of the finite motion property of a mechanism using this approach. Remedies like the finite motion conditions([8]) or the single loop kinematic chain method([30], [29]) proposed by the authors are descriptive in nature and work usually on a case-by-case basis.

In order to fully automate the entire design process, a rigorous and yet precise mathematical theory of the synthesis problem is needed, a theory that provides:

(1) A geometric framework for precise and unambiguous description of the key mechanism concepts and important design objectives;
(2) A systematic method for generating all feasible topologies or architectures based on the design objectives, and computational tools for verifying correctness of the solutions;
(3) Performance metrics on the various task spaces based on which dimensional optimization problems could be formulated and efficiently solved.

Reuleaux [46] introduced the notion of lower pairs to model some commonly used joints by subgroups of the special Euclidean group $SE(3)$. Some commonly occurring task spaces are also modelled by subgroups of $SE(3)$. There are, however, certain joints such as the parallelogram joint ([5], [6]) in the

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Delta robot and the $\mathcal{U}^*$-joint ([1], [10]), and certain task spaces such as that of a 5-axis machining tool or a flight simulator that lack a group structure and defy descriptions by subgroups.

J. Hervé initiated a major program to study systematic generation of parallel mechanism topologies that produce translational type of motions, known as translational parallel manipulators (TPM). In their pioneering works ([20], [17], [21], [33]), Hervé and his co-workers realized that the key to synthesize such a mechanism is the generation of suitable limb (leg or subchain) topologies. Using the Lie-group-algebraic properties of $SE(3)$, Lee and Hervé [33] studied the product of two planar subgroups and their 21 distinct realizations using two 1-dimensional Reuleaux pairs, the revolute joint $\mathcal{R}$ and the prismatic joint $\mathcal{T}$. They showed that the parallel setting of three such limbs result in a TPM and gave a systematic explanation on the generation of some familiar mechanism topologies. Related effort can also be found in the works of J. Angeles [1], and Li, Huang and Hervé [34].

Inspired by Hervé's work, this paper aims to develop a geometric theory for more precise and complete treatment of the synthesis problem. Using differentiable manifold and Lie group theory (differential as well as group aspects of the problem), the paper aims to provide:

(i) A unified framework for modelling commonly used (conventional and extended) joints and task spaces using Lie subgroups and submanifolds of $SE(3)$; A mathematically precise definition for the notion of motion type;

(ii) A list of all feasible subchain (or limb) topologies using the products of two Lie subgroups of $SE(3)$, and realizations of these subchain structures using the notion of *quotient manipulators*;

(iii) A systematic method for specifying all possible subchain structures given the desired motion type of the parallel mechanism;

(iv) Simple computational techniques for verifying finite motion properties of the derived results.

The paper is organized as follows: In Section II, we develop geometric models for constrained rigid motions, with commonly used primitive joints and desired task spaces as examples. We introduce two important classes of regular submanifolds of $SE(3)$ for modelling joints and task spaces that lack a group structure; in Section III, we give a precise definition of motion type and study the synthesis problem for serial manipulators having a desired motion type; in Section IV, we study the parallel manipulator synthesis problem; and in Section V, we compare the synthesis method developed in this paper with the screw (or reciprocal screw) theory based method and the Lie-group-algebraic method of Lee and Hervé [33], and offer a few comments for future works.

II. Modelling of Constrained Rigid Motions

In this section, we develop mathematical models for rigid motions generated by primitive motion generators (PMGs or joints), and by end-effectors of both serial and parallel manipulators. We show that Lie subgroups and submanifolds of the *special Euclidean group* $SE(3)$ provide a natural framework for this problem and explore the geometric properties of these spaces for the synthesis problems in later sections. Readers are referred to ([25], [28], [40], [47]) for most of the concepts treated in this section.

A. $SE(3)$ and its Lie Subgroups

We assume the reader is familiar with the basic notions of differentiable manifold (see e.g., [4]). S^n, the unit sphere in $\mathbb{R}^{n+1}$, and $M(n,\mathbb{R})$, the set of $n\times n$ real matrices, are examples of a differentiable manifold of dimension n and n^2, respectively. $GL(n,\mathbb{R})$, the set of $n\times n$ nonsingular matrices, being an open subset of $M(n,\mathbb{R})$, is also a differentiable manifold of dimension n^2. In addition to its differential structure, $GL(n,\mathbb{R})$ is also an algebraic group with matrix multiplication as the group operation, and the group multiplication and inverse operations are both smooth mappings. It is an example of what is known as a *Lie group*, a differentiable manifold with a compatible group structure (see [4] and [28] for more details). Other examples of a Lie group include $SO(n)$, the special orthogonal group, and any algebraic subgroup (i.e., closed under group operation) of $GL(n,\mathbb{R})$ (or any Lie group) that is a closed subset (see Remark 6.19, p.88, [4]).

Our primary interest lies in the special Euclidean group $SE(3)$, classically defined as the symmetry group of the three dimensional affine space E^3. By regarding E^3 as an orientable Riemannian manifold with a preferred orientation and a choice of length scale, an element of $SE(3)$ is an orientation preserving isometry of E^3. This coordinate-free description of $SE(3)$ is used in Hervé [19]. It is customary in the robotics literature, however, to associate a Cartesian coordinate frame with E^3 and identify it with $\mathbb{R}^3$. An element g of $SE(3)$ is then identified with an (orientation preserving) isometry of $\mathbb{R}^3$,

$$g(q) = Rq + p$$

where $R\in SO(3)$ and $p\in\mathbb{R}^3$. This allows us to identify $SE(3)$ with the semidirect product of $SO(3)$ with $\mathbb{R}^3$, denoted $SO(3)\ltimes\mathbb{R}^3$, which is defined topologically as the product of $SO(3)$ with $\mathbb{R}^3$ but with the group operation $g_1\cdot g_2 = (R_1R_2, R_1p_2+p_1)$. If we use the homogeneous representation for points in $\mathbb{R}^3$, then $SE(3)$ can be further expressed as homogeneous transformations of the form

$$SE(3) = \left\{ \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \mid p\in\mathbb{R}^3, R\in SO(3) \right\} \subset GL(4,\mathbb{R}).$$

Clearly, $SE(3)$ is a closed subgroup of $GL(4,\mathbb{R})$ as it is defined by polynomial equations, and is thus a Lie group of dimension 6 (with its relative topology). The manifold structure of $SE(3)$ is diffeomorphic to that of $SO(3)\ltimes\mathbb{R}^3$ with the diffeomorphism

$$\Psi : SO(3)\ltimes\mathbb{R}^3 \to SE(3) : (R,p) \mapsto \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}. \qquad (1)$$

It is well known that an element $g\in SE(3)$ also represent a displacement (or a rigid motion) of a rigid body relative to a reference or nominal configuration represented by the identify element e. Consequently, $SE(3)$ is also referred to as the configuration space of the rigid body. This fact enables us to carry out kinematic analysis and synthesis based on the Lie group $SE(3)$.

An immediate benefit of the homogeneous or matrix representation (of $SE(3)$) is the relative simplicity in which geometric properties of rigid motions and mechanism kinematics can be computed. Since there is no natural, or geometrically determined way for identifying E^3 with $\mathbb{R}^3$, one has to be cautious when interpreting the computed results. A conclusion or result is meaningful only when it is coordinate invariant. Thus, in one hand, we need to use coordinates for performing computations, but in another hand, we need to ensure that results are coordinate independent.

We collect here several important facts concerning a Lie group G that will be used in later sections.

Associated with a Lie group G is its Lie algebra $\mathfrak{g}$, defined as the tangent space to G at the identity e, i.e, $\mathfrak{g} \triangleq T_eG$, together with a Lie bracket operation that is bilinear and satisfies skew-symmetry and Jacobi's identity. For example, the Lie algebra $so(3)$ of the rotation group $SO(3)$ consists of all 3×3 skew-symmetric matrices,

$$so(3) = \left\{S \in \mathbb{R}^{3\times 3} | S^T = -S\right\}$$

together with the Lie bracket operation given by the matrix commutator, $[S_1, S_2] = S_1S_2 - S_2S_1$. Clearly, $so(3)$ can be identified with $\mathbb{R}^3$ via the map

$$\wedge : \mathbb{R}^3 \longrightarrow so(3) : \omega \longmapsto \hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

The Lie algebra $se(3)$ of $SE(3)$ consists of 4×4 matrices of the form

$$se(3) = \left\{ \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} | \, \omega, v \in \mathbb{R}^3 \right\}$$

together with the matrix commutator as its Lie bracket operation. With a slight abuse of notation, we identify $\mathbb{R}^6$ with $se(3)$ via the map

$$\wedge : \mathbb{R}^6 \to se(3) : \xi = \begin{pmatrix} v \\ \omega \end{pmatrix} \mapsto \hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in se(3).$$

An element $\hat{\xi}$ of $se(3)$ is called a twist, with twist coordinates $\xi = \begin{pmatrix} v \\ \omega \end{pmatrix} \in \mathbb{R}^6$, and pitch ρ as

$$\rho = \begin{cases} \frac{\omega^T v}{||\omega||^2}, & \text{if } \omega \neq 0 \\ \infty, & \text{if } \omega = 0. \end{cases}$$

The exponential map

$$\exp : se(3) \to SE(3) : \hat{\xi} \mapsto e^{\hat{\xi}} \tag{2}$$

is a surjective map and defines a local diffeomorphism taking the zero vector of $se(3)$ to the identity element of $SE(3)$(see [40] for an explicit formula of (2)). Physically for fixed ξ, $e^{\theta\hat{\xi}}, \theta \in \mathbb{R}$, corresponds to a screw motion along the axis of ξ [40]. From the property of the exponential map (2), if $(\hat{v}_1, \cdots, \hat{v}_6)$ is a basis of $se(3)$, then the map

$$(\phi_1, \cdots \phi_6) \longmapsto \exp(\phi_1\hat{v}_1 + \cdots + \phi_6\hat{v}_6) \tag{3}$$

carries a sufficiently small cube $\{(\phi_1, \cdots \phi_6) \in \mathbb{R}^6 \mid |\phi_i| < \epsilon\}$ about 0 in $\mathbb{R}^6$ diffeomorphically onto an open neighborhood of e in $SE(3)$. $(\phi_1, \cdots \phi_6)$ defines *canonical coordinates of the first kind* around the identity element of $SE(3)$. In a similar manner, the map

$$(\theta_1, \cdots \theta_6) \longmapsto e^{\theta_1\hat{v}_1} \cdots e^{\theta_6\hat{v}_6} \tag{4}$$

defines the *canonical coordinates of the second kind* around the identity element of $SE(3)$. Canonical coordinates of the second kind are commonly used in robotics since the forward kinematic map of an open kinematic chain can be written as the product of exponentials.

Definition *1:* **Lie subgroups and Lie subalgebras** A subset H of a Lie group G is a Lie subgroup if it is an algebraic subgroup, i.e., closed under the group operation, and a regular submanifold of G. A subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$ if it is closed under the Lie bracket operation.

Example *1:* Let $\{e_i\}_{i=1}^6$ be the canonical basis of $\mathbb{R}^6 \cong se(3)$, and denote by $\{e_i, e_j\}$ the span of e_i and e_j. Then, the following subspaces of $se(3)$ constitute a Lie subalgebra of $se(3)$: $t(\mathbf{z}) = \{\hat{e}_3\}$, $t(3) = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$, $so(3) = \{\hat{e}_4, \hat{e}_5, \hat{e}_6\}$ and $x(\mathbf{z}) = \{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_6\}$. □

From the theory of differential geometry (see [4]), each Lie subalgebra of $se(3)$ corresponds uniquely to a connected component through the identity of a *Lie subgroup* of $SE(3)$.

Example *2:* Fix $\hat{\xi} \in se(3)$, then

$$H = \{e^{\hat{\xi}\theta} \mid \theta \in \mathbb{R}\}$$

is a one-parameter subgroup of $SE(3)$, generated by $\hat{\xi}$. Identify $T(3)$ with $\{I\} \times \mathbb{R}^3$ through (1), and $SO(3)$ with $SO(3) \times \{0\}$, then

$$T(3) = \{\exp(\hat{e}_1\phi_1 + \hat{e}_2\phi_2 + \hat{e}_3\phi_3) \mid \phi_i \in \mathbb{R}, i = 1, 2, 3\}$$

and

$$SO(3) = \{\exp(\hat{e}_4\phi_1 + \hat{e}_5\phi_2 + \hat{e}_6\phi_3) \mid \phi_i \in \mathbb{R}, i = 1, 2, 3\}.$$

$T(3)$ and $SO(3)$ are referred to as the spatial translational and rotational subgroup of $SE(3)$, respectively. The Schoenflies subgroup

$$X(\mathbf{z}) = \{\exp(\hat{e}_1\phi_1 + \hat{e}_2\phi_2 + \hat{e}_3\phi_3 + \hat{e}_6\phi_4) \mid \phi_i \in \mathbb{R}, i = 1, \cdots 4\}$$

that consists of three translations and one rotation about a fixed axis is a 4-dimensional subgroup. □

Note that the Lie algebra of the Lie subgroup $T(3), SO(3)$ and $X(\mathbf{z})$ is, respectively, $t(3), so(3)$ and $x(\mathbf{z})$. Let $\mathfrak{h}$ be the Lie algebra of a Lie subgroup H, and hence a Lie subalgebra of $se(3)$. *The exponential map (2) maps diffeomorphically a neighborhood of* $0 \in \mathfrak{h}$ *onto a neighborhood of* $e \in H$.

Given $g \in SE(3)$, denote by L_g and R_g, the left and the right translation map, respectively, and $I_g = L_g \circ R_{g^{-1}} = R_{g^{-1}} \circ L_g$ the conjugation map,

$$I_g : SE(3) \to SE(3) : h \mapsto ghg^{-1}. \tag{5}$$

The differential ${L_{g^{-1}}}_*$ of $L_{g^{-1}}$ defines the body velocity of a rigid motion $g(t) \in SE(3), t \in (-\epsilon, \epsilon)$,

$$\begin{aligned} \hat{V}^b &= L_{g(t)^{-1}*} \cdot \dot{g}(t) = \begin{bmatrix} R^T\dot{R} & R^T\dot{p} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \end{aligned}$$

and the differential ${R_{g^{-1}}}_*$ of $R_{g^{-1}}$ defines the spatial velocity of the rigid motion, i.e., $\hat{V}^s = R_{g^{-1}*} \cdot \dot{g}$. The spatial and body velocities are related to each other by the differential I_{g_*} of the conjugation map,

$$\hat{V}^s = I_{g_*}(\hat{V}^b) = g \cdot \hat{V}^b \cdot g^{-1} \quad \text{or} \quad V^s = Ad_g V^b$$

where for $g = (R, p)$, the Adjoint map

$$Ad_g = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}$$

is the matrix representation of I_{g_*} with respect to the twist coordinates.

Example *3:* **Change of coordinate frames**
Let $g_i(t) \in SE(3), t \in (-\epsilon, \epsilon), i = 1, 2$ be the representation of a rigid motion with respect to two different coordinate frames that are related to each other by $g_0 \in SE(3)$. Then, the following relations hold

$$g_2(t) = g_0 g_1(t) g_0^{-1}$$

and

$${V_2}^b = Ad_{g_0} V_1^b.$$

□

Definition *2:* **Conjugate subgroups**
Let H be a Lie subgroup of $SE(3)$. Then, for a given $g \in SE(3)$, the conjugate subgroup of H is a subgroup defined by

$$I_g(H) = \{ghg^{-1} \mid h \in H\}.$$

Two Lie subgroups H_1 and H_2 are said to be equivalent (or belong to the same conjugacy class) if $\exists g \in SE(3)$ such that $H_1 = I_g(H_2)$. Clearly, if H_1 and H_2 are equivalent, then their Lie algebras are related by $\mathfrak{h}_1 = I_{g*}(\mathfrak{h}_2) = Ad_g(\mathfrak{h}_2)$.

Example *4:* Let $H = \{e^{\hat{\xi}\theta} \mid \theta \in \mathbb{R}\}$ be a one-parameter subgroup of screw motions about the axis of a fixed $\hat{\xi} \in se(3)$. Then, given $g = (R, p)$,

$$I_g(H) = \{e^{(Ad_g \xi)^\wedge \theta} | \theta \in \mathbb{R}\}$$

corresponds to the one-parameter subgroup of screw motions about a displaced twist axis $(Ad_g\xi)^\wedge \in se(3)$. Since the pitch of a twist is Ad_g-invariant, i.e., $\rho(\xi) = \rho(Ad_g\xi), \forall g \in SE(3)$, screw motions with the same pitch are thus equivalent. For example, let

$$T(\mathbf{z}) = \{e^{\hat{e}_3\alpha} \mid \alpha \in \mathbb{R}\}$$

denote the set of 1-DoF translations along direction $\mathbf{z}$, then $T(\mathbf{v}) := I_g(T(\mathbf{z})), \mathbf{v} = R\mathbf{z}$, represents 1-DoF translations along direction $\mathbf{v}$. Similarly, if

$$R(o, \mathbf{z}) = \{e^{\hat{e}_6\theta} \mid \theta \in [0, 2\pi]\}$$

models the set of pure rotations about the z-axis (passing through the origin), then $R(p, \boldsymbol{\omega}) := I_g(R(o, \mathbf{z}))$, $\boldsymbol{\omega} = R\mathbf{z}$, represents pure rotations about an axis with direction $\boldsymbol{\omega} = R\mathbf{z}$, and passing through p. □

Example *5:* Let

$$C(o, \mathbf{z}) = \{\exp(\hat{e}_3\phi_1 + \hat{e}_6\phi_2) \mid \phi_i \in \mathbb{R}, i = 1, 2\},$$

then, $C(p, \boldsymbol{\omega}) := I_g(C(o, \mathbf{z}))$ represents the set of cylindrical motions about an axis passing through p and with direction $\boldsymbol{\omega} = R\mathbf{z}$. Similarly, let

$$PL(\mathbf{z}) = \{\exp(\hat{e}_1\phi_1 + \hat{e}_2\phi_2 + \hat{e}_6\phi_3) \mid \phi_i \in \mathbb{R}, i = 1, 2, 3\}$$

be the set of planar rigid motions about the xy-plane, then $PL(\boldsymbol{\omega}) := I_g(PL(\mathbf{z}))$, $\boldsymbol{\omega} = R\mathbf{z}$, represents the set of planar rigid motions about the plane with normal direction $\boldsymbol{\omega}$. In Example 2, let $S(o) := SO(3)$ be the set of rotations about the origin, and $g = (I, N)$. Then, $S(N) := I_g(S(o))$ represents all rotations about point N. □

Remark *1:* **Comments on Notations**
In the above examples, $R(o, \mathbf{z})$ denotes rotations about the $z-$axis (passing through the origin), and $R(p, \boldsymbol{\omega})$ a generic member of its conjugacy class. We will use the notation $SO(2)$ (or simply R) for the conjugacy class of $R(o, \mathbf{z})$ or $R(p_0, \boldsymbol{\omega})$. $R(o, \mathbf{z})$ is also referred to as the *normal form* of $SO(2)$. When there is no confusion, we will use $SO(2)$ to denote both the conjugacy class and its normal form subgroup. Likewise, the meanings of $\{T(\mathbf{z}), T(\mathbf{v}), T(1)$ or T$\}$, $\{C(o, \mathbf{z}), C(p, \boldsymbol{\omega}), C\}$, $\{PL(\mathbf{z}), PL(\boldsymbol{\omega}), SE(2)\}$ and $\{S(o), S(N), SO(3)\}$ are self-explanatory.

Lie subgroups of $SE(3)$ provide an important class of model spaces for rigid motions generated by mechanisms. An interesting question to ask is, can we classify, *up to a conjugation*, all Lie subgroups of $SE(3)$? J.M. Selig [47] gave a classification directly at the group level. Hervé ([20], [21]) enumerated all subgroups of $SE(3)$, and gave a complete description for all motion types modelled by these subgroups. One can also derive this classification, as in ([2], [19]), by first classifying all Lie subalgebras of $se(3)$ and then applying the exponential map to the results. Figure 1 shows a listing of all 10 types[1] of proper Lie subgroups of $SE(3)$, with dimensions ranging from 1 to 4. Within each box, the upper part denotes the Lie subalgebra of the corresponding Lie subgroup in its normal form; the lower part denotes the conjugacy class of the Lie subgroup. Enclosed in the parenthesis are generic members of the conjugacy class. A line connecting a Lie subgroup to a lower dimensional subgroup indicates inclusion under normal form situations (Note that the relations may fail to hold if the subgroups are not in their normal forms, e.g., $T(\mathbf{v}) \notin C(o, \mathbf{z})$ if $\mathbf{v} \neq \mathbf{z}$).

Of the 10 proper Lie subgroups, there are three 1-dimensional subgroups, namely, $SO(2)$, $T(1)$ and H_ρ that represent screw motions of zero, infinity, and non-zero finite pitches, and provide a model for rigid motions generated by a revolute, prismatic and helical joint of pitch ρ, respectively. These joints are referred to as lower pairs by Reuleaux [46]. Other lower pairs include the cylindrical subgroup C, the

[1]Helical motions of distinct pitches belong to different conjugacy classes. They are related to each other by rescaling. Thus, for simplicity, we will not distinguish them here.

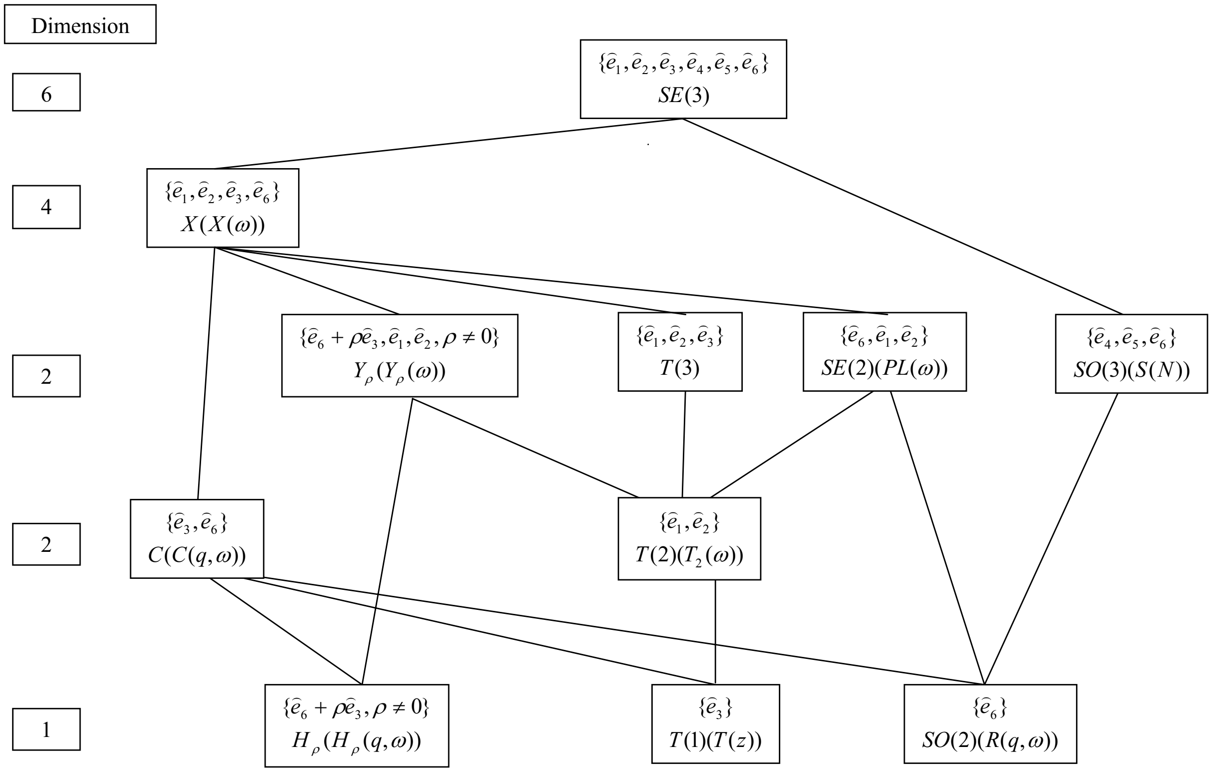

Fig. 1. A classification of Lie subgroups of $SE(3)$. The upper part of each box denotes the Lie subalgebra of the corresponding Lie subgroup in its normal form, and the lower part denotes the conjugacy class. Enclosed in the parenthesis is a generic member of the conjugacy class.

planar subgroup $SE(2)$ and the rotation subgroup $SO(3)$. The four subgroups $T(3)$, $SO(3)$, $SE(2)$ and X are used to model end-effector motions of some common manipulators, e.g, $T(3)$ for Cartesian or the Delta manipulator, and X for the SCARA and the $H4$-parallel manipulator.

There is often a need to determine the intersection of two Lie subgroups (e.g., Hervé [20]). From the property of the exponential map, if the intersection of the Lie algebras of these two Lie subgroups is equal to the Lie algebra of another subgroup, then the intersection agrees with the latter Lie subgroup. We illustrate this point with an example.

Example *6:* Show that the following identity holds, at least in a neighborhood of the identity element e:

$$PL(\boldsymbol{\omega}) \cap S(N) = R(N, \boldsymbol{\omega}).$$

The Lie algebras of $PL(\boldsymbol{\omega})$ and $S(N)$ are given by, respectively,

$$pl(\boldsymbol{\omega}) = \left\{ \begin{bmatrix} \mathbf{u} \\ 0 \end{bmatrix}, \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \boldsymbol{\omega} \end{bmatrix} \right\}$$

and

$$s(N) = \left\{ \begin{bmatrix} N \times \mathbf{u} \\ \mathbf{u} \end{bmatrix}, \begin{bmatrix} N \times \mathbf{v} \\ \mathbf{v} \end{bmatrix}, \begin{bmatrix} N \times \boldsymbol{\omega} \\ \boldsymbol{\omega} \end{bmatrix} \right\},$$

where $\mathbf{u}, \mathbf{v}$ form a basis of the plane perpendicular to $\boldsymbol{\omega}$. Their intersection is simply

$$pl(\boldsymbol{\omega}) \cap s(N) = \left\{ \begin{bmatrix} N \times \boldsymbol{\omega} \\ \boldsymbol{\omega} \end{bmatrix} \right\}$$

which is just the Lie algebra $r(N, \boldsymbol{\omega})$ of $R(N, \boldsymbol{\omega})$. □

Example *7:* **Isotropy Subgroups**
Let $\mathcal{X}$ be a subset of $\mathbb{R}^3$ (e.g., a point, a line or a plane), and define the isotropy subgroup $G_\mathcal{X}$ of $\mathcal{X}$ by

$$G_\mathcal{X} = \{g \in SE(3) \mid g\mathcal{X} = \mathcal{X}\}$$

i.e., $G_\mathcal{X}$ consists of rigid transformations leaving $\mathcal{X}$ fixed or invariant. Let $\mathcal{X}$ be the point N, then $G_\mathcal{X} = S(N)$; $\mathcal{X}$ the line passing through p and with direction $\mathbf{v}$, then $G_\mathcal{X} = C(p, \mathbf{v})$, and $\mathcal{X}$ the plane with unit normal $\boldsymbol{\omega}$, then $G_\mathcal{X} = PL(\boldsymbol{\omega})$. The Schoenflies subgroup $X(\boldsymbol{\omega})$ can be viewed as the isotropy subgroup of a point at infinity. □

Example *8:* $Sim^+(3)$**: The Group of Similarity Transformations**
The group of similarity transformations of $\mathbb{R}^3$, denoted $Sim^+(3)$, consists of matrices of the form

$$\{g \cdot s_\lambda := g \begin{bmatrix} \frac{1}{\lambda} I & 0 \\ 0 & 1 \end{bmatrix} \mid g \in SE(3), \lambda > 0\}.$$

Here, s_λ is a scaling transformation. Let $g_\lambda \in Sim^+(3)$, then $\|g_\lambda(q_1 - q_2)\| = \frac{1}{\lambda}\|q_1 - q_2\|$, and g_λ transforms any Euclidean figure (subset of points) into a proportional (or similar) figure. It is clear that $SE(3)$ is a normal subgroup of $Sim^+(3)$ and differs from it by a scaling transformation. We will need the similarity group to give a precise definition of *motion type* later. □

Example *9:* **Conjugcy Subgroups Under Similarity Transformations**
Note that a transformation g in $SE(3)$ leaves the pitch of a screw motion invariant. Thus, helical motions of distinct pitches belong to different conjugacy classes under elements

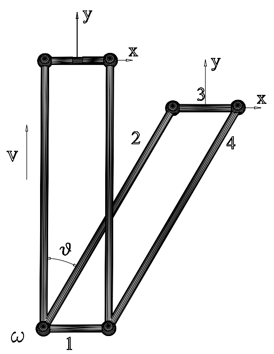

Fig. 2. Composite joint: A plane-hinged Parallelogram

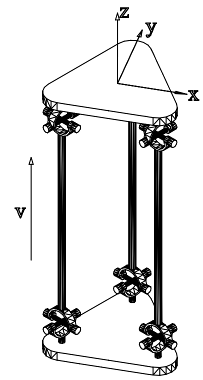

Fig. 3. Composite joint: A $\mathcal{U}^*$-joint

of $SE(3)$. On the other hand, a similarity transformation $g_\lambda \in Sim^+(3)$ leaves all but the helical H_ρ and the Y_ρ subgroups of $SE(3)$ (see Figure 1) invariant. In the latter cases, we have

$$g_\lambda H_\rho(o, \mathbf{z}) g_\lambda^{-1} = H_{\rho/\lambda}(p, \boldsymbol{\omega}), \quad \text{and} \quad g_\lambda Y_\rho(\mathbf{z}) g_\lambda^{-1} = Y_{\rho/\lambda}(\boldsymbol{\omega}).$$

Hence, there are exactly 10 conjugacy subgroups of $SE(3)$ *under the group of similarity transformations.* □

B. Submanifolds and Quotient Spaces of $SE(3)$

The Lie subgroups of $SE(3)$ shown in Figure 1 can be used to model the majority of rigid motions generated by primitive joints and end-effectors of robotic manipulators. There are, however, important exceptions in which the generated rigid motions can not be modelled by Lie subgroups. There are obvious needs to broaden the class of modelling spaces to include *submanifolds* of $SE(3)$, e.g., (1) End-effector motions of a 5-axis machine tool; (2) End-effector motions of a helicopter simulator with 1-DoF translation and 3-DoF rotations; and (3) rigid motions generated by composite joints, such as the plane-hinged parallelogram shown in Figure 2, and the $\mathcal{U}^*$-joint shown in Figure 3. In the first case, observe from Figure 1 that there is no Lie subgroup of dimension 5 at all. Hence, no 5-DoF motions can be modelled by Lie subgroups of $SE(3)$. In the remaining cases, we will see that the model spaces for these rigid motions form only submanifolds of $SE(3)$.

According to [4], there are three types of submanifolds (immersed, embedded, and regular), with the last being the most natural and important (actually the last two types of submanifolds are the same). We will use *regular submanifolds* as our extended model spaces to complement that of Lie subgroups. Hence, a submanifold in the sequel will be understood as a regular submanifold.

Example *10:* Consider the plane-hinged parallelogram in Figure 2, that is utilized in the Delta manipulator design. Let $\boldsymbol{\omega}$ be the unit vector normal to the parallelogram, and $\mathbf{v}$ the vector of link 2 or 4 at the home position, then the set of rigid motions generated by the parallelogram has the form

$$P_a(\boldsymbol{\omega}, \mathbf{v}) = \left\{ \begin{bmatrix} I & (e^{\hat{\omega}\theta} - I)\mathbf{v} \\ 0 & 1 \end{bmatrix} \mid \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \right\}.$$

One can show that $P_a(\boldsymbol{\omega}, \mathbf{v})$ is diffeomorphic to an open subset of S^1 in $\mathbb{R}^3$, and thus an 1-dimensional submanifold of $SE(3)$. A parallelogram produces linear motions along an open subset of the unit circle S^1 if $\|\mathbf{v}\| = 1$. □

Example *11:* A U-joint consists of two consecutive $\mathcal{R}$ joints with orthogonally intersecting axes. The set of rigid motions generated by a U-joint that has its axes initially aligned with the $x-$ and $y-$ axes has the form

$$\begin{aligned} U(o, x, y) &= R(o, \mathbf{x}) \cdot R(o, \mathbf{y}) \\ &= \left\{ \begin{bmatrix} e^{\hat{x}\alpha} e^{\hat{y}\beta} & 0 \\ 0 & 1 \end{bmatrix}, \alpha, \beta \in [0, 2\pi] \right\}, \end{aligned}$$

which is a 2-dimensional submanifold of $SO(3)$ and thus $SE(3)$. □

Example *12:* A generalization of the plane-hinged parallelogram to 3 dimensions is the $\mathcal{U}^*$-joint introduced in [1] and [10]. The $\mathcal{U}^*$-joint consists of three identical subchains symmetrically arranged in parallel as shown in Figure 3. Each subchain in turn consists of two U-joints, where the second and the third $\mathcal{R}$ joints and the first and the fourth $\mathcal{R}$ joints are, respectively, parallel to each other. The configuration space of the $\mathcal{U}^*$-joint has the form

$$\begin{aligned} U^*(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \mathbf{v}) \quad = \quad &\{ \begin{bmatrix} I & (e^{\hat{\omega}_1\alpha} e^{\hat{\omega}_2\beta} - I)\mathbf{v} \\ 0 & 1 \end{bmatrix}, \\ &\alpha \in (0, 2\pi), \beta \in (-\pi/2, \pi/2)\} \end{aligned}$$

where $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ are the direction vectors of the first two axes of the subchain, and $\mathbf{v}$ is the direction vector of the link that connects the two U-joints of the subchain, all at the home position. For simplicity, we assume that $\boldsymbol{\omega}_1 \perp \boldsymbol{\omega}_2 \perp \mathbf{v}$. Note that $U^*(\omega_1, , \omega_2, \mathbf{v})$ is diffeomorphic to an open subset of S^2 in $\mathbb{R}^3$, and thus a 2-dimensional submanifold of $SE(3)$. The $\mathcal{U}^*$-joint generates 2-DoF linear motions along the unit sphere S^2 provided that $\|\mathbf{v}\| = 1$. □

There are two special families of regular submanifolds that are of paramount importance in our study. They are referred to as *category I and II submanifolds*, respectively, and are given by Proposition 1 and 2.

Proposition *1:* **Category I Submanifolds**

Let M_1 and M_2 be a regular submanifold of $T(3)$ and $SO(3)$, of dimension n_1 and n_2, respectively. Then, $M_1 \cdot M_2$ (or $M_2 \cdot M_1$) is a regular submanifold of $SE(3)$, of dimension $n_1 + n_2$.

Remark: The $\cdot$ product above is to be viewed as the product of two sets of homogeneous transformations. In general, $M_1 \cdot$

Subgroups:				**Submanifolds:**		
DoF	Normal form	Conjug.	Applications	Normal form	Conjug.	Applications
2	$T_2(\mathbf{z})$	$T_2(\mathbf{v})$	Planar Cart. motion	$R(o,\mathbf{z})T(\mathbf{x})$	$R(p,\boldsymbol{\omega})T(\mathbf{v})$	
	$C(o,\mathbf{z})$	$C(p,\boldsymbol{\omega})$	Cylindrical motion	$U(o,\mathbf{x},\mathbf{y})$	$U(p,\boldsymbol{\omega}_1,\boldsymbol{\omega}_2)$	Pan-tilting motion
3	$T(3)$	$T(3)$	Spatial Cart. motion	$U(o,\mathbf{x},\mathbf{y})T(\mathbf{z})$	$U(p,\boldsymbol{\omega}_1,\boldsymbol{\omega}_2)T(\mathbf{v})$	Tricept's proximal 3-DoF module
	$PL(\mathbf{z})$	$PL(\boldsymbol{\omega})$	Planar rigid motion	$U(o,\mathbf{x},\mathbf{y})T(\mathbf{z})$	$U(p,\boldsymbol{\omega}_1,\boldsymbol{\omega}_2)T(\mathbf{v})$	Telescopic leg
	$S(o)$	$S(N)$	Rotational motions	$R(o,\mathbf{z})T_2(\mathbf{y})$	$R(p,\boldsymbol{\omega})T_2(\mathbf{v})$	Wafer handling
4	$X(\mathbf{z})$	$X(\boldsymbol{\omega})$	SCARA motion	$T_2(\mathbf{z})U(o,\mathbf{x},\mathbf{y})$	$T_2(\mathbf{v})U(p,\boldsymbol{\omega}_1,\boldsymbol{\omega}_2)$	Haptic device
				$T_2(\mathbf{z})U(o,\mathbf{z},\mathbf{x})$	$T_2(\boldsymbol{\omega}_1)U(p,\boldsymbol{\omega}_1,\boldsymbol{\omega}_2)$	
				$T(\mathbf{z})S(o)$	$T(\mathbf{v})S(N)$	Helicopter simulator
5				$T(3)U(o,\mathbf{x},\mathbf{y})$	$T(3)U(p,\boldsymbol{\omega}_1,\boldsymbol{\omega}_2)$	5-axis machining
				$T_2(\mathbf{z})S(o)$	$T_2(\mathbf{v})S(N)$	

TABLE I

MODELS OF END-EFFECTOR MOTIONS BY LIE SUBGROUPS AND CATEGORY I SUBMANIFOLDS OF $SE(3)$

$M_2 \neq M_2 \cdot M_1$.

Proof: See Appendix A.

In conjunction to Lie subgroups, certain category I submanifolds of $SE(3)$ are used to provide models for end-effector (or task space) motions of a manipulator. For example, by taking the product of the submanifolds $(T(\mathbf{v}), T_2(\mathbf{u}), T(3))$ of $T(3)$ with the submanifolds $(R(o,\boldsymbol{\omega}), U(o,\boldsymbol{\omega}_1,\boldsymbol{\omega}_2), S(o))$ of $SO(3)$, we arrive at the model spaces given by the right half column of Table I, whereas the left half column displays commonly used Lie subgroup motion of dimensions ranging from 2 to 4.

Example *13:* **Model spaces for task motions** The set of rigid motions generated by a five-axis machine tool can be described by

$$T(3) \cdot U(o,\mathbf{x},\mathbf{y})$$

or in a general configuration by

$$I_g(T(3) \cdot U(o,\mathbf{x},\mathbf{y})) = T(3) \cdot U(q,\boldsymbol{\omega}_1,\boldsymbol{\omega}_2), \quad \boldsymbol{\omega}_1 \perp \boldsymbol{\omega}_2.$$

Similarly, the set of rigid motions generated by a wafer handling robot can be modelled by

$$T(\mathbf{z})R(o,\mathbf{z})T(\mathbf{x}) = R(o,\mathbf{z})T_2(\mathbf{y})$$

□

Proposition *2:* **Category II Submanifolds** *Let H_1 and H_2 be a closed Lie subgroup of $SE(3)$, of dimension n_1 and n_2, respectively. Let $H = H_1 \cap H_2$, and $n = dim(H)$. Then, the product $H_1 \cdot H_2$ is a regular submanifold of $SE(3)$ of dimension $n_1 + n_2 - n$.*

Proof: See appendix B.

Note that if H_1 and H_2 commute, then $H_1 \cdot H_2$ is a Lie subgroup of $SE(3)$.

Example *14:* From Figure 1, the following Lie subgroups can be factorized into

$$\begin{aligned}
&C(o,\mathbf{z}) = T(\mathbf{z})R(o,\mathbf{z}), \quad T_2(\mathbf{z}) = T(\mathbf{x})T(\mathbf{y}),\\
&Y_\rho(\mathbf{z}) = H_\rho(o,\mathbf{z})T_2(\mathbf{z}), \quad T(3) = T_2(\mathbf{z})T(\mathbf{z}),\\
&PL(\mathbf{z}) = T_2(\mathbf{z})R(o,\mathbf{z}),\\
&X(\mathbf{z}) = T(3)R(o,\mathbf{z}) = PL(\mathbf{z})T(\mathbf{z}) = C(\mathbf{z})T_2(\mathbf{z})
\end{aligned}$$

The order of the products above can be reversed. □

If H_1 and H_2 do not commute, then $H_1 \cdot H_2$ and $H_2 \cdot H_1$ are distinct submanifolds. If $H_1 \cap H_2$ is nontrivial, one may use the factorizations of Example 14 to eliminate the common component and obtain a simpler ("irreducible", or regular [17]) representation of the product.

Example *15:* The product $C(o,\mathbf{z})X(\mathbf{x})$ can be simplified into

$$\begin{aligned}
C(o,\mathbf{z})X(\mathbf{x}) &= (R(o,\mathbf{z})T(\mathbf{z})) \cdot (T(3)R(o,\mathbf{z}))\\
&= R(o,\mathbf{z})T(3)R(o,\mathbf{x})\\
&= T(3)R(o,\mathbf{z})R(o,\mathbf{x})\\
&= T(3)U(o,\mathbf{z},\mathbf{x}),
\end{aligned}$$

or in a general position, for $\mathbf{u} \neq \mathbf{v}$,

$$C(p,\mathbf{u})X(\mathbf{v}) = T(3)U(p,\mathbf{u},\mathbf{v}).$$

The product $S(o)S(N)$ is a five dimensional submanifold since $S(o) \cap S(N) = R(o, N/\|N\|)$. However, this product can not be further simplified as $S(o)$ is not trivial. □

In general, *the product of three or more subgroups may not be a regular submanifold because of possible singularities*. For example, $S(o)$ is not equal to $R(o,\mathbf{x}) \cdot R(o,\mathbf{y}) \cdot R(o,\mathbf{z})$ because the latter is not a regular submanifold.

Two subgroups are said to be dependant if their intersection is nontrivial. The product of two dependant subgroups is referred to, by Hervé [17], as a *dependant product*. Dependant products were first studied in ([9], [18]) and applied to synthesis of parallel pan-tilt wrists in [17]. As shown in [17], there are all together 25 types of dependant products, which in their normal forms are displayed in Table II. To derive this list, one can use the inclusion relations of Figure 1 to identify all subgroups containing the desired intersection subgroups and then take their products.

Example *16:* To derive the dependant products $P14 \sim P18$, we see from Figure 1 that $C(o,\mathbf{z}), S(o), S(N)$ ($N = \alpha z$), $PL(\mathbf{z})$ and $X(\mathbf{z})$ contain $R(o,\mathbf{z})$. Take the products of these subgroups leading to the desired results. □

Finally, we introduce the notion of quotient spaces, or manifolds, of a Lie group $G \subset SE(3)$ by a subgroup $H \subset G$. Two elements g_1 and g_2 in G are said to be equivalent, $g_1 \sim g_2$, if $g_1 g_2^{-1} \in H$. The equivalent classes are called cosets, and the space of cosets for such a relation is called the *quotient space* of G by H, denoted G/H. G/H has the structure of a differentiable manifold of dimension (dim(G) - dim(H)), and is also referred to as a homogeneous space. When H is a normal subgroup, i.e., $gHg^{-1} = H, \quad \forall g \in G$,

No.	DoF	Product	Irreduc. rep.	Realiz.
Intersection subgroup: $T(\mathbf{z})$				
$P1$	3	$T_2(\mathbf{x})T_2(\mathbf{y})$	$T(3)$	1(8)
P2		$C(o,\mathbf{z})C(p,\mathbf{z})$	$C(o,\mathbf{z})R(p,\mathbf{z})$	24(32)
P3		$C(o,\mathbf{z})T_2(\mathbf{x})$	$R(o,\mathbf{z})T_2(\mathbf{x})$	7(18)
p4	4	$C(o,\mathbf{z})Y_\rho(\mathbf{x})$	$R(o,\mathbf{z})Y_\rho(\mathbf{x})$	29(57)
P5		$C(o,\mathbf{z})T(3)$	$X(\mathbf{z})$	7(34)
P6		$C(o,\mathbf{z})PL(x)$	$U(o,\mathbf{z},\mathbf{x})T_2(\mathbf{x})$	29(57)
P7		$T_2(\mathbf{x})PL(\mathbf{y})$	$X(\mathbf{y})$	7(42)
P8		$T_2(\mathbf{x})Y_\rho(\mathbf{y})$		7(42)
P9	5	$C(o,\mathbf{z})X(\mathbf{x})$	$T(3)U(o,\mathbf{z},\mathbf{x})$	288(708)
P10		$PL(\mathbf{x})Y_\rho(\mathbf{y})$	$T(3)U(o,\mathbf{x},\mathbf{y})$	42(160)
P11		$PL(\mathbf{x})PL(\mathbf{y})$		
P12		$Y_{\rho_1}(\mathbf{x})Y_{\rho_2}(\mathbf{y})$		
Intersection subgroup: $H_\rho(o,\mathbf{z})$				
P13	4	$C(o,\mathbf{z})Y_\rho(\mathbf{z})$	$X(\mathbf{z})$	34(77)
Intersection subgroup: $R(o,\mathbf{z})$				
P14	4	$C(o,\mathbf{z})S(o)$	$T(\mathbf{z})S(o)$	7
P15		$C(o,\mathbf{z})PL(\mathbf{z})$	$X(\mathbf{z})$	34(77)
P16	5	$PL(\mathbf{z})S(o)$	$T_2(\mathbf{z})S(o)$	11(25)
P17		$S(o)S(N)$		2
P18	6	$S(o)X(\mathbf{z})$	$SE(3)$	79(219)
Intersection subgroup: $T_2(\mathbf{z})$				
P19	4	$Y_{\rho_1}(\mathbf{z})Y_{\rho_2}(\mathbf{z})$	$X(\mathbf{z})$	17(37)
P20		$Y_\rho(\mathbf{z})T(3)$		9(25)
P21		$Y_\rho(\mathbf{z})PL(\mathbf{z})$		17(37)
P22		$T(3)PL(\mathbf{z})$		9(25)
P23	5	$PL(\mathbf{z})X(\mathbf{x})$	$T(3)U(o,\mathbf{z},\mathbf{x})$	127(363)
P24		$Y_\rho(\mathbf{z})X(\mathbf{x})$		
Intersection subgroup: $T(3)$				
P25	5	$X(\mathbf{z})X(\mathbf{x})$	$T(3)U(o,\mathbf{z},\mathbf{x})$	564(1452)

TABLE II
CATEGORY II SUBMANIFOLDS OF $SE(3)$: 25 PRODUCTS OF DEPENDANT SUBGROUPS

G/H becomes a Lie group. For example, with $G = SE(3)$ and $H = T(3)$, G/H can be identified with $SO(3)$. Note that G/H is not a submanifold of $SE(3)$. But, it can be locally identified with a submanifold M of $SE(3)$ in a neighborhood of e. To be more precise, we let $G = PL(\mathbf{z})$ and $H = T(\mathbf{x})$, and explore the various ways in which G/H can be identified with submanifolds of G and hence $SE(3)$. Other cases as implied by Figure 1 can be dealt with in a similar manner.

We are interested in finding 2-dimensional submanifolds M containing e such that the map

$$\phi : M \longrightarrow G/H : m \longmapsto m \cdot H \qquad (6)$$

is a local diffeomorphism near e. We call such M a local complement of H in G. By the inverse mapping theorem, (6) is a local diffeomorphism near e if and only if

$$\phi_* : T_e M \longrightarrow T_{e\cdot H} G/H$$

is an isomorphism. Identify $T_{e\cdot H}G/H$ with $\mathfrak{g}/\mathfrak{h}$, and let $V := T_e M \subset \mathfrak{g}$. Then, ϕ is a local diffeomorphism if and only if

$$V \longrightarrow \mathfrak{g}/\mathfrak{h} : v \longmapsto v + h$$

is a vector space isomorphism, i.e., V is a complement of $\mathfrak{h}$ in $\mathfrak{g}$. On the other hand, given a complement V of $\mathfrak{h}$ in $\mathfrak{g}$, there can be many $M \subset G$ such that $T_e M = V$. Thus, it is not possible to classify all local complements of H in G. However, with a basis (v_1, v_2) of V, we can use coordinates of the second kind to obtain a local complement M by letting M to be the image of the map

$$\mathbb{R}^2 \longrightarrow G : (\theta_1, \theta_2) \longmapsto \exp(\theta_1 v_1)\exp(\theta_2 v_2)$$

for (θ_1, θ_2) lying in an open neighborhood of $(0,0) \subset \mathbb{R}^2$. For this reason, we only need to classify pairs $(V, (v_1, v_2))$, where V is a complement of $\mathfrak{h}$ in $\mathfrak{g}$, and (v_1, v_2) is an (ordered) basis of V. It is clear that if (v_1, v_2) is a basis of V, and $(\lambda_1, \lambda_2) \in \mathbb{R}^2$, with $\lambda_1 \neq 0, \lambda_2 \neq 0$, then $M_{(V,(v_1,v_2))}$ coincides with $M_{(V,(\lambda_1 v_1, \lambda_2 v_2))}$ near e. Thus, we only need to classify pairs $(V, [v_1, v_2])$, where V is a complement of $\mathfrak{h}$ in $\mathfrak{g}$ and $[v_1, v_2]$ is an equivalent class of basis defined up to rescaling.

Let $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$ be the normalizer subgroup of H. $N_G(H) = T_2(z)$ in our case. Clearly, if V is a complement of $\mathfrak{h}$ in $\mathfrak{g}$ with a basis (v_1, v_2) and $g \in N_G(H)$, then $Ad_g V$ is another such complement and $(Ad_g v_1, Ad_g v_2)$ is a basis of $Ad_g V$. Thus, it suffices to classify pairs $(V, [v_1, v_2])$ up to conjugations by elements in $T_2(\mathbf{z})$.

For $p = (p_x, p_y) \in \mathbb{R}^2$, let $T_p \in T_2(\mathbf{z})$ denote the transformation of translation in the $xy-$plane by p. Using the canonical basis $(\widehat{e}_1, \cdots \widehat{e}_6)$ of $se(3)$, we have

$$\mathfrak{g} = \{\widehat{e}_1, \widehat{e}_2, \widehat{e}_6\}, \mathfrak{h} = \{\widehat{e}_1\}, t_2(z) = \{\widehat{e}_1, \widehat{e}_2\}$$

and $\widehat{e}_6$ generates $R(o, \mathbf{z})$. We state the classification results for G/H through a series of four lemmas, which are proved in Appendix C.

Lemma *1:* Every complement V of $\mathfrak{h}$ in $\mathfrak{g}$ has a basis of the form

$$v_1 = \widehat{e}_6 + a\widehat{e}_1, \quad v_2 = \widehat{e}_2 + b\widehat{e}_1$$

for some $a, b \in \mathbb{R}$.

Lemma *2:* Every complement of $\mathfrak{h}$ in $\mathfrak{g}$ is conjugate by an element in $T_2(\mathbf{z})$ to an V_b of the form

$$V_b = \{\widehat{e}_6, \widehat{e}_2 + b\widehat{e}_1\}$$

for some $b \in \mathbb{R}$.

Lemma *3:* (i) For $b, c \in \mathbb{R}$. If $Ad_{T_p} V_b = V_c$ for some $p \in \mathbb{R}^2$, then $b = c$. (ii) For $b \in \mathbb{R}$ and $p = (p_x, p_y) \in \mathbb{R}^2$, $Ad_{T_p} V_b = V_b$ if and only if $p_y = -bp_x$.

Lemma *4:* Given $b \in \mathbb{R}$ and under scaling and conjugation by elements of the form T_p, where $p = \lambda(1, -b)$, $\lambda \in \mathbb{R}$, every basis of V_b can be transformed into one of the following three cases:

$$(\widehat{e}_2 + b\widehat{e}_1, \widehat{e}_6), \quad (\widehat{e}_6, \widehat{e}_2 + b\widehat{e}_1)$$
$$(\widehat{e}_6, \widehat{e}_6 + \lambda(\widehat{e}_2 + b\lambda\widehat{e}_1))$$

To summarize, we conclude that, by using coordinates of the second kind, we obtain the following three types of local complements of $H = T(\mathbf{x})$ in $G = PL(\mathbf{z})$:

$$\begin{array}{lrcl} 1) & M & = & \exp(\theta_1(\widehat{e}_2 + b\widehat{e}_1))\exp(\theta_2\widehat{e}_6) \\ & & = & T(\mathbf{u})R(o,\mathbf{z}), \quad \text{with} \quad \mathbf{u} = \mathbf{y} + b\mathbf{x}; \\ 2) & M & = & R(o,\mathbf{z})T(\mathbf{u}); \\ 3) & M & = & \exp(\theta_1\widehat{e}_6)\exp(\theta_2(\widehat{e}_6 + \lambda(\widehat{e}_2 + b\widehat{e}_1)), \quad \lambda \neq 0 \\ & & = & R(o,\mathbf{z})R(N,\mathbf{z}), \quad \text{with} \quad N = \lambda(-1, b, 0). \end{array}$$

In an analogous manner, we can derive classifications for other types of quotient manifolds as implied by Figure 1. The results are shown in Table V.

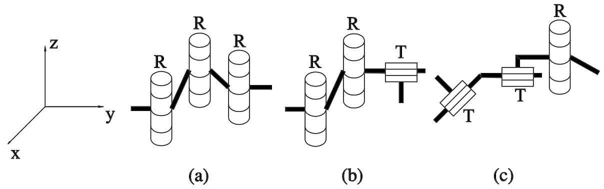

Fig. 4. Equivalent Generators of $PL(\mathbf{z})$

DoF	Notation (G)	Twist Rep. ($\mathcal{G}$)	Remark
1	$\mathcal{T}(\mathbf{v})$ ($\mathcal{T}$)	$(\mathbf{v}, 0)$	Prismatic joint
	$\mathcal{R}(p, \boldsymbol{\omega})$ ($\mathcal{R}$)	$(p \times \boldsymbol{\omega}, \boldsymbol{\omega})$	Revolute joint
	$\mathcal{H}_\rho(p, \boldsymbol{\omega})$ ($\mathcal{H}$)	$(p \times \boldsymbol{\omega} + \rho\boldsymbol{\omega}, \boldsymbol{\omega})$	Helical joint
	$\mathcal{P}_{\mathfrak{a}}(\boldsymbol{\omega}, \mathbf{v})$ ($\mathcal{P}_{\mathfrak{a}}$)	$(\boldsymbol{\omega} \times \mathbf{v}, 0)$	Parallelogram
2	$\mathcal{U}^*(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \mathbf{v})$ ($\mathcal{U}^*$)	$(\boldsymbol{\omega}_1 \times \mathbf{v}, 0)$, $(\boldsymbol{\omega}_2 \times \mathbf{v}, 0)$	$\mathcal{U}^*$-joint

TABLE III

PRIMITIVE MOTION GENERATORS

III. KINEMATIC SYNTHESIS OF SERIAL MANIPULATORS

Using the mathematical tools developed in the previous section, we present in this section a formal geometric theory for kinematic synthesis of serial manipulators(or subchains). This includes a precise characterization of *motion type* and primitive motion generators (PMGs), and synthesis of subgroup and submanifold motion generators. The results of this section will be applied in the next section to the synthesis of parallel manipulators.

A. Motion Type and Primitive Motion Generators

The notion of *motion type* has been widely used in the mechanism community but never precisely defined. Intuitively, the *motion type* of a mechanism in a neighborhood of a reference configuration describes the *pattern* of the set of motions generated by the mechanism over this neighborhood. This notion should be independent of the choice of length scales (or sizes) and the coordinate frame used to represent the set of motions. In other words, rather than describing a particular set of motion, a *motion type* describes a class of motions or more exactly a conjugacy class of motions under the group $Sim^+(3)$ of similarity transformations (invariant under change of length scales and coordinate frames).

Definition *3:* **Motion Type**
Let $\mathcal{M}$ be a mechanism that consists of a system of coupled rigid bodies, one of which is identified as the base and one as the end-effector. Choose a reference configuration of $\mathcal{M}$ and identify the joint variables with zero. Attach a coordinate frame to the end-effector and denote by $C_\mathcal{M}$ the set of rigid motions generated (or attainable) by the end-effector relative to the reference configuration, i.e., $e \in C_\mathcal{M}$. Let Q_0 be a normal form subgroup or submanifold of $SE(3)$ and $Q = g_r Q_0 g_r^{-1}$, the conjugacy class of Q_0 under $Sim^+(3)$. $\mathcal{M}$ is said to have the *motion type* (or *finite motion property*) of Q if there exists $g_r \in Sim^+(3)$ such that $g_r C_\mathcal{M} g_r^{-1}$ agrees with Q_0 in an open neighborhood $U \subset SE(3)$ of e, i.e.,

$$(g_r C_\mathcal{M} g_r^{-1}) \cap U = Q_0 \cap U \tag{7}$$

Remark *2:* (i) We require that Q_0 be a regular submanifold [2] so that $\tilde{U} := Q_0 \cap U$ is an open set (of Q_0) around e , with the differentiable structure consistent with that of Q_0; (ii) If Q_0 is a Lie subgroup, and $\mathcal{M}$ has the motion type of Q, then the inverse mechanism $\mathcal{M}^{-1}$ also has the motion type of Q; (iii) For notational simplicity, we will use Q for Q_0 when its meaning is understood from the context, and call $\mathcal{M}$ a motion generator of Q.

[2]Recall that Lie subgroups are also regular submanifolds.

Example *17:* **Primitive Motion Generators (PMGs)**
Rigid motions generated by a revolute pair $\mathcal{R}(p, \boldsymbol{\omega})$ with joint limits $-\theta_\mathcal{M} < \theta < \theta_\mathcal{M}$ is given by

$$C_\mathcal{M} = \{\exp(\widehat{\xi}\theta) \mid -\theta_\mathcal{M} < \theta < \theta_\mathcal{M}\}$$

where $\xi = (p \times \omega, \omega) \in \mathbb{R}^6$ is the twist associated with the joint axis. Clearly, $C_\mathcal{M}$ can be transformed by an element of $SE(3)$ to agree with $R(o, \mathbf{z})$ in an open neighborhood of the identity. Thus, a revolute pair is said to have the motion type of $SO(2)$ or simply R. Similarly, rigid motions of a helical pair $\mathcal{H}_\rho(p, \boldsymbol{\omega})$ with non-zero pitch ρ can be transformed by an element $g_r \in Sim^+(3)$ to agree with $H_1(o, \mathbf{z})$, the helical subgroup of unit pitch, in an open neighborhood of the identity. Hence, a helical pair is said to have the motion type of H. $\mathcal{R}, \mathcal{H}$ and the prismatic joint $\mathcal{T}$ are referred to as the conventional PMGs. □

Example *18:* **Extended PMGs**
As shown in Example 10, rigid motions generated by a plane-hinged parallelogram has the form $P_a(\boldsymbol{\omega}, \mathbf{v})$. It corresponds to translations along a circle of radius $\|\mathbf{v}\|$. This set can be transformed by a similarity transformation to agree with $P_a(\mathbf{z}, \mathbf{y})$ in an open neighborhood of the identity. Thus, a plane-hinged parallelogram $\mathcal{P}_{\mathfrak{a}}(\boldsymbol{\omega}, \mathbf{v})$ is said to have the motion type of P_a. Similarly, the $\mathcal{U}^*$-joint of Example 12 is said to have the motion type of U^*, the conjugacy class of the submanifold $U^*(\mathbf{x}, \mathbf{y}, \mathbf{z})$. $\mathcal{P}_{\mathfrak{a}}$- and $\mathcal{U}^*$-joints are referred to as the extended PMGs. □
In this paper, we will study mechanism synthesis using both conventional set of PMGs $(\mathcal{R}, \mathcal{T}, \mathcal{H})$, and extended set of PMGs $(\mathcal{R}, \mathcal{T}, \mathcal{H}, \mathcal{P}_{\mathfrak{a}}, \mathcal{U}^*)$. The twist representation of a PMG G is denoted by $\mathcal{G}$. One can refer to Table III for an explicit expression of $\mathcal{G}$ for all PMGs.

Example *19:* $SE(2)-$**Motion Generators**
Let $\mathscr{P} = (\mathcal{T}(\mathbf{v}), \mathcal{R}(p, \mathbf{z}))$ be a set of PMGs that lie in $PL(\mathbf{z})$. Consider cascading in series three such PMGs to form a serial kinematic chain

$$\mathcal{M} = G_1 \cdot G_2 \cdot G_3 \tag{8}$$

where $G_i \in \mathscr{P}, i = 1, 2, 3$. The product above is the product of three conjugacy classes. The mechanism is said to have the motion type of $SE(2)$ if there exists a configuration in which the set of rigid motions generated by $\mathcal{M}$ agrees with $PL(\mathbf{z})$ in a neighborhood of the identity. Clearly, the mechanism

$$\mathcal{M}_1 = \mathcal{T}(\mathbf{u}) \cdot \mathcal{T}(\mathbf{v}) \cdot \mathcal{R}(p, \mathbf{z})$$

with the geometric condition that $\mathbf{u} \nparallel \mathbf{v}$ is a $SE(2)-$ motion generator because

$$\overline{M}_1 := \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\} = \{t(u), t(v), r(p, z)\} = pl(z). \quad (9)$$

Here, the geometric condition guarantees that the mechanism is not at a singular configuration, a situation which may happen with the product of more than three PMGs. In a similar manner, the mechanism $\mathcal{M}_2 = \mathcal{T}(\mathbf{u}) \cdot \mathcal{R}(p, \mathbf{z}) \cdot \mathcal{T}(\mathbf{v})$ with a similar geometric condition is also a $SE(2)$-motion generator. In general, the problem of finding all possible $SE(2)$-motion generators (topologies or realizations) with a given set of PMGs is a combinatorial problem, subjecting to the geometric condition imposed by (9). Eliminating the case with all three prismatic joints, we obtain a total of seven topological configurations that generate $SE(2)$ motions. Figure 4 displays three such cases. These topological configurations are invariant under conjugations by elements of $PL(\mathbf{z})$ and rescaling (or the group $Sim^+(2)$). □

For the mechanism in Figure 4(b), the prismatic joint of the $(\mathcal{R} \cdot \mathcal{R} \cdot \mathcal{T})$-chain can be replaced by a parallelogram joint as long as the unit circle of the $\mathcal{P}_{\mathfrak{a}}$-joint lies in the $xy-$plane and the following geometric condition is maintained at the chosen configuration

$$\{\xi_1, \xi_2, \overline{P_a}\} = pl(\mathbf{z}).$$

By a similar argument, we can replace any (or both) of the $\mathcal{T}$ joints in Figure 4(c) by a $\mathcal{P}_{\mathfrak{a}}$-joint(s) without affecting the results. In other words, there are a total of 21 distinct topologies having the motion type of $SE(2)$ with $\mathscr{P}^e = (\mathcal{T}, \mathcal{R}, \mathcal{P}_{\mathfrak{a}})$.

We now formulate the general synthesis problem of serial manipulators having the motion type of Q. Let $\mathscr{P}$ be a pre-specified set of PMGs in Q.

Problem *1:* **Serial Kinematic Synthesis**
Find the set of all serial mechanisms of the form $\mathcal{M} = G_1 \cdot G_2 \cdots G_l$, with $G_i \in \mathscr{P}, i = 1, \cdots l$, and $DoF(\mathcal{M}) := \sum_{i=1}^{l} DoF(G_i) = Dim(Q)$, such that at a certain configuration M has the motion type of Q.
Note that redundant manipulators are not considered here, with the requirement that $DoF(\mathcal{M}) = Dim(Q)$.

We separate solutions of the synthesis problem into a Lie subgroup case and a regular submanifold case.

B. Subgroup Motion Generators

Proposition *3: Let Q be a Lie subgroup of dimension n. $\mathcal{M}$ is a motion generator of Q if and only if $DoF(\mathcal{M}) = n$ and there exists a configuration such that $\overline{\mathcal{M}} := \{\mathcal{G}_1, \cdots, \mathcal{G}_l\} = T_e Q$.*

Synthesis of some of the Lie subgroups given in Figure 1 are discussed through the following examples.

Example *20:* **Synthesis of $C(o, \mathbf{z})$-subchains**
The Lie algebra of $C(o, \mathbf{z})$ is spanned by $(\mathbf{z}^T, 0)^T$ and $(0, \mathbf{z}^T)^T$. Thus, by Proposition 3, all the serial mechanisms in Figure 5 generate $C(o, \mathbf{z})$. Rearranging the order of the primitive generators in Figure 5 yields a total of 6 realizations for $C(o, \mathbf{z})$. Note that here the parallelogram joint can not be used to replace the prismatic joint as the unit circle S^1 of the $\mathcal{P}_{\mathfrak{a}}$-joint can not be embedded in $C(o, \mathbf{z})$. □

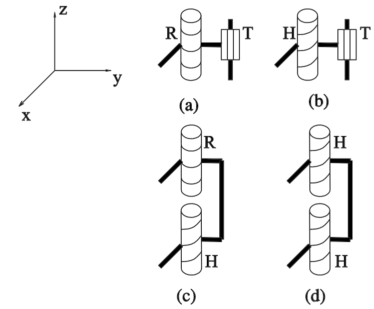

Fig. 5. Equivalent Generators of $C(o, \mathbf{z})$

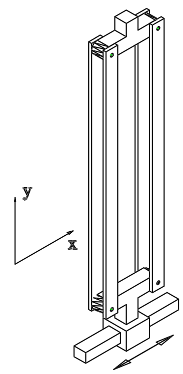

Fig. 6. A Generator of $T_2(\mathbf{z})$

Example *21:* **Synthesis of $T_2(\mathbf{z})$ and $T(3)$ subchains**
It is easy to see that cascading two prismatic pairs, or a prismatic pair with a parallelogram joint, as in Figure 6, generates $T_2(\mathbf{z})$ (Note that the S^1 of the parallelogram joint should lie in $T_2(\mathbf{z})$). Similarly, cascading three prismatic pairs, or a prismatic pair with a $\mathcal{U}^*$-pair, as in Figure 7, generates $T(3)$. The order of the constituting pairs can be interchanged to yield new realizations. □

Example *22:* **Synthesis of $X(\mathbf{z})$-subchains**
Since

$$X(\mathbf{z}) = T(3) \cdot R(p, \mathbf{z}) = PL(\mathbf{z}) \cdot T(\mathbf{z}) = T_2(\mathbf{z}) \cdot C(p, \mathbf{z}),$$

the simplest way to synthesize an $X(\mathbf{z})$ subchain is by cascading in series a $T(3)$ generator with a revolute joint $\mathcal{R}(p, \mathbf{z})$, a $PL(\mathbf{z})$ generator with a prismatic joint $\mathcal{T}(\mathbf{z})$, or a $T_2(\mathbf{z})$ generator with a $C(p, \mathbf{z})$ generator. Using the conventional

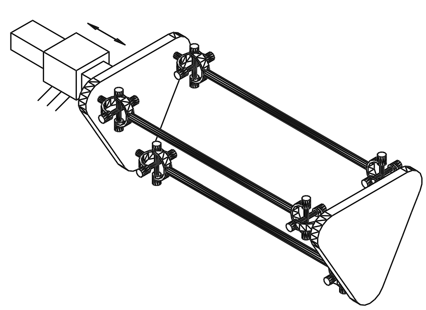

Fig. 7. A Generator of $T(3)$

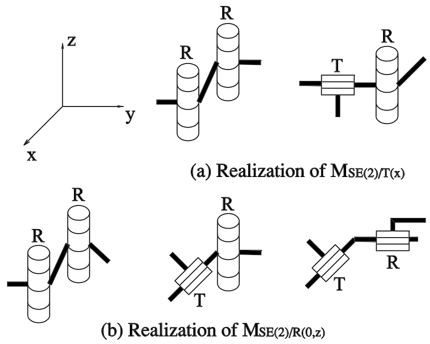

Fig. 8. Realizations of Quotient Manifolds of $SE(2)$

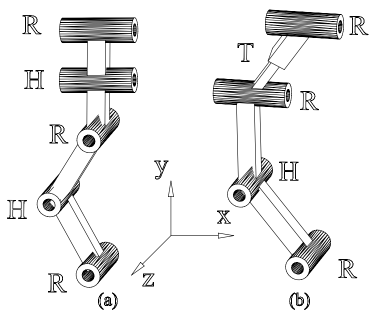

Fig. 9. Realizations of product submanifold $T(3) \cdot U(o, \mathbf{z}, \mathbf{x})$

set of primitive generators $(\mathcal{T}, \mathcal{R}, \mathcal{H})$, there are a total of 79 realizations of $X(\mathbf{z})$ (see also Hervé and Sparacino [21]), some of which are shown in Table IV.

Observe that any of the prismatic joint $\mathcal{T}$ in Table IV can be replaced by a $\mathcal{P}_{\mathfrak{a}}$-joint, and any pair of two successive $\mathcal{T}$'s can be replaced by a $\mathcal{U}^*-$joint. For example, replacing the $\mathcal{T}$ by $\mathcal{P}_{\mathfrak{a}}$ in the $(\mathcal{R} \cdot \mathcal{R} \cdot \mathcal{T} \cdot \mathcal{R})$-subchain gives rise to the subchain of the Delta manipulator. Using the extended set of primitive generators $(\mathcal{T}, \mathcal{R}, \mathcal{H}, \mathcal{P}_{\mathfrak{a}}, \mathcal{U}^*)$, there are a total of 219 realizations of $X(\mathbf{z})$. □

We summarize in Table IV the various subgroup motion generators and their total number in terms of the conventional set of PMGs, the number in the parenthesis indicates that obtained using the extended set of PMGs.

C. Submanifold Motion Generators

First, we consider motion generators for category I submanifolds.

Proposition *4: Given a desired motion type $Q = N_1 \cdot N_2$, with $N_1 \subset T(3)$ and $N_2 \subset SO(3)$, suppose that $\mathcal{M}_1$ generates N_1 and $\mathcal{M}_2$ generates N_2, then $\mathcal{M} = \mathcal{M}_1 \cdot \mathcal{M}_2$ generates $Q = N_1 \cdot N_2$.*

Proof. See Appendix D.

Next, we consider motion generators for a category II submanifold Q of the form $Q = H_1 \cdot H_2$, with H_1 and H_2 being a Lie subgroup of $SE(3)$.

Definition *4:* **Quotient Manipulators**

Let G be a Lie subgroup of $SE(3)$ and H a Lie subgroup of G. Let $\mathcal{M}_H$ be a motion generator of H. Then, $\mathcal{M}_{G/H}$ is said to be a *quotient manipulator* if $DoF(\mathcal{M}_{G/H}) = dim(G) - dim(H)$ and $\mathcal{M}_{G/H} \cdot \mathcal{M}_H$ generates G, i.e,

$$\overline{\mathcal{M}}_{G/H} \oplus \overline{\mathcal{M}}_H = T_e(G).$$

Example *23:* Realizations of the quotient manifold $PL(\mathbf{z})/T(\mathbf{x})$ were discussed in detail at the end of the previous section. As we have seen, up to a conjugation and rescaling, there are a total of three distinct topologies having the motion type of $PL(\mathbf{z})/T(\mathbf{x})$. By a similar analysis, there are four distinct realizations of $PL(\mathbf{z})/R(o, \mathbf{x})$, some of which are shown in Figure 8. □

Realizations of other quotient manifolds of $SE(3)$ are given in Table V.

Proposition *5: Given a desired motion type $Q = H_1 \cdot H_2$, with H_1 and H_2 being a Lie subgroup of $SE(3)$. Let $H = H_1 \cap H_2$. (i) if $H = \{e\}$, then $\mathcal{M} = \mathcal{M}_{H_1} \cdot \mathcal{M}_{H_2}$ generates Q; (ii) if $H > \{e\}$, then $\mathcal{M}_1 = \mathcal{M}_{H_1/H} \cdot \mathcal{M}_{H_2}$, $\mathcal{M}_2 = \mathcal{M}_{H_1} \cdot \mathcal{M}_{H_2/H}$ and $\mathcal{M}_3 = \mathcal{M}_{H_1/L_1} \cdot \mathcal{M}_{H_2/L_2}$, with $H = L_1 \cdot L_2$, all generate Q.*

Proof. See Appendix E.

Example *24:* **Motion generators of** $PL(\mathbf{z}) \cdot PL(\mathbf{x})$

Let $Q = PL(\mathbf{z}) \cdot PL(\mathbf{x})$. As $H = PL(\mathbf{z}) \cap PL(\mathbf{x}) = T(\mathbf{y})$, from Proposition 5, the product submanifold has the realizations

$$\begin{aligned} \mathcal{M} &= \mathcal{M}_{PL(\mathbf{z})/T(\mathbf{y})} \cdot \mathcal{M}_{PL(\mathbf{x})} \\ &= \mathcal{M}_{PL(\mathbf{z})} \cdot \mathcal{M}_{PL(\mathbf{x})/T(\mathbf{y})} \end{aligned} \quad . \tag{10}$$

Realizations for the various terms contained in Eq.(10) can be found from Table IV and V. $PL(\mathbf{x})$ has a total of 7(16), and $PL(\mathbf{z})/T(\mathbf{y})$ a total of 3(5) realizations. Thus, there are a total of 2 × 3 × 7= 42 (160) realizations for Q. □

Example *25:* **Motion generators of** $C(o, \mathbf{z}) \cdot S(o)$

Let $Q = C(o, \mathbf{z}) \cdot S(o)$, $H = C(o, \mathbf{z}) \cap S(o) = R(o, \mathbf{z})$, and we have

$$\begin{aligned} \mathcal{M} &= \mathcal{M}_{C(o,\mathbf{z})/R(o,\mathbf{z})} \cdot \mathcal{M}_{S(o)} \\ &= \mathcal{M}_{C(o,\mathbf{z})} \cdot \mathcal{M}_{S(o)/R(o,\mathbf{z})} \end{aligned} \tag{11}$$

From Table IV and V, there are 2 and 7 realizations for $C(o, \mathbf{z})/R(o, \mathbf{z})$ and $C(o, \mathbf{z})$, respectively. As only one realization exists for both $S(o)$ and $S(o)/R(o, \mathbf{z})$, there are a total of 9 possible realizations of Q. However, the two realizations $\mathcal{T}(\mathbf{z}) \cdot \mathcal{R}(o, \mathbf{z}) \cdot \mathcal{M}_{S(o)/R(o,\mathbf{z})}$ and $\mathcal{H}_\rho(o, \mathbf{z}) \cdot \mathcal{R}(o, \mathbf{z}) \cdot \mathcal{M}_{S(o)/R(o,\mathbf{z})}$ from the second product are identical to these in the first product since $\mathcal{R}(o, \mathbf{z}) \cdot \mathcal{M}_{S(o)/R(o,\mathbf{z})} = \mathcal{M}_{S(o)}$. Discounting the identical ones leads to a total of 7 distinct realizations of Q. □

Example *26:* **Motion generators of** $X(\mathbf{z}) \cdot X(\mathbf{x})$

Let $Q = X(\mathbf{z}) \cdot X(\mathbf{x})$. As $H = X(\mathbf{z}) \cap X(\mathbf{x}) = T(3) = T(\mathbf{u}) \cdot T_2(\mathbf{v})$, by Proposition 5, the product submanifold has the realizations

$$\begin{aligned} \mathcal{M} &= \mathcal{M}_{X(\mathbf{z})} \cdot \mathcal{M}_{X(\mathbf{x})/T(3)} \\ &= \mathcal{M}_{X(\mathbf{z})/T(3)} \cdot \mathcal{M}_{X(\mathbf{x})} \\ &= \mathcal{M}_{X(\mathbf{z})/T(\mathbf{u})} \cdot \mathcal{M}_{X(\mathbf{x})/T_2(\mathbf{v})} \\ &= \mathcal{M}_{X(\mathbf{z})/T_2(\mathbf{u})} \cdot \mathcal{M}_{X(\mathbf{x})/T(\mathbf{v})} \end{aligned} \tag{12}$$

Realizations for the various terms contained in Eq.(12) can be found from Table IV and V. Eliminating all identical realizations, we obtain a total of 564(1452) motion generators for Q.

DoF	Subgroup	Representative generators	Total
4	$X(\mathbf{z})$	$\mathcal{R}(q,\mathbf{z})\cdot\mathcal{T}(\mathbf{u})\cdot\mathcal{T}(\mathbf{v})\cdot\mathcal{T}(w)$, $\mathcal{H}_\rho(q,\mathbf{z})\cdot\mathcal{T}(\mathbf{u})\cdot\mathcal{T}(\mathbf{v})\cdot\mathcal{T}(w)$, $\mathcal{R}(q_1,\mathbf{z})\cdot\mathcal{R}(q_2,\mathbf{z})\cdot\mathcal{T}(\mathbf{u})\cdot\mathcal{T}(\mathbf{v})$, $\mathcal{R}(q_1,\mathbf{z})\cdot\mathcal{H}_\rho(q_2,\mathbf{z})\cdot\mathcal{T}(\mathbf{u})\cdot\mathcal{T}(\mathbf{v})$, $\mathcal{H}_{\rho_1}(q_1,\mathbf{z})\cdot\mathcal{H}_{\rho_2}(q_2,\mathbf{z})\cdot\mathcal{T}(\mathbf{u})\cdot\mathcal{T}(\mathbf{v})$, $\mathcal{H}_{\rho_1}(q_1,\mathbf{z})\cdot\mathcal{H}_{\rho_2}(q_2,\mathbf{z})\cdot\mathcal{H}_{\rho_3}(q_3,\mathbf{z})\cdot\mathcal{T}(\mathbf{u})$, $\mathcal{R}(q_1,\mathbf{z})\cdot\mathcal{R}(q_2,\mathbf{z})\cdot\mathcal{R}(q_3,\mathbf{z})\cdot\mathcal{T}(\mathbf{u})$, $\mathcal{R}(q_1,\mathbf{z})\cdot\mathcal{R}(q_2,\mathbf{z})\cdot\mathcal{H}_\rho(q_3,\mathbf{z})\cdot\mathcal{T}(\mathbf{u})$, $\mathcal{R}(q_1,\mathbf{z})\cdot\mathcal{H}_{\rho_1}(q_2,\mathbf{z})\cdot\mathcal{H}_{\rho_2}(q_3,\mathbf{z})\cdot\mathcal{T}(\mathbf{u})$, $\mathcal{H}_{\rho_1}(q_1,\mathbf{z})\cdot\mathcal{H}_{\rho_2}(q_2,\mathbf{z})\cdot\mathcal{H}_{\rho_3}(q_3,\mathbf{z})\cdot\mathcal{R}(q_4,\mathbf{z})$, $\mathcal{H}_\rho(q_1,\mathbf{z})\cdot\mathcal{R}(q_2,\mathbf{z})\cdot\mathcal{R}(q_3,\mathbf{z})\cdot\mathcal{R}(q_4,\mathbf{z})$, $\mathcal{H}_{\rho_1}(q_1,\mathbf{z})\cdot\mathcal{H}_{\rho_2}(q_2,\mathbf{z})\cdot\mathcal{R}(q_3,\mathbf{z})\cdot\mathcal{R}(q_4,\mathbf{z})$, $\mathcal{H}_{\rho_1}(q_1,\mathbf{z})\cdot\mathcal{H}_{\rho_2}(q_2,\mathbf{z})\cdot\mathcal{H}_{\rho_3}(q_3,\mathbf{z})\cdot\mathcal{H}_{\rho_4}(q_4,\mathbf{z})$	79(219)
3	$T(3)$	$\mathcal{T}(\mathbf{u})\cdot\mathcal{T}(\mathbf{v})\cdot\mathcal{T}(\mathbf{w})$	1(8)
3	$PL(\mathbf{z})$	$\mathcal{T}(\mathbf{u})\cdot\mathcal{T}(\mathbf{v})\cdot\mathcal{R}(q,\mathbf{z})$, $\mathcal{T}(\mathbf{u})\cdot\mathcal{R}(q_1,\mathbf{z})\cdot\mathcal{R}(q_2,\mathbf{z})$, $\mathcal{R}(q_1,\mathbf{z})\cdot\mathcal{R}(q_2,\mathbf{z})\cdot\mathcal{R}(q_3,\mathbf{z})$	7(16)
3	$Y_\rho(\mathbf{z})$	$\mathcal{T}(\mathbf{u})\cdot\mathcal{T}(\mathbf{v})\cdot\mathcal{H}_\rho(q,\mathbf{z})$, $\mathcal{T}(\mathbf{u})\cdot\mathcal{H}_\rho(q_1,\mathbf{z})\cdot\mathcal{H}_\rho(q_2,\mathbf{z})$, $\mathcal{H}_\rho(q_1,\mathbf{z})\cdot\mathcal{H}_\rho(q_2,\mathbf{z})\cdot\mathcal{H}_\rho(q_3,\mathbf{z})$	7(16)
3	$S(o)$	$\mathcal{R}(o,\mathbf{u})\cdot\mathcal{R}(o,\mathbf{v})\cdot\mathcal{R}(o,\mathbf{w})$	1
2	$C(o,\mathbf{z})$	$\mathcal{T}(\mathbf{z})\cdot\mathcal{R}(o,\mathbf{z})$, $\mathcal{T}(\mathbf{z})\cdot\mathcal{H}_\rho(o,\mathbf{z})$, $\mathcal{H}_\rho(o,\mathbf{z})\cdot\mathcal{R}(o,\mathbf{z})$, $\mathcal{H}_{\rho_1}(o,\mathbf{z})\cdot\mathcal{H}_{\rho_2}(o,\mathbf{z})$	7
2	$T_2(\mathbf{z})$	$\mathcal{T}(\mathbf{u})\cdot\mathcal{T}(\mathbf{v})$	1(4)

TABLE IV

EQUIVALENT GENERATORS FOR LIE SUBGROUPS OF $SE(3)$

Figure 9(a) displays a realization for $M_{X(\mathbf{z})/T(\mathbf{z})}\cdot M_{X(\mathbf{x})/T_2(\mathbf{z})}$ $(\mathbf{u}=\mathbf{v}=\mathbf{z})$, and Figure 9(b) a realization for $M_{X(\mathbf{z})/T_2(\mathbf{x})}\cdot M_{X(\mathbf{x})/T(\mathbf{x})}(\mathbf{u}=\mathbf{v}=\mathbf{x})$. Note that the translational direction of the prismatic joint in Figure 9(b) is not necessarily perpendicular to its two adjacent revolute joints. $\square$

The total number of realizations for other Category II submanifolds are shown in the last column of Table II.

Propositions 4 and 5 provide us with a set of tools needed for finding systematically motion generators of category I and II submanifolds. These tools can be applied separately or in combination when a submanifold is expressed in different forms.

We now illustrate the application of these tools by finding all motion generators for some of the category I submanifolds.

Example *27:* **Synthesis of $T_2(\mathbf{z})\cdot U(o,\mathbf{z},\mathbf{x})$-subchains** From Table II, the submanifold can be written as

$$T_2(\mathbf{z})\cdot U(o,\mathbf{z},\mathbf{x}) = PL(\mathbf{z})\cdot C(o,\mathbf{x})$$

with $H = PL(\mathbf{z})\cap C(o,\mathbf{x}) = T(\mathbf{x})$. Thus, the submanifold has the following realizations

$$\begin{aligned}\mathcal{M}_{T_2(\mathbf{z})}\cdot\mathcal{M}_{U(o,\mathbf{z},\mathbf{x})} &= \mathcal{M}_{PL(\mathbf{z})}\cdot\mathcal{M}_{C(o,\mathbf{x})/T(\mathbf{x})}\\ &= \mathcal{M}_{PL(\mathbf{z})/T(\mathbf{x})}\cdot\mathcal{M}_{C(o,\mathbf{x})}\ .\end{aligned}$$

Realizations of the various terms in the above products can be found from Table IV and V (that of $U(o,\mathbf{z},\mathbf{x})$ is fairly straightforward). Discounting identical realizations, there are a total of 29 realizations for $T_2(\mathbf{z})\cdot U(o,\mathbf{z},\mathbf{x})$. If we include the parallelogram joint, there are a total of 57 realizations, see Table VI. $\square$

Example *28:* **Synthesis of $T_2(\mathbf{z})\cdot S(o)$-subchains** Let $(\mathbf{u},\mathbf{v})$ be a basis of the xy-plane, and observe the following equivalences,

$$\begin{aligned}\mathcal{M}_{T_2(\mathbf{z})}\cdot\mathcal{M}_{S(o)} &= \mathcal{M}_{PL(\mathbf{z})}\cdot\mathcal{M}_{S(o)/R(o,z)}\\ &= \mathcal{M}_{PL(\mathbf{z})/R(o,\mathbf{z})}\cdot\mathcal{M}_{S(o)}\\ &= \mathcal{M}_{T_2(\mathbf{z})\cdot U(o,\mathbf{z},\mathbf{x})}\cdot\mathcal{R}(o,\mathbf{y})\\ &= \mathcal{T}(\mathbf{u})\cdot\mathcal{M}_{T(\mathbf{v})\cdot S(o)}\\ &= \mathcal{R}(q,\mathbf{z})\cdot\mathcal{M}_{T(\mathbf{v})\cdot S(o)}\ .\end{aligned}$$

Realizations for the various terms contained in the above equations can be found from Table IV and V, leading to a total

DoF	Quotient Manifold	Representative realizations	Total
1	$C(o,\mathbf{x})/T(\mathbf{x})$	$\mathcal{R}(o,\mathbf{x})$, $\mathcal{H}_\rho(o,\mathbf{x})$	2
1	$C(o,\mathbf{z})/R(o,\mathbf{z})$	$\mathcal{T}(\mathbf{z})$, $\mathcal{H}_\rho(o,\mathbf{z})$	2
2	$PL(\mathbf{z})/T(\mathbf{x})$	$\mathcal{T}(\mathbf{u})\cdot\mathcal{R}(q,\mathbf{z})$, $\mathcal{R}(q_1,\mathbf{z})\cdot\mathcal{R}(q_2,\mathbf{z})$	3(5)
2	$PL(\mathbf{z})/R(o,\mathbf{z})$	$\mathcal{T}(\mathbf{u})\cdot\mathcal{T}(\mathbf{v})$, $\mathcal{R}(q,\mathbf{z})\cdot\mathcal{T}(\mathbf{v})$, $\mathcal{R}(q_1,\mathbf{z})\cdot\mathcal{R}(q_2,\mathbf{z})$	4(9)
1	$PL(\mathbf{z}))/T_2(\mathbf{z}))$	$\mathcal{R}(q,\mathbf{z})$	1
2	$Y_\rho(\mathbf{z})/T(\mathbf{x})$	$\mathcal{T}(\mathbf{u})\cdot\mathcal{H}_\rho(q,\mathbf{z})$, $\mathcal{H}_\rho(q_1,\mathbf{z})\cdot\mathcal{H}_\rho(q_2,\mathbf{z})$	3(5)
2	$Y_\rho(\mathbf{z})/H_\rho(o,\mathbf{z})$	$\mathcal{T}(\mathbf{u})\cdot\mathcal{T}(\mathbf{v})$, $\mathcal{H}_\rho(q,\mathbf{z})\cdot\mathcal{T}(\mathbf{v})$, $\mathcal{H}_\rho(q_1,\mathbf{z})\cdot\mathcal{H}_\rho(q_2,\mathbf{z})$	4(9)
1	$Y_\rho(\mathbf{z})/T_2(\mathbf{z})$	$\mathcal{H}_\rho(q,\mathbf{z})$	1
2	$S(o)/R(o,\mathbf{z})$	$\mathcal{R}(o,\mathbf{u})\cdot\mathcal{R}(o,\mathbf{v})$	1
1	$X(\mathbf{z})/T(3)$	$\mathcal{R}(q,\mathbf{z})$, $\mathcal{H}_\rho(q,x)$	2
2	$X(\mathbf{z})/T_2(\mathbf{v})$	$\mathcal{R}(q,\mathbf{z})\cdot\mathcal{T}(\mathbf{u})$, $\mathcal{H}_\rho(q,\mathbf{z})\cdot\mathcal{T}(\mathbf{u})$, $\mathcal{R}(q_1,\mathbf{z})\cdot\mathcal{R}(q_2,\mathbf{z})$, $\mathcal{R}(q_1,\mathbf{z})\cdot\mathcal{H}_\rho(q_2,\mathbf{z})$, $\mathcal{H}_{\rho_1}(q_1,\mathbf{z})\cdot\mathcal{H}_{\rho_2}(q_2,\mathbf{z})$	8(12)
3	$X(\mathbf{z})/T(\mathbf{u})$	$\mathcal{R}(q,\mathbf{z})\cdot\mathcal{T}(\mathbf{v})\cdot\mathcal{T}(\mathbf{w})$, $H_\rho(q,\mathbf{z})\cdot\mathcal{T}(\mathbf{v})\cdot\mathcal{T}(\mathbf{w})$, $\mathcal{R}(q_1,\mathbf{z})\cdot\mathcal{R}(q_2,\mathbf{z})\cdot\mathcal{T}(\mathbf{v})$, $\mathcal{R}(q_1,\mathbf{z})\cdot\mathcal{H}_\rho(q_2,\mathbf{z})\cdot\mathcal{T}(\mathbf{v})$, $\mathcal{H}_{\rho_1}(q_1,\mathbf{z})\cdot\mathcal{H}_{\rho_2}(q_2,\mathbf{z})\cdot\mathcal{T}(\mathbf{v})$, $\mathcal{H}_{\rho_1}(q_1,\mathbf{z})\cdot\mathcal{H}_{\rho_2}(q_2,\mathbf{z})\cdot\mathcal{H}_{\rho_3}(q_3,\mathbf{z})$, $\mathcal{R}(q_1,\mathbf{z})\cdot\mathcal{R}(q_2,\mathbf{z})\cdot\mathcal{R}(q_3,\mathbf{z})$, $\mathcal{R}(q_1,\mathbf{z})\cdot\mathcal{R}(q_2,\mathbf{z})\cdot\mathcal{H}_\rho(q_3,\mathbf{z})$, $\mathcal{R}(q_1,\mathbf{z})\cdot\mathcal{H}_{\rho_1}(q_2,\mathbf{z})\cdot\mathcal{H}_{\rho_2}(q_3,\mathbf{z})$,	26(54)

TABLE V

REPRESENTATIVE REALIZATIONS FOR SOME QUOTIENT MANIFOLDS

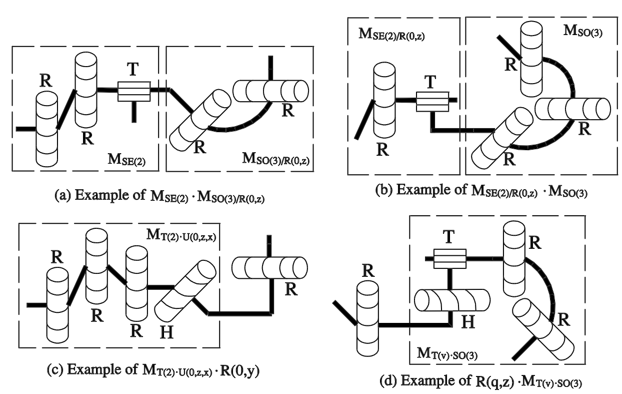

Fig. 10. Realizations of Product Submanifold $T(2)\cdot SO(3)$

DoF	Submanifold	Representative generators	Total
3	$T_2(\mathbf{z}) \cdot R(o, \mathbf{x})$	$\mathcal{M}_{T_2(\mathbf{z})} \cdot \mathcal{R}(o, \mathbf{x})$	1(4)
3	$T(\mathbf{z}) \cdot U(o, \mathbf{x}, \mathbf{y})$	$\mathcal{T}(\mathbf{z}) \cdot \mathcal{M}_{U(o,\mathbf{x},\mathbf{y})}$	1
4	$T_2(\mathbf{z}) \cdot U(o, \mathbf{z}, \mathbf{x})$ $= PL(\mathbf{z}) \cdot C(o, \mathbf{x})$	$\mathcal{M}_{PL(\mathbf{z})} \cdot \mathcal{M}_{C(o,\mathbf{x})/T(\mathbf{x})}$, $\mathcal{M}_{PL(\mathbf{z})/T(\mathbf{x})} \cdot \mathcal{M}_{C(o,\mathbf{x})}$	29 (57)
4	$T(\mathbf{z}) \cdot S(o)$ $= C(o, \mathbf{z}) \cdot S(o)$	$\mathcal{M}_{C(o,\mathbf{z})/R(o,\mathbf{z})} \cdot \mathcal{M}_{S(o)}$, Δ $\mathcal{M}_{C(o,\mathbf{z})} \cdot \mathcal{M}_{S(o)/R(o,\mathbf{z})}$	7
5	$T_2(\mathbf{z}) \cdot S(o)$ $= PL(\mathbf{z}) \cdot S(o)$	$\mathcal{M}_{PL(\mathbf{z})} \cdot \mathcal{M}_{S(o)/R(o,\mathbf{z})}$, $\mathcal{M}_{PL(\mathbf{z})/R(o,\mathbf{z})} \cdot \mathcal{M}_{S(o)}$, Δ $\mathcal{M}_{T_2(\mathbf{z})\cdot U(o,\mathbf{z},\mathbf{x})} \cdot \mathcal{R}(o, \mathbf{y})$, $\mathcal{T}(\mathbf{u}) \cdot \mathcal{M}_{T(\mathbf{v})\cdot S(o)}$, Δ $\mathcal{R}(q, \mathbf{z}) \cdot \mathcal{M}_{T(\mathbf{v})\cdot S(o)}$ Δ	45 (84)
5	$T(3) \cdot U(o, \mathbf{z}, \mathbf{x})$ $= X(\mathbf{z}) \cdot X(\mathbf{x})$ $= X(\mathbf{z}) \cdot C(q, \mathbf{x})$ $= X(\mathbf{z}) \cdot PL(\mathbf{x})$ $= X(\mathbf{z}) \cdot Y_\rho(\mathbf{x})$	$\mathcal{M}_{T(3)} \cdot \mathcal{M}_{U(o,\mathbf{z},\mathbf{x})}$, $\mathcal{M}_{X(\mathbf{z})} \cdot \mathcal{M}_{X(\mathbf{x})/T(3)}$, Δ $\mathcal{M}_{X(\mathbf{z})/T(3)} \cdot \mathcal{M}_{X(\mathbf{x})}$, $\mathcal{M}_{X(\mathbf{z})/T(\mathbf{u})} \cdot \mathcal{M}_{X(\mathbf{x})/T_2(\mathbf{v})}$, $\mathcal{M}_{X(\mathbf{z})/T_2(\mathbf{u})} \cdot \mathcal{M}_{X(\mathbf{x})/T(\mathbf{v})}$, $\mathcal{M}_{X(\mathbf{z})/T(\mathbf{x})} \cdot \mathcal{M}_{C(q,\mathbf{x})}$, $\mathcal{M}_{X(\mathbf{z})} \cdot \mathcal{M}_{C(q,\mathbf{x})/T(\mathbf{x})}$ Δ $\mathcal{M}_{X(\mathbf{z})/T_2(\mathbf{x})} \cdot \mathcal{M}_{PL(\mathbf{x})}$, $\mathcal{M}_{X(\mathbf{z})} \cdot \mathcal{M}_{PL(\mathbf{x})/T_2(\mathbf{x})}$, Δ $\mathcal{M}_{X(\mathbf{z})/T(\mathbf{s})} \cdot \mathcal{M}_{PL(\mathbf{x})/T(\mathbf{t})}$, $\mathcal{M}_{X(\mathbf{z})/T_2(\mathbf{x})} \cdot \mathcal{M}_{Y_\rho(\mathbf{x})}$, $\mathcal{M}_{X(\mathbf{z})} \cdot \mathcal{M}_{Y_\rho(\mathbf{x})/T_2(\mathbf{x})}$, Δ $\mathcal{M}_{X(\mathbf{z})/T(\mathbf{s})} \cdot \mathcal{M}_{Y_\rho(\mathbf{x})/T(\mathbf{t})}$,	564 (1452)
5	$S(N) \cdot S(o)$	$\mathcal{M}_{S(N)} \cdot \mathcal{M}_{S(o)/R(0,\mathbf{t})}$, $\mathcal{M}_{S(N)/R(0,\mathbf{t})} \cdot \mathcal{M}_{S(o)}$ Δ	2

TABLE VI

EQUIVALENT GENERATORS FOR SOME PRODUCT SUBMANIFOLDS

of 45 realizations for the submanifold. Taking into account the parallelogram joint, there are a total of 84 realizations.

Some examples of the realizations are shown in Figure 10. □

Example *29:* **Synthesis of $T(3) \cdot U(o, \mathbf{z}, \mathbf{x})$ subchain** From Table II, we can get the following identities

$$\begin{aligned} T(3) \cdot U(o, \mathbf{z}, \mathbf{x}) &= X(\mathbf{z}) \cdot X(\mathbf{x}) \\ &= X(\mathbf{z}) \cdot C(q, \mathbf{x}) \text{ (or } C(q, \mathbf{z}) \cdot X(\mathbf{x})) \\ &= X(\mathbf{z}) \cdot PL(x) \text{ (or } PL(z) \cdot X(\mathbf{x})) \\ &= X(\mathbf{z}) \cdot Y_\rho(\mathbf{x}) \text{ (or } Y_\rho(\mathbf{z}) \cdot X(\mathbf{x})) \\ &= PL(\mathbf{z}) \cdot PL(\mathbf{x}) \\ &= PL(\mathbf{z}) \cdot Y_\rho(\mathbf{x}) \text{ (or } Y_\rho(\mathbf{z}) \cdot PL(\mathbf{x})) \\ &= Y_{\rho_1}(\mathbf{z}) \cdot Y_{\rho_2}(\mathbf{x}) \text{ (or } Y_{\rho_1}(\mathbf{z}) \cdot Y_{\rho_2}(\mathbf{x})) \end{aligned} \quad (13)$$

Note that the set of realizations of $X(\mathbf{z}) \cdot X(\mathbf{x})$ contains the realizations for all other product submanifolds on the right hand side of (13). Hence from Example 26, all together there are a total of 564 realizations in terms of $(\mathcal{T}, \mathcal{R}, \mathcal{H})$, and 1452 realizations in terms of the extended set of primitive generators. □

IV. SYNTHESIS OF PARALLEL MANIPULATORS

From now on, the set of rigid motions generated by a serial manipulator $\mathcal{M}_i$ will be represented by its motion type, or more precisely by an open subset of $C_{\mathcal{M}_i}$. Which open subset to use for $C_{\mathcal{M}_i}$ is immaterial within the context of mechanism synthesis. For any $g \in C_{\mathcal{M}_i}$, $R_{g^{-1}*} T_g C_{\mathcal{M}_i} \subset se(3)$ is well defined and represents the set of infinitesimal motions of the end-effector at g. Apparently, this set of infinitesimal motions is equal to $\overline{\mathcal{M}}_i$ if the manipulator is at the home configuration e.

Definition *5:* **Parallel Motion (PM) Generator (or Parallel Realization)** Let $\mathcal{M} = \mathcal{M}_1 \| \cdots \| \mathcal{M}_k$ be a mechanism consisting of the parallel connection of k (serial) manipulator subchains $\mathcal{M}_1, \cdots \mathcal{M}_k$, that share a common base and a common end-effector. Denote the set of end-effector motions by

$$C_{\mathcal{M}} := C_{\mathcal{M}_1} \cap C_{\mathcal{M}_2} \cap \cdots \cap C_{\mathcal{M}_k}, \quad (14)$$

and the set of infinitesimal motions attainable by M at $g \in C_M$ by

$$R_{g^{-1}*} T_g C_{\mathcal{M}} := R_{g^{-1}*} T_g C_{\mathcal{M}_1} \cap \cdots \cap R_{g^{-1}*} T_g C_{\mathcal{M}_k}. \quad (15)$$

$\mathcal{M}$ is called a *parallel motion generator* or *a parallel realization* of a Lie subgroup or submanifold Q if $\mathcal{M}$ has the motion type of Q, i.e., $C_{\mathcal{M}}$ agrees with Q in an open neighborhood U of $SE(3)$ around the identity element.

We will assume that $k = dim(Q)$ in order to place one actuator at the base of each subchain.

Problem *2:* **Parallel Manipulator Synthesis** Given a desired motion type Q, as specified in Table I, find all parallel motion (PM) generators of Q.

The following proposition, which is a special case of a more general theorem proved in Appendix F, will be used to find parallel realizations of Q.

Proposition *6: Given a desired motion type $Q \subset SE(3)$, with $k = dim(Q)$. Assume that each $C_{\mathcal{M}_j}, j = 1, \cdots k$ contains a connected open subset Q_U of Q around e,*

$$Q_U \subseteq C_{\mathcal{M}_j}, j = 1, \cdots, k \quad (16)$$

and consequently $Q_U \subseteq C_{\mathcal{M}}$. If the condition

$$T_e Q = \overline{\mathcal{M}}_1 \cap \cdots \cap \overline{\mathcal{M}}_k \quad (17)$$

or the dual condition

$$(T_e^* Q)^\perp = (T_e^* C_{\mathcal{M}_1})^\perp + \cdots + (T_e^* C_{\mathcal{M}_k})^\perp \quad (18)$$

holds, where

$$(T_e^* Q)^\perp = \{f \in \mathbb{R}^6 \| \langle f, \xi \rangle = 0, \forall \xi \in T_e Q\}$$

denotes the set of constraint forces for $T_e Q$, then, $\mathcal{M} = \mathcal{M}_1 \| \cdots \| \mathcal{M}_k$ is a PM generator of Q. Furthermore, if for every $g \in Q_U$,

$$R_{g^{-1}*} T_g Q = R_{g^{-1}*} T_g (C_{\mathcal{M}_1}) \cap \cdots \cap R_{g^{-1}*} T_g (C_{\mathcal{M}_k}) \quad (19)$$

(or its dual condition holds), then there exists a connected open subset W of $SE(3)$ around e such that $Q_U = C_{\mathcal{M}} \cap W$, i.e, $C_{\mathcal{M}}$ agrees with Q_U in W.

Remark *3:* (i) (17) and its dual (18) are referred to as the *velocity matching condition* (VMC) and the *force matching condition* (FMC), respectively; (ii) (19) is referred to as the global VMC (GVMC). It can be used to determine the maximal extent in which C_M agrees with Q, i.e., to find the maximal Q_U such that $Q_U = C_{\mathcal{M}} \cap W$, with W an open subset of $SE(3)$.

Let us elaborate on the conditions of Proposition 6 for Q being a Lie subgroup. From the previous section, let

$$C_{\mathcal{M}_j} = H_j \cdot Q, \quad j = 1, \cdots, k \quad (20)$$

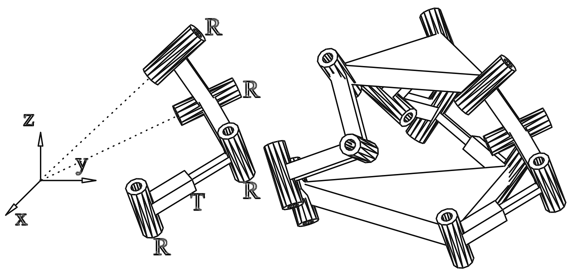

Fig. 11. Example of $SO(3)$ PM Motion Generator with $H_j = PL(\boldsymbol{\omega}_j)$

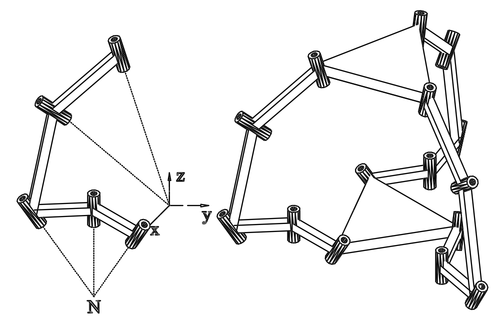

Fig. 12. Example of $SO(3)$ PM Motion Generator with $H_j = S(N_j)$

(with the equality to be understood in the sense of motion type) for some subgroup H_j. It is easy to see that $Q_U \subset C_{\mathcal{M}_j}$ for some open subset Q_U of Q around e. The trivial case with $H_j = \{e\}$ will not be considered here as the intersection in (14) is not robust. Choice of H_j will be based on the VMC (17) (or its dual), and the so-called *dead-locking* phenomena.

Lemma *5:* **Dead-locking**
When a serial manipulator (or a subchain) $\mathcal{M} = G_1 \cdots G_n$, where G_i, $i = 1, \cdots n$, are PMGs, is constrained (e.g., by closure-constraints) to move in a Lie subgroup or submanifold Q with $dim(Q) < DoF(\mathcal{M})$, and a subset k of the n primitive generators generate Q, e.g, $G_1 \cdots G_k$ generates Q, then the joint coordinates corresponding to the remaining joints will be dead-locked, i.e., $\theta_i = 0, i = k+1, \cdots n$, and these joints can be removed without affecting the constrained motions.

If H_j has a trivial intersection with Q, i.e, $H_j \cap Q = \{e\}$, then from Proposition 5, the realization of $H_j \cdot Q$ is given by $\mathcal{M}_j = \mathcal{M}_{H_j} \cdot \mathcal{M}_Q$. Thus by Lemma 5, the primitive generators in $\mathcal{M}_{H_j}$ will be dead-locked when $\mathcal{M}_j$ is constrained to move in the subgroup Q.

Corollary *1: A parallel manipulator $\mathcal{M}$ with its subchains $\mathcal{M}_j, j = 1, \cdots k$ satisfying (20) for H_j having non-trivial intersections with Q, and the VMC (17) or its dual (18) is a PM generator of Q.*

Corollary *2:* **Reordering Method**
Suppose that $\mathcal{M} = \mathcal{M}_1 \| \cdots \| \mathcal{M}_l$, with $\mathcal{M}_j = \mathcal{M}_{H_j} \cdot \mathcal{M}_{Q/(Q\cap H_j)} := (P_{j_1} \cdots P_{j_m}) \cdot (P_{j_{m+1}} \cdots P_{j_n})$, and $P_{j_k}, k = 1, \cdots n$ PMGs, is a PM generator of Q. Then, reordering the joints of $\mathcal{M}_{Q/(Q\cap H_j)}$ (but not that of $\mathcal{M}_{H_j}$) leads to another realization of Q, i.e., $\mathcal{N} = \mathcal{N}_1 \| \cdots \| \mathcal{N}_l$, with

$$\mathcal{N}_j = P_{j_{m+1}} \cdot (P_{j_1} \cdots P_{j_m}) \cdot P_{j_{m+2}} \cdots P_{j_n}$$

is also a PM generator of Q.
Note that by reordering, the conditions of Proposition 6 remain unchanged. Corollary 2 allows us to derive additional realizations from existing ones.

We now apply Proposition 6 and Corollary 1 and 2 to study subgroup PM generators for the following examples.

Example *30: Synthesis of $SO(3)$ PM Generators*
From Table II, there are three Lie subgroups that have non-trivial intersections with $S(o)$

$$C_{\mathcal{M}_j} = \begin{cases} PL(\boldsymbol{\omega}_j) \cdot S(o) \\ S(N_j) \cdot S(o) \\ C(o, \boldsymbol{\omega}_j) \cdot S(o) \end{cases} \tag{21}$$

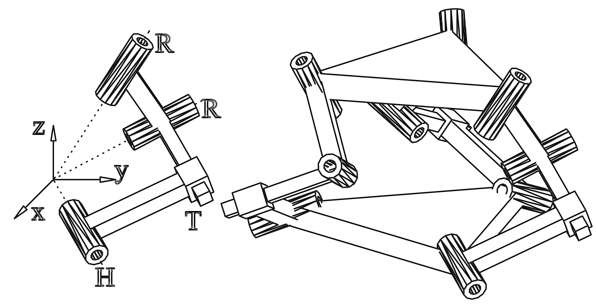

Fig. 13. Example of $SO(3)$ PM Motion Generator with $H_j = C(o, \boldsymbol{\omega}_j)$

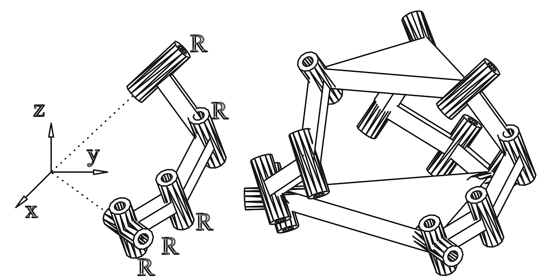

Fig. 14. Example of a $SO(3)$ PM Motion Generator obtained by the Reordering Method (A new interpretation of Fig.9 in [26])

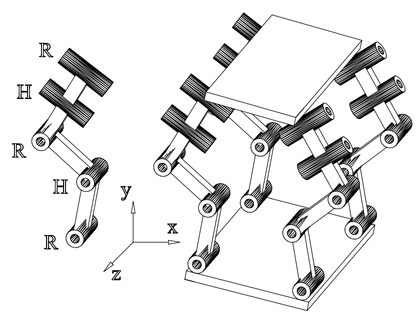

Fig. 15. Example of a $X(\mathbf{z})$ PM Motion Generator

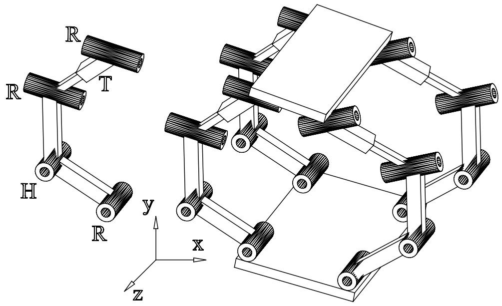

Fig. 16. Example of a $X(\mathbf{z})$ PM Motion Generator

$C_{\mathcal{M}_j}$ is, respectively, a 5- and 4-dimensional submanifold of $SE(3)$ containing $S(o)$. Realizations of the three product terms in (21) are given in Table VI. The constraint force for a subchain generating $PL(\boldsymbol{\omega})\cdot S(o)$ at e is given by $(\boldsymbol{\omega}^T,0)$, and that for $S(N)\cdot SO(3)$ by $(\mathbf{t}^T,0)$, where $\mathbf{t}$ is the unit direction vector from the origin to N. Thus, if

$$C_{\mathcal{M}_j} = PL(\boldsymbol{\omega}_j)\cdot S(o) \qquad j=1,\cdots,3$$

and with

$$(T_e^* C_{\mathcal{M}_j})^\perp = \{(\boldsymbol{\omega}_j^T,0)\}, \quad j=1,\cdots 3$$

the FMC shows that if the $\boldsymbol{\omega}_j$'s, $j=1,\cdots,3$ are linearly independent, then $\mathcal{M}=\mathcal{M}_1\|\mathcal{M}_2\|\mathcal{M}_3$ is a PM generator of $SO(3)$.

Similarly, if

$$C_{\mathcal{M}_j} = S(N_j)\cdot S(o) \qquad j=1,\cdots,3$$

then

$$(T_e^* C_{\mathcal{M}_j})^\perp = \{(\mathbf{t}_j^T,0)\}, \quad j=1,\cdots 3$$

and $\mathcal{M}=\mathcal{M}_1\|\mathcal{M}_2\|\mathcal{M}_3$ is a PM generator of $SO(3)$ if the $\mathbf{t}_j$'s are linearly independent.

Figure 11 and 12 give examples of these two types of PM generators.

The third type of PM generators of $SO(3)$ is obtained when

$$C_{\mathcal{M}_j} = C(o,\boldsymbol{\omega}_j)\cdot S(o) \qquad j=1,\cdots,3$$

Note that

$$\overline{\mathcal{M}}_j = \{\xi_{j,1},\cdots,\xi_{j,4}\}$$

where

$$\begin{aligned}\xi_{j,1} &= (\boldsymbol{\omega}_j^T,0)^T, \quad \xi_{j,2}=(\mathbf{0},(1,0,0))^T\\ \xi_{j,3} &= (\mathbf{0},(0,1,0))^T, \quad \xi_{j,4}=(\mathbf{0},(0,0,1))^T.\end{aligned}$$

Hence, in order to satisfy the VMC, at least two of $\boldsymbol{\omega}_j$'s, $j=1,\cdots,3$ need to be linearly independent. Examples of this types of $SO(3)$ PM generators are shown in Figure 13.

From Table II, there are 45(84) realizations for $PL(\boldsymbol{\omega})\cdot S(o)$, 2 for $S(N)\cdot S(o)$ and 7 for $C(o,\boldsymbol{\omega})\cdot S(o)$. Thus, there are in principle a total of 54(93) PM generators of $SO(3)$, provided that the subchains' motion types are equivalent and in the form of Eq.(21). Because of dead-locking, however, not all of these realizations are feasible. For example, the realization

$$\mathcal{M}_j = \mathcal{M}_{H_j/(H_j\cap S(o))}\cdot\mathcal{M}_{S(o)}$$

will result in dead-locking of all joints for $\mathcal{M}_{H_j/(H_j\cap S(o))}$ when the loop constraints are imposed. On the other hand, consider a realization of the form

$$\mathcal{M}_{H_j}\cdot\mathcal{M}_{S(o)/(H_j\cap S(o))}$$

If $\mathcal{M}_{H_j}$ contains some joints that form a $H_j\cap S(o)$ generator, then by Lemma 5, the remaining joints of $\mathcal{M}_{H_j}$ will become dead-locked. For instance, $\mathcal{M}_{PL(\boldsymbol{\omega}_j)}$ in the realization of $\mathcal{M}_{PL(\boldsymbol{\omega}_j)}\cdot\mathcal{M}_{S(o)/R(o,\boldsymbol{\omega}_j)}$ should not contain a $\mathcal{R}(o,\boldsymbol{\omega}_j)$ joint to avoid dead-locking. Eliminating all infeasible solutions which are marked by Δ in Table VI, there are a total of 35(63) parallel realizations for $SO(3)$.

Additional realizations can be derived from existing ones using the reordering method. For instance, let $\mathcal{M}=\mathcal{M}_1||\mathcal{M}_2||\mathcal{M}_3$ be a feasible solution with $\mathcal{M}_j$ consisting of

$$\begin{aligned}\mathcal{M}_j = {} & \mathcal{M}_{PL(\boldsymbol{\omega}_j)}\cdot\mathcal{M}_{S(o)/R(o,\boldsymbol{\omega}_j)}\\ = {} & (\mathcal{R}(p_{j1},\boldsymbol{\omega}_j)\cdot\mathcal{R}(p_{j2},\boldsymbol{\omega}_j)\cdot\mathcal{R}(p_{j3},\boldsymbol{\omega}_j))\cdot\mathcal{R}(o,\mathbf{u}_j)\cdot\mathcal{R}(o,\mathbf{v}_j)\end{aligned}$$

where $\boldsymbol{\omega}_j,\mathbf{u}_j$ and $\mathbf{v}_j$ are linearly independent. By Corollary 2, reordering the joints of $\mathcal{M}_{S(o)/(R(o,\boldsymbol{\omega}_j))}$ (but not that of $\mathcal{M}_{PL(\boldsymbol{\omega}_j)}$) leads to another feasible realization of $SO(3)$, i.e., $\mathcal{N}=\mathcal{N}_1\|\mathcal{N}_2\|\mathcal{N}_3$ with

$$\mathcal{N}_j=\mathcal{R}(o,\mathbf{u}_j)\cdot(\mathcal{R}(p_{j1},\boldsymbol{\omega}_j)\cdot\mathcal{R}(p_{j2},\boldsymbol{\omega}_j)\cdot\mathcal{R}(p_{j3},\boldsymbol{\omega}_j))\cdot\mathcal{R}(o,\mathbf{v}_j)$$

is also a PM generator of $SO(3)$ (Fig.14). Similarly, if $\mathcal{M}_j=\mathcal{M}_{S(N_j)}\cdot\mathcal{M}_{S(o)/R(o,\mathbf{t}_j)}$, applying the reordering method to the joints of $\mathcal{M}_{S(o)/R(o,\mathbf{t}_j)}$ yields more $SO(3)$ PM generators. Note that these two types of spherical mechanisms have also been studied in [26], [8] using pure group and screw theory. However, the finite motion property of such mechanisms could not be *precisely* verified in these papers.

Finally, note that PM generators of $SO(3)$ obtained by Eq.(21) characterize most solutions to Problem 2, but *not all*. Kong and Gosselin [30] studied a $SO(3)$ parallel mechanism with subchains given by

$$\mathcal{M}_j = X_{j1}\cdot X_{j2}\cdot X_{j3}\cdot\mathcal{R}(o,\mathbf{u}_j)\cdot\mathcal{R}(o,\mathbf{v}_j),\ j=1,2,3 \quad (22)$$

where X_{j1},X_{j2},X_{j3} are three revolute joints obtained by removing a revolute joint $\mathcal{R}(o,\boldsymbol{\omega}_j)$ from a Bennett mechanism [3]. Clearly, $\mathcal{M}_j$ in Eq.(22) has the motion type different from Eq.(21), and hence can not be synthesized by the method described above. However, we can derive this mechanism using Proposition 6 as follows. $X_{j1}\cdot X_{j2}\cdot X_{j3}$ contains an open subset of $R(o,\boldsymbol{\omega}_j)$, and hence $\mathcal{M}_j$ contains an open subset of $S(o)$. Furthermore, the constraint force of each subchain at home configuration e is given by $(f_j^T,0)$, where f_j is determined by two axes of X_{j1},X_{j2},X_{j3} and the origin. Thus

$$(T_e^* C_{\mathcal{M}_j})^\perp = \{(f_j^T,0)\}, \quad j=1,\cdots 3$$

From FMC, we see that if the f_j's, $j=1,\cdots,3$ are linearly independent, then $\mathcal{M}=\mathcal{M}_1\|\mathcal{M}_2\|\mathcal{M}_3$ is a PM generator of $SO(3)$. □

Example *31: Synthesis of $SE(2)$ PM Generators*
The two Lie subgroups that have non-trivial intersections with $PL(\mathbf{z})$ are $S(N)$ and $C(o,\boldsymbol{\omega})$. Observe that the order of H_j and Q in Eq.(21) is immaterial, we have

$$C_{\mathcal{M}_j} = \begin{cases} PL(\mathbf{z})\cdot S(N_j)\\ PL(\mathbf{z})\cdot C(o,\boldsymbol{\omega}_j)\end{cases}$$

If

$$C_{\mathcal{M}_j} = PL(\mathbf{z}) \cdot S(N_j) \qquad j = 1, \cdots, 3$$

then the constraint force space of each subchain at e is spanned by $f_j = (\mathbf{z}^T, (N_j \times \mathbf{z})^T)$. Applying the FMC to f_j, we see that $\mathcal{M} = \mathcal{M}_1 \| \mathcal{M}_2 \| \mathcal{M}_3$ is a $SE(2)$ PM generator if the N_j's are different from each other.

If

$$C_{\mathcal{M}_j} = PL(\mathbf{z}) \cdot C(o, \boldsymbol{\omega}_j) \qquad j = 1, \cdots, 3$$

then by the VMC, we see that $\mathcal{M}$ is a PM generator of $SE(2)$ if at least two of the $\boldsymbol{\omega}_j$'s, $j = 1, \cdots, 3$ are linearly independent.

Note that in order to avoid joint dead-locking, we need to restrict our realizations of $\mathcal{M}_j$ to $\mathcal{M}_{PL(\mathbf{z})/R(N_j, \mathbf{z})} \cdot \mathcal{M}_{S(N_j)}$ and $\mathcal{M}_{PL(\mathbf{z})/T(\boldsymbol{\omega}_j)} \cdot \mathcal{M}_{C(o, \boldsymbol{\omega}_j)}$.

Eliminating unfeasible realizations, we obtain a total of 31(52) realizations for PM generators of $SE(2)$. Using the reordering method on the joints of $\mathcal{M}_{PL(\mathbf{z})/R(N_j, \mathbf{z})}$ and $\mathcal{M}_{PL(\mathbf{z})/T(\boldsymbol{\omega}_j)}$ (but not that of $\mathcal{M}_{S(N_j)}$ and $\mathcal{M}_{C(o, \boldsymbol{\omega}_j)}$), we may obtain additional realizations. □

Example *32: Synthesis of $X(\mathbf{z})$ PM Generators*
We have the following options for the subchains

$$C_{\mathcal{M}_j} = \begin{cases} X(\mathbf{z}) \cdot X(\boldsymbol{\omega}_j) \\ X(\mathbf{z}) \cdot C(q, \boldsymbol{\omega}_j) \\ X(\mathbf{z}) \cdot PL(\boldsymbol{\omega}_j) \\ X(\mathbf{z}) \cdot Y_\rho(\boldsymbol{\omega}_j) \end{cases}$$

The subchains that generate $X(\mathbf{z}) \cdot X(\boldsymbol{\omega}_j)$, $X(\mathbf{z}) \cdot C(q, \boldsymbol{\omega}_j)$, $X(\mathbf{z}) \cdot PL(\boldsymbol{\omega}_j)$ and $X(\mathbf{z}) \cdot Y_\rho(\boldsymbol{\omega}_j)$ are all equivalent to $\mathcal{M}_{T(3) \cdot U(o, \mathbf{z}, \boldsymbol{\omega}_j)}$ (see Example 29).

With

$$(T_e^* C_{\mathcal{M}_j})^\perp = \{(\mathbf{0}, \boldsymbol{\mu}_j)\} \quad j = 1, \cdots, 4$$

where $\boldsymbol{\mu}_j$ is the normal of the plane determined by $\mathbf{z}$ and $\boldsymbol{\omega}_j$, and applying the FMC, we see that $\mathcal{M} = \mathcal{M}_1 \| \cdots \| \mathcal{M}_4$ is an $X(\mathbf{z})$ PM generator if at least two of the $\boldsymbol{\omega}_j' s$ are linearly independent.

Eliminating all the dead-locking realizations marked by Δ in Table VI, we have a total of 406 (1014) PM generators of $X(\mathbf{z})$. An example using subchain of Figure 9(a) is shown in Fig.15, and that of Fig.9(b) in Figure 16. Applying the reordering method to the joints of $\mathcal{M}_{X(\mathbf{z})/T(3)}$, $\mathcal{M}_{X(\mathbf{z})/T(\mathbf{u}_j)}$, and $\mathcal{M}_{X(\mathbf{z})/T_2(\mathbf{u}_j)}$ yields, amazingly, more topological structures of PM generators of $X(\mathbf{z})$. For example, the 3T1R parallel mechanisms in Fig.7 of [24] and Fig.10(b),(c),(e),(h) of [29] can all be viewed as $X(\mathbf{z})$ PM generators obtained by the reordering method. Note that the realizations derived by Kong [29] and Fang [7] comprised of $\mathcal{R}$ and $\mathcal{T}$ joints only are contained as a subset of our solutions. □

Example *33: Synthesis of $T(3)$ PM Generators*
For synthesis of $T(3)$ parallel manipulators, we need to go back to Proposition 6, by finding subchains that satisfy (16), or Lie subgroups or submanifolds that contain $T(3)$. An obvious solution is given by

$$C_{\mathcal{M}_j} = X(\boldsymbol{\omega}_j), \quad j = 1, \cdots 3$$

Applying the VMC, we see that $\mathcal{M} = \mathcal{M}_1 \| \mathcal{M}_2 \| \mathcal{M}_3$ is a $T(3)$ PM generator if at least two of the $\boldsymbol{\omega}_j$'s are linearly independent. The Delta manipulator is a good example of this type of PM generators.

We can also use for the subchain the following products: $X(\boldsymbol{\omega}_{j1}) \cdot X(\boldsymbol{\omega}_{j2})$, $X(\boldsymbol{\omega}_{j1}) \cdot C(q, \boldsymbol{\omega}_{j2})$, $X(\boldsymbol{\omega}_{j1}) \cdot PL(\boldsymbol{\omega}_{j2})$, or $X(\boldsymbol{\omega}_{j1}) \cdot Y_\rho(\boldsymbol{\omega}_{j2})$, as they are all equivalent to $T(3) \cdot U(o, \boldsymbol{\omega}_{j1}, \boldsymbol{\omega}_{j2})$.

Since

$$(T_e^* C_{\mathcal{M}_j})^\perp = \{(\mathbf{0}, \boldsymbol{\mu}_j)\} \quad j = 1, \cdots, 3$$

where $\boldsymbol{\mu}_j$ is the normal of the plane determined by $\boldsymbol{\omega}_{j1}$ and $\boldsymbol{\omega}_{j2}$, it is easy to see that $\mathbf{u}_j, j = 1, \cdots, 3$ must be linearly independent in order for $\mathcal{M}$ to be a PM generator of $T(3)$.

By Lemma 5, dead-locking occurs when $\mathcal{M}$ consists of three prismatic joints (or three parallelogram joint, or one parallelogram and one $\mathcal{U}^*$ joint, etc). Eliminating all the dead-locking realizations, we have a total of 536 (1196) PM generators of $T(3)$ in terms of $(\mathcal{T}, \mathcal{R}, \mathcal{H})$ (or the five extended primitive generators). Lee and Hervé [33] used the product $PL(\boldsymbol{\omega}_{j1}) \cdot PL(\boldsymbol{\omega}_{j2})$, which is also equivalent to $T(3) \cdot U(o, \boldsymbol{\omega}_{j1}, \boldsymbol{\omega}_{j2})$, as the subchain structure and obtained 21 distinct realizations in terms of $(\mathcal{T}, \mathcal{R})$. Since generators for $PL(\boldsymbol{\omega}_{j1}) \cdot PL(\boldsymbol{\omega}_{j2})$ are contained in $X(\boldsymbol{\omega}_{j1}) \cdot X(\boldsymbol{\omega}_{j2})$, we are able to generate a much larger set of feasible solutions than Lee and Hervé.

Applying the reordering method to the joints of $\mathcal{M}_{X(\boldsymbol{\omega}_{ji})/T(3)}$, $\mathcal{M}_{X(\boldsymbol{\omega}_{ji})/T(\mathbf{u}_j)}$, and $\mathcal{M}_{X(\boldsymbol{\omega}_{ji})/T_2(\mathbf{v}_j)}$, $i = 1, 2$, $j = 1, \cdots, 3$ leads to many more structures of PM generators of $T(3)$. For example, let $\mathcal{M} = \mathcal{M}_1 || \mathcal{M}_2 || \mathcal{M}_3$ be a feasible $T(3)$ generator with $\mathcal{M}_j = \mathcal{M}_{X(\boldsymbol{\omega}_{j1})/T(\mathbf{u}_j)} \cdot \mathcal{M}_{X(\boldsymbol{\omega}_{j2})/T_2(\mathbf{v}_j)}$, where $\mathbf{u}_j, \mathbf{v}_j$ are such that $T(3) = T(\mathbf{u}_j) \cdot T_2(\mathbf{v}_j)$. Reordering the joints of either $\mathcal{M}_{X(\boldsymbol{\omega}_{j1})/T(\mathbf{u})}$ or $\mathcal{M}_{X(\boldsymbol{\omega}_{j2})/T_2(\mathbf{v})}$ (not both) will not change the conditions of Proposition 6, and hence yield more translational parallel mechanisms. The commutation of factors method in [33] is a special case of the reordering method, which is used when $\mathcal{M}_{X(\omega_{j1})/T(\mathbf{u})}$ and $M_{X(\omega_{j2})/T_2(\mathbf{v})}$ comprise of purely $\mathcal{R}$ and $\mathcal{T}$ joints. Tsai's manipulator [49] can be derived by the commutation of factors(or more generally, the reordering method). □

A. Submanifold PM Generators

Proposition 6 and the synthesis procedure for subgroup PM generators can be extended to regular submanifolds. In this section, we show how to synthesize PM generators that are asymmetric and have the motion types of $T_2(\mathbf{z}) \cdot R(o, \mathbf{x})$ and $T(\mathbf{z}) \cdot U(o, \mathbf{x}, \mathbf{y})$, respectively. These cases are less studied in the literature but nonetheless have importance of their own, especially in design of hybrid parallel mechanisms [50]. For example, a $T(\mathbf{z}) \cdot U(o, \mathbf{x}, \mathbf{y})$ PM mechanism in serial with a $T(2)$ mechanism constitute a hybrid five-axis machine tool [44].

Example *34:* **Synthesis of $T_2(\mathbf{z}) \cdot R(o, \mathbf{x})$ PM Generators**
From Table I, there are five submanifolds or Lie subgroups that

DoFs of subchains	First subchain	Second subchain	Third subchain	Possible total	Exmaples
3-3-4	$T_2(\mathbf{z}) \cdot R(o, \mathbf{x})$	$T_2(\mathbf{z}) \cdot R(o, \mathbf{x})$	$X(\mathbf{x})$	79(3504)	HALF-II (Fig.7 and Fig.9 of [36])
3-3-5	$T_2(\mathbf{z}) \cdot R(o, \mathbf{x})$	$T_2(\mathbf{z}) \cdot R(o, \mathbf{x})$	$T(3) \cdot U(o, \mathbf{z}, \mathbf{x})$	564(23232)	
3-4-4	$T_2(\mathbf{z}) \cdot R(o, \mathbf{x})$	$T_2(\mathbf{z}) \cdot U(o, \mathbf{z}, \mathbf{x})$	$X(\mathbf{x})$	2291(49932)	Modified HALF-II (Fig.8(a) of [36])
3-4-5	$T_2(\mathbf{z}) \cdot R(o, \mathbf{x})$	$T_2(\mathbf{z}) \cdot U(o, \mathbf{z}, \mathbf{x})$	$T(3) \cdot U(o, \mathbf{z}, \mathbf{x})$	16356(331056)	
3-4-4	$T_2(\mathbf{z}) \cdot R(o, \mathbf{x})$	$X(\mathbf{x})$	$X(\mathbf{x})$	6241(191844)	
3-4-5	$T_2(\mathbf{z}) \cdot R(o, \mathbf{x})$	$X(\mathbf{x})$	$T(3) \cdot U(o, \mathbf{z}, \mathbf{x})$	44556(1271952)	
3-5-5	$T_2(\mathbf{z}) \cdot R(o, \mathbf{x})$	$T(3) \cdot U(o, \mathbf{z}, \mathbf{x})$	$T(3) \cdot U(o, \mathbf{z}, \mathbf{x})$	318096(8433216)	
3-4-5	$T_2(\mathbf{z}) \cdot R(o, \mathbf{x})$	$X(\mathbf{x})$	$T_2(\mathbf{z}) \cdot S(o)$	2291(49932)	
3-5-5	$T_2(\mathbf{z}) \cdot R(o, \mathbf{x})$	$T(3) \cdot U(o, \mathbf{z}, \mathbf{x})$	$T_2(\mathbf{z}) \cdot S(o)$	2291(49932)	
4-4-4	$T_2(\mathbf{z}) \cdot U(o, \mathbf{z}, \mathbf{x})$	$T_2(\mathbf{z}) \cdot U(o, \mathbf{z}, \mathbf{x})$	$X(\mathbf{x})$	66439(711531)	HALF-I (Fig.4 of [36]), and second family of Hervé's asymmetrical PKM (Fig.2 of [45])
4-4-5	$T_2(\mathbf{z}) \cdot U(o, \mathbf{z}, \mathbf{x})$	$T_2(\mathbf{z}) \cdot U(o, \mathbf{z}, \mathbf{x})$	$T(3) \cdot U(o, \mathbf{y}, \mathbf{x})$	474324(4717548)	First family of Hervé's asymmetrical PKM (Fig.1 of [45])
4-4-4	$T_2(\mathbf{z}) \cdot U(o, \mathbf{z}, \mathbf{x})$	$X(\mathbf{x})$	$X(\mathbf{x})$	180989(2733777)	Hana (Fig.6 of [36])
4-4-4	$T_2(\mathbf{z}) \cdot U(o, \mathbf{z}, \mathbf{x})$	$T(3) \cdot U(o, \mathbf{y}, \mathbf{x})$	$T(3) \cdot U(o, \mathbf{y}, \mathbf{x})$	9224784(120173328)	
4-4-5	$T_2(\mathbf{z}) \cdot U(o, \mathbf{z}, \mathbf{x})$	$X(\mathbf{x})$	$T(3) \cdot U(o, \mathbf{y}, \mathbf{x})$	474324(4717548)	
4-4-5	$T_2(\mathbf{z}) \cdot U(o, \mathbf{z}, \mathbf{x})$	$X(\mathbf{x})$	$T_2(\mathbf{z}) \cdot S(o)$	103095(1048572)	
4-5-5	$T_2(\mathbf{z}) \cdot U(o, \mathbf{z}, \mathbf{x})$	$T(3) \cdot U(o, \mathbf{y}, \mathbf{x})$	$T_2(\mathbf{z}) \cdot S(o)$	736020(6952176)	
4-4-5	$X(\mathbf{x})$	$X(\mathbf{x})$	$T_2(\mathbf{z}) \cdot S(o)$	280845(4028724)	
4-5-5	$X(\mathbf{x})$	$T_2(\mathbf{z}) \cdot S(o)$	$T_2(\mathbf{z}) \cdot S(o)$	159975(1545264)	

TABLE VIII

EQUIVALENT PM GENERATORS FOR SUBMANIFOLD $T_2(\mathbf{z}) \cdot R(o, \mathbf{x})$

DoF	Subchain Type	Constraint Force at e
3	$T_2(\mathbf{z}) \cdot R(o, \mathbf{x})$	$\{(\mathbf{z}^T, 0), (0, \mathbf{y}^T), (0, \mathbf{z}^T)\}$
4	$X(\mathbf{x})$	$\{(0, \mathbf{y}^T), (0, \mathbf{z}^T)\}$
4	$T_2(\mathbf{z}) \cdot U(o, \mathbf{z}, \mathbf{x})$	$\{(\mathbf{z}^T, 0), (0, \mathbf{y}^T)\}$
5	$T(3) \cdot U(o, \mathbf{z}, \mathbf{x})$	$\{(0, \mathbf{y}^T)\}$
5	$T_2(\mathbf{z}) \cdot S(o)$	$\{(\mathbf{z}^T, 0)\}$

TABLE VII

CONSTRAINT FORCE SPACE AT e FOR THE SUBCHAINS THAT CONTAIN $T_2(\mathbf{z}) \cdot R(o, \mathbf{x})$

contain $T_2(\mathbf{z}) \cdot R(o, \mathbf{x})$

$$C_{\mathcal{M}_j} = \begin{cases} T_2(\mathbf{z}) \cdot R(o, \mathbf{x}) & (DoF = 3) \\ X(\mathbf{x}) & (DoF = 4) \\ T_2(\mathbf{z}) \cdot U(o, \mathbf{z}, \mathbf{x}) & (DoF = 4) \\ T(3) \cdot U(o, \mathbf{z}, \mathbf{x}) & (DoF = 5) \\ T_2(\mathbf{z}) \cdot S(o) & (DoF = 5) \end{cases} \tag{23}$$

Subchains that have the above motion types are summarized in Table IV and VI, whereas the constraint force space for each of the subchains at e is shown in Table VII. The FMC indicates that if

$$(T_e^* C_{\mathcal{M}_1})^{\perp} + \cdots + (T_e^* C_{\mathcal{M}_3})^{\perp} = \{(\mathbf{z}^T, 0), (0, \mathbf{y}^T), (0, \mathbf{z}^T)\}$$

then $\mathcal{M} = \mathcal{M}_1 \| \mathcal{M}_2 \| \mathcal{M}_3$ is a PM generator of $T_2(\mathbf{z}) \cdot R(o, \mathbf{x})$. Using this condition, the possible subchain combinations which form a $T_2(\mathbf{z}) \cdot R(o, \mathbf{x})$ PM generator can be easily derived. The results are summarized in Table VIII, where the fifth column gives the possible number of realizations using the conventional set of PMGs and the number in the parenthesis for the extended set of PMGs. These numbers are obtained by simply multiplying the number of realizations for each subchains from Table IV and VI. Some of these realizations may not be feasible due to dead-locking, but can be eliminated using techniques similar to that of the subgroup case. The last column of Table VIII shows some of these mechanisms studied in [36] and [45]. One can see that these mechanisms are contained as a subset of our solutions.

Additional realizations can be obtained by applying the reordering method to the subchains with the motion type of $T(3) \cdot U(o, \mathbf{y}, \mathbf{x})$ or $T(3) \cdot U(o, \mathbf{z}, \mathbf{x})$. For example, let $C_{\mathcal{M}_1} = C_{\mathcal{M}_2} = T_2(\mathbf{z}) \cdot U(o, \mathbf{z}, \mathbf{x})$, $C_{\mathcal{M}_3} = T(3) \cdot U(o, \mathbf{y}, \mathbf{x})$ and the realization

$$\begin{aligned} \mathcal{M}_1 &= \mathcal{M}_{PL(\mathbf{z})} \cdot \mathcal{M}_{R(o,\mathbf{x})} \\ &= \mathcal{R}(p_1, \mathbf{z}) \cdot \mathcal{R}(p_2, \mathbf{z}) \cdot \mathcal{R}(p_3, \mathbf{z}) \cdot \mathcal{R}(o, \mathbf{x}) \\ \mathcal{M}_2 &= \mathcal{M}_{PL(\mathbf{z})} \cdot \mathcal{M}_{R(o,\mathbf{x})} \\ &= \mathcal{R}(q_1, \mathbf{z}) \cdot \mathcal{R}(q_2, \mathbf{z}) \cdot \mathcal{R}(q_3, \mathbf{z}) \cdot \mathcal{R}(o, \mathbf{x}) \\ \mathcal{M}_3 &= \mathcal{M}_{PL(\mathbf{y})} \cdot \mathcal{M}_{PL(\mathbf{x})/T(\mathbf{z})} \\ &= \mathcal{R}(s_1, \mathbf{y}) \cdot \mathcal{R}(s_2, \mathbf{y}) \cdot \mathcal{R}(s_3, \mathbf{y}) \cdot \mathcal{R}(t_1, \mathbf{x}) \cdot \mathcal{R}(t_2, \mathbf{x}) \end{aligned}$$

This corresponds to the first family of Hervé's asymmetrical parallel mechanisms (Fig. 1 of [45]). Applying the reordering method to the joints of $\mathcal{M}_{PL(\mathbf{x})/T(\mathbf{z})}$ in $\mathcal{M}_3$ yields a new subchain

$$\mathcal{M}_3' = \mathcal{R}(t_2, \mathbf{x}) \cdot \mathcal{R}(s_1, \mathbf{y}) \cdot \mathcal{R}(s_2, \mathbf{y}) \cdot \mathcal{R}(s_3, \mathbf{y}) \cdot \mathcal{R}(t_1, \mathbf{x})$$

Clearly, $C'_{\mathcal{M}_3}$ also contains an open neighborhood of $T_2(\mathbf{z}) \cdot R(o, \mathbf{x})$ around e. Thus, by the FMC, the mechanism $\mathcal{M}_1 \| \mathcal{M}_2 \| \mathcal{M}_3'$ shown in Fig.17 is a new realization of $T_2(\mathbf{z}) \cdot R(o, \mathbf{x})$. The modified version of HALF-II in Fig.8(b) of [36] can also be interpreted as one obtained by the reordering method. □

Example *35:* **Synthesis of $T(\mathbf{z}) \cdot U(o, \mathbf{x}, \mathbf{y})$ PM Generators**

The submanifolds that contain $T(\mathbf{z}) \cdot U(o, \mathbf{x}, \mathbf{y})$ are

$$C_{\mathcal{M}_j} = \begin{cases} T(\mathbf{z}) \cdot U(o, \mathbf{x}, \mathbf{y}) & (DoF = 3) \\ T_2(\mathbf{x}) \cdot U(o, \mathbf{x}, \mathbf{y}) & (DoF = 4) \\ T(\mathbf{z}) \cdot S(o) & (DoF = 4) \\ T_2(\mathbf{y}) \cdot S(o) & (DoF = 5) \\ T(3) \cdot U(o, \mathbf{x}, \mathbf{y}) & (DoF = 5) \end{cases} \tag{24}$$

DoFs of subchains	First subchain	Second subchain	Third subchain	Possible Total	Exmaples
3-4-4	$T(\mathbf{z})\cdot U(o,\mathbf{x},\mathbf{y})$	$T(\mathbf{z})\cdot S(o)$	$T(\mathbf{z})\cdot S(o)$	49	
3-5-5	$T(\mathbf{z})\cdot U(o,\mathbf{x},\mathbf{y})$	$T_2(\mathbf{y})\cdot S(o)$	$T_2(\mathbf{y})\cdot S(o)$	2025(7056)	
3-4-5	$T(\mathbf{z})\cdot U(o,\mathbf{x},\mathbf{y})$	$T(\mathbf{z})\cdot S(o)$	$T_2(\mathbf{y})\cdot S(o)$	315(588)	
3-5-5	$T(\mathbf{z})\cdot U(o,\mathbf{x},\mathbf{y})$	$T(3)\cdot U(o,\mathbf{x},\mathbf{y})$	$T(3)\cdot U(o,\mathbf{x},\mathbf{y})$	318096(2108304)	
3-4-5	$T(\mathbf{z})\cdot U(o,\mathbf{x},\mathbf{y})$	$T(\mathbf{z})\cdot S(o)$	$T(3)\cdot U(o,\mathbf{x},\mathbf{y})$	3948(10164)	
3-5-5	$T(\mathbf{z})\cdot U(o,\mathbf{x},\mathbf{y})$	$T_2(\mathbf{y})\cdot S(o)$	$T(3)\cdot U(o,\mathbf{x},\mathbf{y})$	25380(121968)	
4-4-5	$T_2(\mathbf{x})\cdot U(o,\mathbf{x},\mathbf{y})$	$T_2(\mathbf{x})\cdot U(o,\mathbf{x},\mathbf{y})$	$T_2(\mathbf{y})\cdot S(o)$	37845(272916)	Third family of Hervé's asymmetrical PKM (Fig.3 of [45])
4-4-4	$T_2(\mathbf{x})\cdot U(o,\mathbf{x},\mathbf{y})$	$T_2(\mathbf{x})\cdot U(o,\mathbf{x},\mathbf{y})$	$T(\mathbf{z})\cdot S(o)$	5887(22743)	Fourth family of Hervé's asymmetrical PKM (Fig.4 of [45])
4-5-5	$T_2(\mathbf{x})\cdot U(o,\mathbf{x},\mathbf{y})$	$T_2(\mathbf{y})\cdot S(o)$	$T_2(\mathbf{y})\cdot S(o)$	58725(402192)	
4-4-4	$T_2(\mathbf{x})\cdot U(o,\mathbf{x},\mathbf{y})$	$T(\mathbf{z})\cdot S(o)$	$T(\mathbf{z})\cdot S(o)$	1421(2793)	
4-4-5	$T_2(\mathbf{x})\cdot U(o,\mathbf{x},\mathbf{y})$	$T(\mathbf{z})\cdot S(o)$	$T_2(\mathbf{y})\cdot S(o)$	9135(33516)	
4-5-5	$T_2(\mathbf{x})\cdot U(o,\mathbf{x},\mathbf{y})$	$T_2(\mathbf{y})\cdot S(o)$	$T(3)\cdot U(o,\mathbf{x},\mathbf{y})$	736020(6952176)	
4-4-5	$T_2(\mathbf{x})\cdot U(o,\mathbf{x},\mathbf{y})$	$T(\mathbf{z})\cdot S(o)$	$T(3)\cdot U(o,\mathbf{x},\mathbf{y})$	114492(579348)	
4-4-5	$T(\mathbf{z})\cdot S(o)$	$T(\mathbf{z})\cdot S(o)$	$T(3)\cdot U(o,\mathbf{x},\mathbf{y})$	27636(71148)	
4-4-5	$T(\mathbf{z})\cdot S(o)$	$T(3)\cdot U(o,\mathbf{x},\mathbf{y})$	$T(3)\cdot U(o,\mathbf{x},\mathbf{y})$	2226672(14758128)	
4-4-5	$T(\mathbf{z})\cdot S(o)$	$T_2(\mathbf{y})\cdot S(o)$	$T(3)\cdot U(o,\mathbf{x},\mathbf{y})$	177660(853776)	

TABLE X

EQUIVALENT PM GENERATORS FOR SUBMANIFOLD $T(\mathbf{z})\cdot U(o,\mathbf{x},\mathbf{y})$

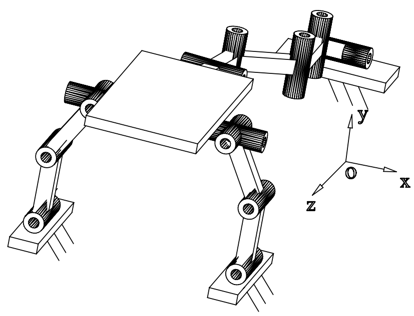

Fig. 17. Example of a $T_2(\mathbf{z})\cdot R(o,\mathbf{x})$ PM Motion Generator

DoF	Subchain Type	Constraint Force at e
3	$T(\mathbf{z})\cdot U(o,\mathbf{x},\mathbf{y})$	$\{(\mathbf{x}^T,0),(\mathbf{y}^T,0),(0,\mathbf{z}^T)\}$
4	$T_2(\mathbf{x})\cdot U(o,\mathbf{x},\mathbf{y})$	$\{(\mathbf{x}^T,0),(0,\mathbf{z}^T)\}$
4	$T(\mathbf{z})\cdot S(o)$	$\{(\mathbf{x}^T,0),(\mathbf{y}^T,0)\}$
5	$T_2(\mathbf{y})\cdot S(o)$	$\{(\mathbf{y}^T,0)\}$
5	$T(3)\cdot U(o,\mathbf{x},\mathbf{y})$	$\{(0,\mathbf{z}^T)\}$

TABLE IX

CONSTRAINT FORCE AT e FOR THE SUBCHAINS THAT CONTAIN $T(\mathbf{z})\cdot U(o,\mathbf{x},\mathbf{y})$

Realizations for the above motion types are summarized in Table VI. The constraint force space for each of the subchains at e is given in Table IX. From the FMC, it is easy to see that if

$$(T_e^*C_{\mathcal{M}_1})^\perp+\cdots+(T_e^*C_{\mathcal{M}_3})^\perp=\{(\mathbf{x}^T,0),(\mathbf{y}^T,0),(0,\mathbf{z}^T)\}$$

then, $\mathcal{M}=\mathcal{M}_1\|\mathcal{M}_2\|\mathcal{M}_3$ is a PM generator of $T(\mathbf{z})\cdot U(o,\mathbf{x},\mathbf{y})$. Table X shows possible realizations of $T(\mathbf{z})\cdot U(o,\mathbf{x},\mathbf{y})$ motion generators.

□

Finally, to conclude this section, we show another application of Proposition 6 in deriving the motion type of the parallelogram joint over a maximum set. This is in general a problem of considerable difficulties using any local theory.

Example *36:* **Synthesis of PM Generators for Circular Translations**

Let the set of desired motions be

$$Q=\left\{\left[\begin{array}{cc} I & (e^{\hat\omega\theta}-I)\mathbf{v}\\ 0 & 1\end{array}\right]\ \mid\ \theta\in[0,2\pi]\right\}$$

where ω is the axis of rotation for the circle, and $\mathbf{v}\perp\omega$ a vector from the center of the circle to the origin of the spatial coordinate frame. Q is thus S^1 embedded in $SE(3)$, containing e. We now show that the parallelogram is a motion generator of Q. Let $\mathcal{M}=\mathcal{M}_1\|\mathcal{M}_2$, where for $i=1,2$,

$$\mathcal{M}_i=\mathcal{R}(q_{i1},\omega)\cdot\mathcal{R}(q_{i2},\omega)=\mathcal{R}(q_{i1},\omega)\cdot\mathcal{R}(q_{i1}+\mathbf{v},\omega)\quad(25)$$

and $q_{21}=q_{11}+\mathbf{u}$, see Fig.2. Notice that

$$\left[\begin{array}{cc} e^{\hat\omega\theta_{i1}} & (I-e^{\hat\omega\theta_{i1}})q_{i1}\\ 0 & 1\end{array}\right]\cdot\left[\begin{array}{cc} e^{\hat\omega\theta_{i2}} & (I-e^{\hat\omega\theta_{i2}})q_{i2}\\ 0 & 1\end{array}\right]$$
$$=\left[\begin{array}{cc} I & (e^{\hat\omega\theta_{i1}}-I)\mathbf{v}\\ 0 & 1\end{array}\right]\cdot\left[\begin{array}{cc} e^{\hat\omega(\theta_{i1}+\theta_{i2})} & (I-e^{\hat\omega(\theta_{i1}+\theta_{i2})})q_{i2}\\ 0 & 1\end{array}\right]$$

Hence,

$$C_{\mathcal{M}_1}=R(q_{11},\omega)\cdot Q=Q\cdot R(q_{12},\omega)\quad(26)$$
$$C_{\mathcal{M}_2}=R(q_{21},\omega)\cdot Q=Q\cdot R(q_{22},\omega)\quad(27)$$

from which one derives that,

$$Q\subseteq C_{\mathcal{M}_1}\cap C_{\mathcal{M}_2}.$$

If at the home configuration e, $\mathbf{v}$ and $\mathbf{u}$ do not coincide, then the spatial velocities of the subchains are given by

$$\overline{\mathcal{M}_1}=\{(q_{11}\times\omega,\omega)^T,((\omega\times\mathbf{v})^T,0)^T\}$$
$$\overline{\mathcal{M}_2}=\{(q_{21}\times\omega,\omega)^T,((\omega\times\mathbf{v})^T,0)^T\}$$

with the intersection

$$\{((\omega\times\mathbf{v})^T,0)^T\}\quad(28)$$

as long as $q_{11}\neq q_{21}$. (28) is precisely the tangent space of Q at e. By the first part of Proposition 6, $\mathcal{M}$ generates Q.

Suppose the angle from $\mathbf{u}$ to $\mathbf{v}$ is $\phi = ATAN2(\omega^T(\mathbf{u} \times \mathbf{v})^3, \mathbf{u}^T\mathbf{v}) > 0$, and define an open subset Q_U of Q as

$$Q_U = \{g \in Q \mid \theta \in (-\phi, \pi - \phi)\}$$

Then for all $g \in Q_U$, the spatial velocity spaces of the two subchains are

$$R_{g^{-1}*}T_g C_{\mathcal{M}_1} = \{(-\omega \times q_{11}, \omega)^T, ((e^{\hat{\omega}\theta}(\omega \times \mathbf{v}))^T, 0)^T\}$$
$$R_{g^{-1}*}T_g C_{\mathcal{M}_2} = \{(-\omega \times q_{21}, \omega)^T, ((e^{\hat{\omega}\theta}(\omega \times \mathbf{v})^T, 0)^T\}$$

where $\theta = \theta_{j1} = -\theta_{j2}, j = 1, 2$, and their intersection is

$$\{((e^{\hat{\omega}\theta}\omega \times \mathbf{v})^T, 0)^T\} \quad (29)$$

which is precisely the velocity space of Q at g. However, one can check that this equality will not hold when $\theta = -\phi$ or $\pi - \phi$. Therefore, by the second part of Proposition 6, Q_U is the maximal open subset of Q that $\mathcal{M}$ generates.

Note that if $\mathcal{R}(q_{i1}, \omega)$ and $\mathcal{R}(q_{i2}, \omega)$ in Eq.(25) are replaced by two helical joints with the same pitch, i.e, $\mathcal{H}_{\rho_i}(q_{i1}, \omega)$ and $\mathcal{H}_{\rho_i}(q_{i2}, \omega)$, $i = 1, 2$, using the same approach, we can prove that the new mechanism also has the motion type of Q. □

V. Conclusion

To conclude the paper, we give in this section a brief comparison of the geometric method developed in this paper for mechanism synthesis with the Lie-group-algebraic method of Lee and Hervé [33] and the screw theory based constraint-synthesis method, and highlight several problems for future research along this direction.

Clearly, a synthesis method is gauged by its ability in: (i) providing a precise and unambiguous description of the fundamental concepts of a mechanism, including the instantaneous and global (or finite) motion properties of the input and output spaces of the mechanism as well as that of its constituting subchains; and (ii) generating systematically all feasible topologies based on the design specifications. Since the topological richness of a parallel mechanism is largely dependent on that of its constituting subchains, a good synthesis method should provide a thorough understanding on the design and generation of relevant subchain topologies.

Screw (or reciprocal screw) theory, on which the constraint-synthesis method is based, is a study of the Lie algebra and its subspaces (or its annihilators) of the Lie group $SE(3)$. Thus, the constraint-synthesis method is good at providing simple and intuitive descriptions of the instantaneous motion properties of a mechanism. Finite motion property of the mechanism is inferred when the underlying subspace or the corresponding set of constraint (or annihilating) wrenches remains unchanged after the system undergoes any feasible finite perturbations. Techniques like the finite motion condition [8] or the single-loop kinematic chain method [30] have been developed for this purpose. However, by overlooking the group structure of the motion space, it is rather difficult to develop a complete and effective method for mechanism analysis and synthesis based on this approach.

The Lie-group-algebraic method resolves the finite motion problem by operating directly at the group level. Using the algebraic properties of the Lie subgroups, it enumerates all generators of the planar subgroup $PL(\boldsymbol{\omega})$ in terms of the $\mathcal{T}$ and $\mathcal{R}$ PMGs. The product of two planar subgroups $PL(\mathbf{u}) \cdot PL(\mathbf{v})$ is explored as the limb structure, based on which systematic generation of the various TPM topologies is obtained. The Lie-group-algebraic method does not explore the differential property of $SE(3)$, and computations, e.g., intersection of two subgroups, are often carried out by brutal force at the group level. Although it used the notion of submanifolds for the limb structure, it does not explore systematically the rich submanifold structures of $SE(3)$, nor could it handle primitive joints or task spaces of the submanifold type.

The geometric method developed in this paper can be considered a full extension and completion of the Lie-group-algebraic method. By exploring both the algebraic and differential properties of $SE(3)$ and its Lie subgroups and submanifolds, it provides a unified framework for modelling primitive joints and task spaces having the motion type of a subgroup and/or a submanifold. Computations are reduced to linear algebra problems (and can thus be automated) as they are carried out at the Lie algebra level, and finite motion is guaranteed through the exponential map and the various properties of the product submanifolds. The similarity group and the notion of conjugacy class are used to provide a formal and precise mathematical definition of motion type. For systematic generation of serial manipulator topologies, this paper introduced the notion of quotient manipulators and a set of synthesis tools (Props. 3, 4 and 5). Quotient manipulators provide a formal generalization of the notion $\{G - 1(v)\}$ in [33], and the set of synthesis tools developed in this paper is more powerful and complete than its counterpart (Section 3) in [33]. The method developed in Section IV here for parallel mechanism synthesis is a general method and is applicable to synthesis of any desired motion type (subgroup or submanifold). It combines the velocity matching condition (or the FMC) from the screw theory based method with the neighborhood containment condition (Eq.(16)) of the group based method to give explicit conditions on the required subchains. The notion of dead-locking is formalized here to avoid over-constrained mechanisms, and the method of reordering (or commutation of factors in [33]) is illustrated for systematic generation of additional topologies from existing ones. Generality of the geometric synthesis method is demonstrated through numerous examples for systematic generation of a large class of feasible topologies for parallel mechanisms with not just translational type of motions but motions of any type, a subgroup or a submanifold.

As for future work along this direction, we mention

- Develop a parametric model for each class of topological structures and introduce suitable performance metrics on each motion type so that an appropriate optimization problem on a suitable set of dimensional parameters can be formulated.
- Investigate solutions of the optimization problem to identify suitable topological configurations for specific applications.
- Develop a tolerance (or error) model of the primitive generators and study the error characteristics of the

[3] In Fig 2, $\phi = \pi/2$.

associated realizations. $\square$

Appendix A: Proof of Proposition 1

Since $SE(3)$ is diffeomorphic to $T(3)\times SO(3)$, an element $m_1\cdot m_2$ of $M_1\cdot M_2$ corresponds directly to $(m_1,m_2)\in T(3)\times SO(3)$. Thus, $M_1\cdot M_2$ corresponds with $M_1\times M_2$, a (n_1+n_2)-dimensional regular submanifold of $T(3)\times SO(3)$. Hence, $M_1\cdot M_2$ is a (n_1+n_2)-dimensional regular submanifold of $SE(3)$ by the equivalent manifold structure of $SE(3)$ and $T(3)\times SO(3)$.

Appendix B: Proof of Proposition 2

First of all, we endow $H_1\cdot H_2$ with the subspace topology of $SE(3)$. Second, let $(\hat\eta_1,\cdots,\hat\eta_n)$ be a basis of $\mathfrak{h}$, and complete the basis so that $(\hat\eta_1,\cdots,\hat\eta_n,\hat\eta_{n+1},\cdots,\hat\eta_{n_1})$ forms a basis of $\mathfrak{h}_1$, $(\hat\eta_1,\cdots,\hat\eta_n,\hat\eta_{n_1+1},\cdots,\hat\eta_{n_1+n_2-n})$ a basis of $\mathfrak{h}_2$, and $\eta_1,\cdots,\eta_{n_1+n_2-n},\cdots,\eta_6$ a basis of $se(3)$. Let U and V be, respectively, defined as

$$\begin{aligned} U &= \{e^{\hat\eta_1\theta_1}\cdots e^{\hat\eta_{n_1+n_2-n}\theta_{n_1+n_2-n}}\in H_1\cdot H_2 \;\mid\; |\theta_i|<\epsilon\}\\ V &= \{e^{\hat\eta_1\theta_1}\cdots e^{\hat\eta_6\theta_6}\in SE(3)\;\mid\; |\theta_i|<\epsilon\}\end{aligned}$$

Clearly, U is a slice of V : $U=\{x\in V \mid \theta_{n_1+n_2-n+1}=0,\cdots,\theta_6=0\}$. For sufficiently small ϵ, V is an open coordinate neighborhood of e in $SE(3)$, with local coordinates $(\theta_1,\cdots,\theta_6)$, and $U=V\cap(H_1\cdot H_2)$. Hence, U is an open coordinate neighborhood of e in $H_1\cdot H_2$, with preferred local coordinates $(\theta_1,\cdots,\theta_{n_1+n_2-n})$. Using left and right translation, any point $h_1\cdot h_2$ of $H_1\cdot H_2$, has such a preferred local coordinate neighborhood $h_1\cdot U\cdot h_2$. Thus $H_1\cdot H_2$ has the n-submanifold property, and is a regular submanifold of $SE(3)$.

Appendix C: Proofs of Lemmas 1 ∼ 4

Proof of Lemma 1. Let u_1,u_2 be an arbitrary basis of V, and write

$$\begin{aligned} u_1 &= \lambda_1\widehat{e}_6+\lambda_2\widehat{e}_2+\lambda_3\widehat{e}_3\\ u_2 &= \mu_1\widehat{e}_6+\mu_2\widehat{e}_2+\mu_3\widehat{e}_1\end{aligned}$$

where $\lambda_j,\mu_j\in\mathbb{R}$ for $j=1,2,3$. Since $(u_1,u_2,\widehat{e}_1)$ is a basis of $\mathfrak{g}$, we see that $\det(A)\neq 0$, where

$$A=\begin{bmatrix}\lambda_1 & \lambda_2\\ \mu_1 & \mu_2\end{bmatrix}$$

Let $[v_1,v_2]$ = $[u_1,u_2]A^{-1}$. Since

$$[u_1,u_2]=[\widehat{e}_6,\widehat{e}_2,\widehat{e}_1]\begin{bmatrix} A\\ \lambda_1,\mu_3\end{bmatrix}$$

we get

$$[v_1,v_2]=[u_1,u_2]A^{-1}=[\widehat{e}_6,\widehat{e}_2,\widehat{e}_1]\begin{bmatrix} I\\ (\lambda_3,\mu_3)A^{-1}\end{bmatrix}$$

Let $(a,b)=(\lambda_3,\mu_3)A^{-1}$. Then,

$$v_1=\widehat{e}_6+a\widehat{e}_1,\quad v_2=\widehat{e}_2+b\widehat{e}_1.$$

Thus, every complement V of $\mathfrak{h}$ in $\mathfrak{g}$ is of the form

$$V=\{\widehat{e}_6+a\widehat{e}_1,\widehat{e}_2+b\widehat{e}_1\}$$

$\square$

Proof of Lemma 2. Let U be any complement of $\mathfrak{h}$ in $\mathfrak{g}$. By Lemma 1, $\exists a,b\in\mathbb{R}$ such that

$$U=\{\widehat{e}_6+a\widehat{e}_1,\widehat{e}_2+b\widehat{e}_1\}$$

Let $p=(0,-a)$. Then,

$$\begin{aligned} Ad_{T_p}(\widehat{e}_6+a\widehat{e}_1) &= \widehat{e}_6-a\widehat{e}_1+a\widehat{e}_1=\widehat{e}_6\\ Ad_{T_p}(\widehat{e}_2+b\widehat{e}_1) &= \widehat{e}_2+b\widehat{e}_1\end{aligned}$$

or $V=Ad_{T_p}U$ as required. $\square$

Proof of Lemma 3. Assume $p=(p_x,p_y)\in\mathbb{R}^2$ is such that $Ad_{T_p}V_b=V_c$. Using $\widehat{e}_2+c\widehat{e}_1\in Ad_{T_p}V_b$, we see that $\exists\alpha,\beta\in\mathbb{R}$ such that

$$\widehat{e}_2+c\widehat{e}_1=\alpha(\widehat{e}_6+p_y\widehat{e}_1-p_x\widehat{e}_2)+\beta(\widehat{e}_2+b\widehat{e}_1)$$

which implies that $\alpha=0,\beta=1$ and $b=c$.

If $Ad_{T_p}V_b=V_b$, then $\widehat{e}_6\in Ad_{T_p}V_b$. There exist $\lambda,\mu\in\mathbb{R}$ such that

$$\widehat{e}_6=\lambda(\widehat{e}_6+p_y\widehat{e}_1-p_x\widehat{e}_2)+\mu(\widehat{e}_2+b\widehat{e}_1)$$

which implies that $\lambda=1,p_y+\mu b=0$ and $-p_x+\mu=0$. Thus, $p_y=-p_xb$.

Conversely, assume that $p_y=-p_xb$. Let $\lambda=1,\mu=p_x$. Then,

$$\widehat{e}_6=\lambda Ad_{T_p}\widehat{e}_6+\mu Ad_{T_p}(\widehat{e}_2+b\widehat{e}_1)\in Ad_{T_p}V_b$$

Since $\widehat{e}_2+b\widehat{e}_1=Ad_{T_p}(\widehat{e}_2+b\widehat{e}_1)\in Ad_{T_p}V_b$ we have that $V_b=Ad_{T_p}V_b$. $\square$

Proof of Lemma 4. Let (v_1,v_2) be a basis of V_b. Then,

$$\begin{aligned} v_1 &= \lambda_1\widehat{e}_6+\lambda_2(\widehat{e}_2+b\widehat{e}_1)\\ v_2 &= \mu_1\widehat{e}_6+\mu_2(\widehat{e}_2+b\widehat{e}_1)\end{aligned}\qquad\text{where}\quad \begin{vmatrix}\lambda_1 & \lambda_2\\ \mu_1 & \mu_2\end{vmatrix}\neq 0$$

Case 1: $\lambda_1=0$. Then, $\mu_1\neq -$ and $\lambda_2\neq 0$. Let

$$\begin{aligned} v_1' &= \frac{1}{\lambda_2}v_1=\widehat{e}_2+b\widehat{e}_1\\ v_2' &= \frac{1}{\mu_1}v_2=\widehat{e}_6+\frac{\mu_2}{\mu_1}(\widehat{e}_2+b\widehat{e}_1)\end{aligned}$$

Let $p=(p_x,-p_xb)$. We have

$$\begin{aligned} Ad_{T_p}v_1' &= \widehat{e}_2+b\widehat{e}_1\\ Ad_{T_p}v_2' &= \widehat{e}_6-p_xb\widehat{e}_1-p_x\widehat{e}_2+\frac{\mu_2}{\mu_1}\widehat{e}_2+\frac{\mu_2}{\mu_1}b\widehat{e}_1\end{aligned}$$

By taking $p_x=\frac{\mu_2}{\mu_1}$, we get

$$\begin{aligned} Ad_{T_p}v_1' &= \widehat{e}_2+b\widehat{e}_1\\ Ad_{T_p}v_2' &= \widehat{e}_6\end{aligned}$$

Thus, after rescaling and Ad_{T_p}, the basis (v_1,v_2) becomes $(\widehat{e}_2+b\widehat{e}_1,\widehat{e}_6)$. Similarly, if $\mu_1=0$, then $\lambda_1\neq 0,\mu_2\neq 0$, and every basis of V_b gets transformed to $(\widehat{e}_6,\widehat{e}_2+b\widehat{e}_1)$.

Case 2: $\lambda_1\neq 0,\mu_1\neq 0$. Then,

$$\begin{aligned} \frac{1}{\lambda_1}v_1 &= \widehat{e}_6+\frac{\lambda_2}{\lambda_1}(\widehat{e}_2+b\widehat{e}_1)\\ \frac{1}{\mu_1}v_2 &= \widehat{e}_6+\frac{\mu_2}{\mu_1}(\widehat{e}_2+b\widehat{e}_1)\end{aligned}$$

Let $p = \frac{\lambda_2}{\lambda_1}(1, -b)$. We have

$$\begin{aligned} Ad_{T_p}(\frac{1}{\lambda_1}v_1) &= \hat{e}_6 \\ Ad_{T_p}(\frac{1}{\mu_1}v_2) &= \hat{e}_6 + (\frac{\mu_2}{\mu_1} - \frac{\lambda_2}{\lambda_1})(\hat{e}_2 + b\hat{e}_1) \end{aligned}$$

Note that $\lambda := \frac{\mu_2}{\mu_1} - \frac{\lambda_2}{\lambda_1} \neq 0$. Thus, every basis of V_b gets transformed to $(\hat{e}_6, Ad_{T_p}\hat{e}_6)$, with $p = \lambda(-1, b)$.

APPENDIX D: PROOF OF PROPOSITION 4

Since $C_{\mathcal{M}_1}$ contains an open neighborhood U of e, $C_{\mathcal{M}_2}$ an open neighborhood V of e, and $U \times V$ is an open neighborhood of (e, e) of $N_1 \times N_2$, $U \cdot V$ is an open neighborhood of e of $N_1 \cdot N_2$ because of the equivalent manifold structures of $N_1 \times N_2$ with $N_1 \cdot N_2$. Thus $C_{\mathcal{M}}$ contains an open neighborhood of e in $N_1 \cdot N_2$. Since $DoF(\mathcal{M}) = DoF(\mathcal{M}_1) + DoF(\mathcal{M}_2) = dim(N_1) + dim(N_2) = dim(Q)$, we conclude that $\mathcal{M}$ generates Q.

APPENDIX E: PROOF OF PROPOSITION 5

We only show case (ii) as (i) follows as a special case. Assume that $dim(H_1) = m_1$, $dim(H_2) = m_2$ and $dim(H) = n$. Also assume that $\overline{\mathcal{M}}_{H_1} = \{\zeta_1, \ldots, \zeta_{m_1}\}$, $\overline{\mathcal{M}}_{H_2} = \{\eta_1, \ldots, \eta_{m_2}\}$, $\overline{\mathcal{M}}_{H_1/H} = \{\delta_1, \cdots, \delta_{m_1-n}\}$, $\overline{\mathcal{M}}_{H_2/H} = \{\gamma_1, \cdots, \gamma_{m_2-n}\}$. Thus, the twist representation of $\mathcal{M}_1 = \mathcal{M}_{H_1/H} \cdot \mathcal{M}_{H_2}$ is given by

$$\overline{\mathcal{M}}_1 = \{\delta_1, \cdots, \delta_{m_1-n}, \eta_1, \ldots, \eta_{m_2}\}$$

and the forward kinematic map of the associated manipulator has the form

$$\begin{aligned} f_1 \quad &: \mathbb{R}^{m_1+m_2-n} \to SE(3) \\ &: (\theta_1, \cdots, \theta_{m_1+m_2-n}) \mapsto \quad e^{\delta_1\theta_1} \cdots e^{\delta_{m_1-n}\theta_{m_1-n}} \\ &\qquad \cdot e^{\eta_1\theta_{m_1-n+1}} \cdots e^{\eta_{m_2}\theta_{m_1-n+m_2}} \end{aligned}$$

Since $e^{\delta_i\theta_i} \in H_1$ and $e^{\eta_j\theta_{m_1-n+j}} \in H_2$, by the product closure, the image of f_1 lies in Q. As Q is a regular submanifold of $SE(3)$, by [4], f_1 defines a smooth map into Q

$$\begin{aligned} \hat{f}_1 \quad &: \mathbb{R}^{m_1+m_2-n} \to Q \\ &: (\theta_1, \cdots, \theta_{m_1+m_2-n}) \mapsto \quad e^{\delta_1\theta_1} \cdots e^{\delta_{m_1-n}\theta_{m_1-n}} \\ &\qquad \cdot e^{\eta_1\theta_{m_1-n+1}} \cdots e^{\eta_{m_2}\theta_{m_1-n+m_2}} \end{aligned}$$

Since the twists in $\mathcal{M}_1$ are linearly independent, we have that $\overline{\mathcal{M}}_1 = T_eQ$ and $\hat{f}_{1*0}$ is an isomorphism. Thus by the inverse function theorem, $\hat{f}_1$ is a local diffeomorphism at 0. Hence, its image contains an open neighborhood of e in Q. Furthermore, as $DoF(\mathcal{M}_1) = m_1 + m_2 - n = dim(Q)$, we see that $\mathcal{M}_1$ generates Q.

A similar argument shows that $\mathcal{M}_2$ and $\mathcal{M}_3$ also generates Q.

APPENDIX F : A MORE GENERAL THEOREM THAN PROPOSITION 6

Theorem *1:* Let G be a submanifold of $SE(3)$. Suppose that $C_1, \cdots, C_k$ are connected regular submanifolds of G, each containing a connected regular submanifold $\mathbf{C}$, i.e,

$$\mathbf{C} \subseteq C_j, \quad j = 1, ..., k \tag{30}$$

and thus $\mathbf{C} \subseteq C_1 \cap \cdots \cap C_k$. If for some $x \in \mathbf{C}$, the following condition holds:

$$T_x\mathbf{C} = T_xC_1 \cap T_xC_2 \cap \cdots \cap T_xC_k \tag{31}$$

then, there exists a connected neighborhood U_x of x, where U_x is an open subset of G, such that $U_x \cap \mathbf{C} = U_x \cap (C_1 \cap \cdots \cap C_k)$. If for every $x \in \mathbf{C}$, Eq. (31) holds, then there exist a connected open subset W of G, such that $\mathbf{C} = W \cap (C_1 \cap \cdots \cap C_k)$.

Proof : Without loss of generality we prove only the case $k = 2$. Let $m := Dim(G)$, and $x \in \mathbf{C}$. Choose a coordinate neighborhood (U, ϕ) of x with local coordinates $(x_1, \cdots, x_m)$. In this neighborhood, C_1 can be expressed as

$$C_1 \cap U = \phi^{-1}\left\{(x_1, \cdots, x_m) \mid \begin{array}{l} h_1(x_1, \cdots, x_m) = 0, \cdots, \\ h_s(x_1, \cdots, x_m) = 0 \end{array}\right\}$$

for some functions $h_1, \cdots h_s$, and s the codimension of C_1 in G. Similarly, C_2 can be expressed as

$$C_2 \cap U = \phi^{-1}\left\{(x_1, \cdots, x_m) \mid \begin{array}{l} g_1(x_1, \cdots, x_m) = 0, \cdots, \\ g_l(x_1, \cdots, x_m) = 0 \end{array}\right\}$$

Thus,

$$(C_1 \cap C_2) \cap U = \phi^{-1}\left\{(x_1, \cdots, x_m) \mid \begin{array}{l} h_1 = 0, \cdots, h_s = 0, \\ g_1 = 0, \cdots, g_l = 0 \end{array}\right\}.$$

Since $T_x\mathbf{C} = T_xC_1 \cap T_xC_2$ at $x \in \mathbf{C} \subseteq C_1 \cap C_2$, we have

$$rank \begin{pmatrix} dh_1 \\ \vdots \\ dh_s \\ dg_1 \\ \vdots \\ dg_l \end{pmatrix}_{\phi(x)} = n$$

where n is the codimension of $\mathbf{C}$ in G. Without loss of generality, we assume that the first n rows of the above matrix are linearly independent at $\phi(x) \in \phi(\mathbf{C} \cap U)$. Thus

$$rank \begin{pmatrix} dh_1 \\ \vdots \\ dh_s \\ dg_1 \\ \vdots \\ dg_t \end{pmatrix}_{\phi(x)} = n$$

where $s + t = n$. By the submersion theorem [4], there is an open neighborhood U' of G around x, $U' \subseteq U$, such that

$$\tilde{\mathbf{C}} = \phi^{-1}\left\{(x_1, \cdots, x_m) \in \phi(U') \mid \begin{array}{l} h_1 = 0, \cdots, h_s = 0, \\ g_1 = 0, \cdots, g_t = 0 \end{array}\right\}$$

is an $m-n$ dimensional regular submanifold of G containing x. It is obvious that $(C_1 \cap C_2) \cap U' \subseteq \tilde{\mathbf{C}}$. Since $\mathbf{C} \subseteq (C_1 \cap C_2)$, we also get $\mathbf{C} \cap U' \subseteq (C_1 \cap C_2) \cap U' \subseteq \tilde{\mathbf{C}}$. As both $\mathbf{C} \cap U'$ and $\tilde{\mathbf{C}}$ are $m-n$ dimensional submanifolds of G, the former is contained in the latter. Thus, there exists an open neighborhood V of x in G, $V \subseteq U'$ such that $\mathbf{C} \cap V = (C_1 \cap C_2) \cap V = \overline{\mathbf{C}} \cap V$. V is the U_x we wanted. If for every $x \in \mathbf{C}$, Eq. (31) holds, then every x will have a U_x, and the union of all these U_x's gives the W we wanted, i.e, $W = \cup_{x \in \mathbf{C}} U_x$.

Replacing G in Theorem 1 by $SE(3)$ and C_j the desired end-effector motions of the subchains, we have Proposition 6.

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