THE EFFECTS OF WIDE-BAND SIGNALS ON RADAR ANTENNA DESIGN

Lawrence R. Dausin, 1/Lt, USAF
Kenneth E. Niebuhr, 1/Lt, USAF
Nils J. Nilsson, 2/Lt, USAF
Rome Air Development Center
Griffiss AFB, New York

Summary

The amount of information about the angle of arrival of a plane wave incident on a line aperture is shown to depend on the bandwidth of the incident signal as well as on the extent of the aperture. Conventional antenna theory assumes the angular resolution of an antenna depends only on the aperture extent and illumination. However, by autocorrelation techniques further angular resolution can be obtained. This additional directivity depends on the extent of the signal bandwidth. It is shown for line arrays that the obtainable directivity information depends on the aperture illumination function and the autocorrelation function of the incident signal. In view of the vast amount of information available concerning conventional antennas, the results of this paper have been related to conventional antenna theory. The antenna designer can specify the desired directivity pattern in terms of a conventional single-frequency aperture illumination function. The effects of this single frequency illumination function can then be achieved for wide-band signals by a simpler illumination function. Some examples of the theory are given.

Introduction

The directivity pattern of a colinear antenna array can be calculated by taking the Fourier transform of the aperture illumination function.[1] However, the pattern so obtained applies only to CW sinusoidal signals. In pulse radar applications the incident signal is not a CW sinusoid, and thus the actual pattern will differ from the calculated pattern. This difference becomes important when the reciprocal of the signal bandwidth approaches the time required for the signal to travel from one end of the antenna to the other. It is the purpose of this paper to show how the dependence of the directivity pattern on the type of signal used might lead to interesting consequences in radar antenna design.

Let us illustrate these remarks by a simple example. Consider the two element antenna array shown in Fig. 1. The array consists of two isotropic collectors spaced a distance $2x_1$ apart. A radar echo signal, s(t), of known form is incident upon the array at some angle Θ. Using the ordinary Fourier transform technique for calculating the directivity pattern, g(u), we obtain the standard multi-lobe interferometer pattern. If I(x) is the antenna aperture illumination density function and if g(u) is the antenna directivity pattern for a sine wave of angular frequency, ω_0, then the following Fourier transform pairs relate I(x) to g(u).

$$g(u) = \int_{-\infty}^{\infty} I(x) e^{j\omega_0 u x/c} \, dx \qquad (1)$$

and

$$I(x) = \frac{\omega_0}{2\pi c} \int_{-\infty}^{\infty} g(u) e^{-j\omega_0 u x/c} \, du$$

where

x = distance from aperture center in meters

u = sin Θ

c = velocity of light in meters/sec

In the two-element array shown in Fig. 1, I(x) is a Dirac delta function placing all of the weighting density at $\pm x_1$. By evaluating the Fourier transform, we find that the directivity pattern for this array is given by

1 Superscript numbers refer to the References at the end of the paper.

$$g(u) = \cos \frac{\omega_0 u x_1}{c} \qquad (2)$$

When $2x_1 \gg \lambda_0 = \frac{2\pi c}{\omega_0}$, this pattern has many lobes and thus no unambiguous directivity information can be obtained. Such a pattern is shown in Fig. 2. However, the pattern shown in Fig. 2 is relevant only when the incident signal is a CW sinusoid. It is well known that the directivity ambiguity can be reduced if s(t) is a wide-band signal. Kock and Stone[2] have pointed out that by cross-correlation techniques such two-element arrays can be used for passive locators of wide-band noise sources. Other work by Jacobsen[3] explored the capabilities and limitations of two-element array correlation techniques. For example, if a radar target reflects a short pulse toward the array from an angle θ, this angle can be estimated by measuring the relative delay between the reception of the echoes by elements number one and two. When the reflected echo is received simultaneously by both elements, the target must be perpendicular to the line connecting the elements. If element number one receives the echo first, then the target must be to the left of the antenna, and so forth. If the return echo signal, s(t), is not a short pulse but has the same bandwidth as does a short pulse, then by matched filter techniques it can be compressed to a short pulse and the same principle will apply.*

Mathematical Formulation

Let us refer again to the two element antenna illustrated in Fig. 1. The relative delay between reception of the echoes by elements number one and two will be $\frac{2x_1 \sin\theta}{c} \triangleq \frac{2x_1 u}{c}$. If we pass the sum of the voltages from the elements through a matched filter we will obtain the output

$$\Phi(u,t) = R_s\left(t - \frac{x_1 u}{c}\right) + R_s\left(t + \frac{x_1 u}{c}\right) \quad (3)$$

where

$R_s(\tau)$ = the autocorrelation function of s(t)

Φ (u,t) will peak up for the appropriate values of u and t. In our example, Φ (u,t) will reach a maximum when t = 0 and when u = 0. We can distinguish without difficulty a target producing an output Φ_1(u,t) from one causing the output Φ_2(u,t) only if Φ_1 is small when Φ_2 attains its maximum. Thus, Φ (u,t) is an "ambiguity function" in both angle and range. Its use is similar to the ambiguity function in range and velocity of Woodward.[4] When the range of the target is known exactly, Φ(u,o) is the ambiguity function of the target's angle. Let us call this ambiguity function, Φ (u), the actual pattern of this two-element array relative to the signal s(t). Then,

$$\Phi(u) = R_s\left(-\frac{x_1 u}{c}\right) + R_s\left(\frac{x_1 u}{c}\right) = 2R_s\left(\frac{x_1 u}{c}\right) \quad (4)$$

It should be noted at this point that when s(t) is a CW sinusoid, Φ (u) becomes the interferometer pattern given in Eq.(2). But when s(t) is a band-limited signal whose power is spread evenly over a bandwidth of B cps, then Φ(u) takes on a $\frac{\sin u}{u}$ shape as shown in Fig. 3. The center frequency of the band is f_0. Φ (u) is a maximum when u = 0 or when the target is reflecting toward the array's broadside.* Targets off broadside by an amount such that u is much greater than $\frac{c}{Bx_1}$ can easily be distinguished from broadside targets. The pattern beamwidth then is about $\frac{c}{Bx_1}$ radians.

Let us relate the results we have obtained so far to conventional theory. To obtain the pattern shown in Fig. 3 the aperture illumination would have to be as shown in Fig. 4 if a CW sinusoid, of frequency f_0, were the incident signal. We will call this aperture illumination the artificial aperture, $I_0(y)$. That is, to obtain the same pattern as that shown in Fig. 3 we would have to use the artificial aperture illumination if we restricted ourselves to conventional theory. By using the signal s(t), however, we can

---

* After passing s(t) through a matched filter the signal form will be changed to the auto correlation function of s(t). The autocorrelation function will be a pulse whose width is inversely proportional to the signal's bandwidth.

* By adding the appropriate time delay to the output of one of the elements we can make Φ (u) peak up for targets off broadside. In this way the array can be "steered".

achieve the same effect with two isotropic elements as shown in Fig. 1. The bandwidth of the signal gives an apparent breadth to the elements of the array. Consequently the directivity of a pair of these apparently isotropic elements increases in proportion to the percentage bandwidth of the signal being received.

Let us now consider a more general case--one in which our array itself is directive for single frequencies and is made up of a series of 2N isotropic elements spaced Δ apart. We then have elements at $x = \pm\Delta, \pm 2\Delta, \ldots \pm i\Delta, \ldots$. Suppose we weight the output of the ith element by I_i so that the weighting is an even function about the array center. We then proceed as before, passing the weighted signals through a matched filter and then adding the resulting autocorrelation functions. The result, when the target range is known, is the angle ambiguity function for this multi-element array. That is,

$$\phi(u) = \sum_{i=-N}^{N} I_i R_S\left(\frac{u i \Delta}{c}\right) \qquad (5)$$

We can generalize Eq.(5) to an integral as the point collectors come closer together, and as I_i approaches a continuous density function of x. The result is

$$\phi(u) = 2\int_0^{\infty} I(x) R_S\left(\frac{u x}{c}\right) dx \qquad (6)$$

Eq.(6) is the angle pattern that can be obtained with an actual antenna aperture illumination function I(x), and an incident signal whose autocorrelation function is $R_s(\tau)$. We have assumed however, that the signal form is known exactly--this assumption implies that the reflecting target's range and velocity are known exactly. The result also assumes that the aperture illumination I(x), is an even function of x. Since $R_s(\tau)$ is even, so also will be $\phi(u)$.

Let us now determine what the artificial aperture illumination $I_o(y)$ would have to be to give the pattern $\phi(u)$ if a sinusoidal signal of radian frequency ω_o were used. For this calculation we will use the cosine form of the second formula in Eq.(1) replacing g(u) by the desired pattern $\phi(u)$.

$$I_o(y) = \frac{\omega_o}{\pi c}\int_0^{\infty} \phi(u) \cos \frac{\omega_o u y}{c} du \qquad (7)$$

Substituting Eq.(6) into Eq.(7) we obtain

$$I_o(y) = \frac{2\omega_o}{\pi c}\int_0^{\infty}\int_0^{\infty} I(x) R_S\left(\frac{u x}{c}\right) \cos \frac{\omega_o u y}{c} dx\, du \qquad (8)$$

After changing the order of integration, we obtain

$$I_o(y) = \frac{2\omega_o}{\pi c}\int_0^{\infty} I(x) dx \int_0^{\infty} R_S\left(\frac{u x}{c}\right) \cos \frac{\omega_o u y}{c} du \qquad (9)$$

From the Wiener-Kintchine theorem the power spectral density, $G_s(\omega)$, of the incident signal is related to the autocorrelation function by the Fourier integral:

$$G_s(\omega) = \frac{1}{\pi}\int_0^{\infty} R_S(\tau) \cos \omega\tau \, d\omega \qquad (10)$$

Making use of Eq.(10) in the inner integral of Eq.(9) yields

$$I_o(y) = 2\omega_o \int_0^{\infty} \frac{I(x)}{x} G_S\left(\frac{\omega_o y}{x}\right) dx \qquad (11)$$

Eq.(11)* gives the aperture illumination that would have to be used in conjunction with a sine wave signal to give the same pattern obtained by using some aperture illumination I(x) in conjunction with a signal whose power spectral density is $G_s(\omega)$.

---

* If x and y are independent random variables with density functions $f_X(x)$ and $f_Y(y)$, respectively, and if z = x·y, then the density function $f_Z(z)$ is given by

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{f_X(x)}{|x|} f_Y(z/x) dx$$

The consequences of the analogy between the equation for $f_Z(z)$ and Eq.(11) have not yet been fully explored.

Examples

An expression, Eq.(11), has been derived in which the illumination function of an aperture incorporating a correlation scheme and the power spectrum of the incident signal are related to the illumination function of an artificial aperture operating at a single frequency, ω_o. The use of this relationship can be shown by a simple example.

Consider the case of a uniformly illuminated aperture of length D

$$I(x) = \begin{cases} 1/D & \text{for } |x| < D/2 \\ 0 & \text{otherwise} \end{cases} \quad (12)$$

and a rectangular power spectral density function centered about ω_o and of width $2\pi B$ radians/sec

$$G_s(\omega) = \begin{cases} \frac{1}{4\pi B} & \text{for } \omega_a < |\omega| < \omega_b \\ 0 & \text{otherwise} \end{cases} \quad (13)$$

Then, from Eq.(11)

$$I_o(y) = \frac{2\omega_o}{4\pi B D}\int_{x_1}^{x_2} \frac{dx}{x} = \frac{1}{kD} \ln \frac{x_2}{x_1} \quad (14)$$

where the limits of integration, determined by the boundaries of $G_s(\omega)$, are

$$x_1 = \frac{\omega_o}{\omega_b} y$$

$$x_2 = \begin{cases} D/2 \\ \frac{\omega_o}{\omega_a} y \end{cases} \quad \text{whichever is smaller}$$

and

$k = \frac{2\pi B}{\omega_o}$ = percentage bandwidth of the incident signal

It is noted that $I_o(y) = 0$ for $|y| > \frac{\omega_b}{\omega_o}\frac{D}{2}$. The complete specification of $I_o(y)$ is then

$$I_o(y) = \begin{cases} \frac{1}{kD} \ln \frac{\omega_b}{\omega_a} & \text{for } |y| < \frac{\omega_a}{\omega_o}\frac{D}{2} \\ \frac{1}{kD} \ln\left[\frac{\omega_b}{\omega_o}\frac{D}{2}\frac{1}{y}\right] & \text{for } \frac{D}{2}\frac{\omega_a}{\omega_o} < |y| < \frac{D}{2}\frac{\omega_b}{\omega_o} \\ 0 & \text{for } |y| > \frac{\omega_b}{\omega_o}\frac{D}{2} \end{cases} \quad (15)$$

$I(x)$, $G_s(\omega)$ and $I_o(y)$ for this example are shown in Fig. 5.

It is apparent for this case that unless the percentage bandwidth becomes appreciable the artificial aperture illumination will be only slightly different from the actual illumination function. Thus, the aperture's pattern for wide-band signals will differ from the sinusoidal pattern only when k becomes large. The uniform illumination function used in this example makes efficient use of the length D of the aperture. The use of wide-band signals does little to increase the directivity information available from such an efficiently illuminated aperture unless k becomes large. The two element array considered earlier, however, makes inefficient use of its aperture. It was noted that the use of wide-band signals greatly increased the directivity information available from the two element array.

An interesting example will now be given to show how a uniform artificial aperture illumination can be obtained from an actual illumination of delta functions (discrete elements) with a saving on the number of elements which would be required in the single frequency case. A band-limited spectrum will be used to construct the artificial illumination. If the actual element spacing is equal to or smaller than $\lambda_b/2$ (λ_b being the wavelength of the highest frequency received) it is assumed the resultant pattern will not differ from that of a continuous illumination.

As an introduction to the problem it is advisable to reconsider the two isotropic element case. The elements were located at $x = \pm x_1$ as shown in Fig. 1 and the resultant artificial illumination due to the bandwidth of the received signal is shown in Fig. 4. The addition of two more elements at distance $x = \pm x_2$ makes it possible to increase the extent of the artificial illumination. With the proper choice of x_2, the artificial illumination shown in Fig. 6 can be achieved. The required number of elements needed in the single frequency case to produce the same pattern could be greater than four and thus a saving of elements might be achieved by using wideband signals. The number of elements to be saved will be shown to be dependent upon the extent of the desired illumination and the bandwidth of the received signal.

In order to derive the relationship existing between bandwidth and element spacing, the general case where I(x) is a series of delta functions spaced at $x = x_1, x_2, x_3 --- x_N$ and weighted by $A_1, A_2, \ldots A_N$, respectively, will be considered. The artificial illumination as expressed by Eq.(11) will be of the form

$$I_o(y) = \sum_{i=1}^{N} \frac{A_i}{x_i} G_s\left(\frac{\omega_o y}{x_i}\right) \tag{16}$$

The spectral density of the received waveform has been assumed to be flat, centered about ω_o, and band-limited to W cps and thus can be expressed as

$$G_s\left(\frac{\omega_o y}{x_i}\right) = \begin{cases} 1/4\pi W & \text{for } \omega_a < \left|\frac{\omega_o y}{x_i}\right| < \omega_b \\ 0 & \text{elsewhere} \end{cases} \tag{17}$$

By specifying the desired illumination to be uniform, the spacing of the elements must be chosen judicially and the amplitudes of the individual artificial illumination due to any two elements must be made equal. The second condition can be achieved by adjusting A_i, the gain of the individual elements, to equal x_i.

The artificial illumination now becomes

$$I_o(y) = \sum_{i=1}^{N} G_s\left(\frac{\omega_o y}{x_i}\right) \tag{18}$$

The judicial spacing of the elements can be obtained by requiring the leading edge of the artificial illumination for element pair x_{n-1} to just meet the trailing edge of the artificial illumination for element pair x_n. Since the leading edge due to element pair x_{n-1} is specified by $\omega_b = \frac{\omega_o y}{x_{n-1}}$, and the trailing edge due to element pair x_n by $\omega_a = \frac{\omega_o y}{x_n}$, the above condition will be met when

$$\omega_b x_{n-1} = \omega_a x_n \tag{19}$$

Solving for x_n gives

$$x_n = \frac{\omega_b}{\omega_a} x_{n-1} = \left(\frac{\omega_b}{\omega_a}\right)^2 x_{n-2} = \left(\frac{\omega_b}{\omega_a}\right)^{n-1} x_1 \tag{20}$$

where x_1 is the first element whose trailing edge is matched to the leading edge of the next element. The reason x_1 has been defined in such a vague manner is because of the previous condition that the spacing between elements shall not be smaller than $\lambda_b/2$, and therefore x_1 in general will not be the closest element to the center of the array. The element spacing for all elements between $x = 0$ and $x = x_1$ will be $\lambda_b/2$ (see Fig. 7).

It is now possible to solve for the position of x_1 by first noting that the relative distance between any two elements to the left of x_1, is $\lambda_b/2$. Therefore the position of x_1 from the center of the array is $M\lambda_b/2$ where M is an unknown integer. In order to determine M, the distance between elements to the right of x_1 must be formulated and will yield the expression

$$x_m - x_{m-1} > \lambda_b/2 \tag{21}$$

or

$$x_{m-1}\left(\frac{\omega_b}{\omega_a} - 1\right) > \lambda_b/2 \tag{22}$$

By letting $x_{m-1} = x_1$ in Eq.(22) we obtain

$$x_1\left(\frac{\omega_b}{\omega_a} - 1\right) > \lambda_b/2 \tag{23}$$

or substituting $x = M\frac{\lambda_b}{2}$ into the above expression gives

$$M\left(\frac{\lambda_b}{2}\right)\left(\frac{\omega_b}{\omega_a} - 1\right) > \lambda_b/2 \tag{24}$$

and therefore

$$M > \frac{1}{\frac{\omega_b}{\omega_a} - 1} = \frac{1}{k} - \frac{1}{2} \tag{25}$$

where k is the percentage bandwidth of the received signal. The smallest integer M which satisfies Eq.(25) must be considered the only possible solution if overlapping of individual artificial illumination is to be prevented.

In order to determine the number of elements needed to construct the artificial illumination, the upper extent, y_u, of y must be found. This is determined from the expression

$$\omega_b = \frac{\omega_0}{x_N} y_u \qquad (26)$$

x_N being the last element of the array. Solving for y_u gives

$$y_u = \frac{\omega_b}{\omega_0} x_N = \left(1 + \frac{k}{2}\right) x_N \qquad (27)$$

The total number of elements needed for the array using bandwidth will be 2 (N + M - 1). Through the use of Eqs.(20), (25), and (27); N and M can be determined from specified values of k and y_u. Note that either y_u and k might have to be slightly adjusted in order to insure that N and M will be integers. A numerical example has been solved using $k = 0.1$ and $y_u = 50\,\lambda_0$. The required number of elements was 66 as compared with the 200 which would be required in the single frequency case.

There is a great saving of required elements for the particular numerical example solved. However this percentage decrease should not be expected in smaller arrays. It is further pointed out that there will be a decrease in the signal to noise ratio due to the decrease in the number of receiving elements.

Conclusion

Examples have been given to show that the bandwidth of a signal incident upon a colinear array accounts for the extra directivity information obtainable from such an array. Before the effects of bandwidth become appreciable the length of time represented by the reciprocal of the bandwidth must be less than the propagation time across the aperture. One of the most interesting, and possibly most profitable, ways in which large bandwidths alter the antenna design problem is in the saving of a substantial number of elements required to synthesize a given receive pattern.

Many questions remain to be answered. For instance, is there a generalized Radar Uncertainty Principle, similar to that of Woodward, covering the six parameters of interest: range, range-rate, the two angles, and the two angle-rates? The one dimensional linear arrays considered in this paper should be extended to area or perhaps volume arrays. Also of interest are the very wide-base-leg, wide-band systems in which the target angle, and perhaps the target echo form, are different for the different collecting elements of the array. Finally the problem of the effects of corrupting noise on both the resolution and the accuracy capabilities of these spatial systems should be studied.

Acknowledgment

The authors wish to express their appreciation for the valuable comments and advice of Mr. Fred I. Diamond and Mr. Moses A. Diab of the Rome Air Development Center, and of Dr. Morris Handelsman, formerly of the Rome Air Development Center. Also, helpful suggestions were kindly offered by Dr. Raymond M. Wilmotte of RCA.

References

1. J. D. Kraus, Antennas, McGraw Hill, New York 1950

2. W. E. Kock and J. L. Stone, "Space-Frequency Equivalence," Proc of the IRE, (Letter), Vol. 46, No. 2 p 499, Feb 1958

3. M. J. Jacobsen, "Analysis of a Multiple Receiver Correlation System," JASA, Vol. 29, No. 12, pp 1342-1347, December 1957

4. P. M. Woodward, Probability and Information Theory, with Applications to Radar, McGraw Hill, New York 1953

5. "Available Source Information in Receiving Apertures," E. R. Hiller, Jr., TR #3, M.I.T. Res. Lab. of Electronics, Sept. 1, 1956

6. S. Silver, Microwave Antenna Theory and Design, M.I.T. Radiation Laboratory Series, Vol 12, McGraw Hill, New York 1949

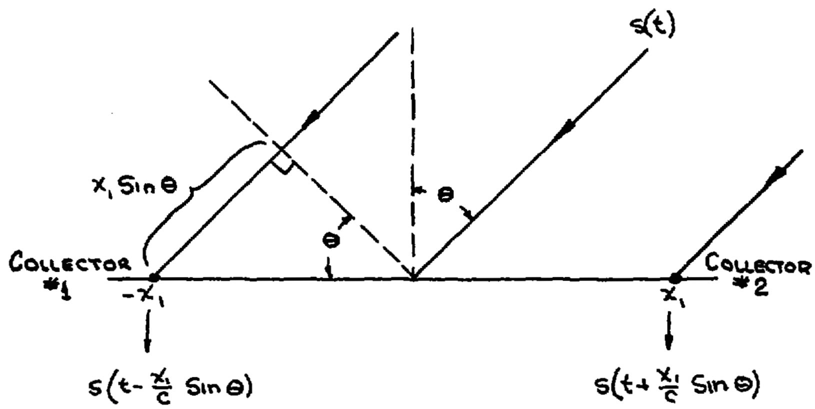

Fig. 1. Two isotropic elements with s(t) incident at an angle θ.

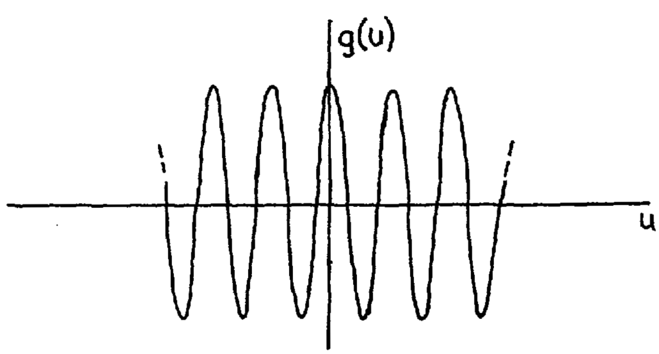

Fig. 2. Multilobed pattern for a two-element array.

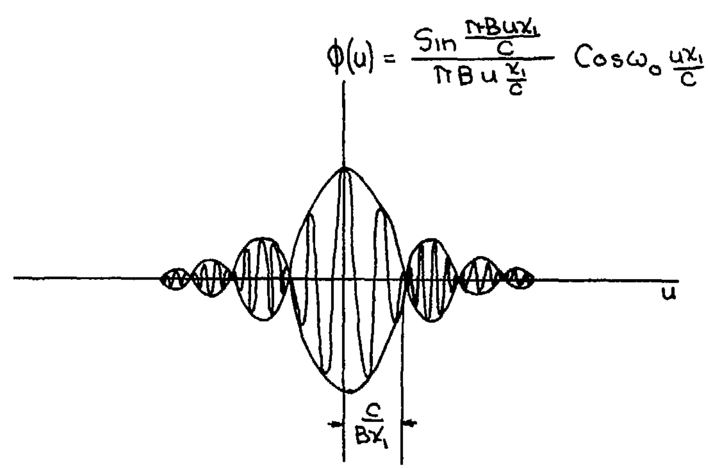

Fig. 3. The actual pattern of the two-element array when s(t) has center frequency of ω_0 and bandwidth of β cps.

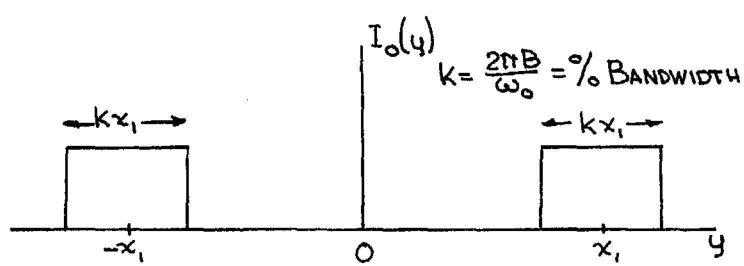

Fig. 4. The artificial aperture.

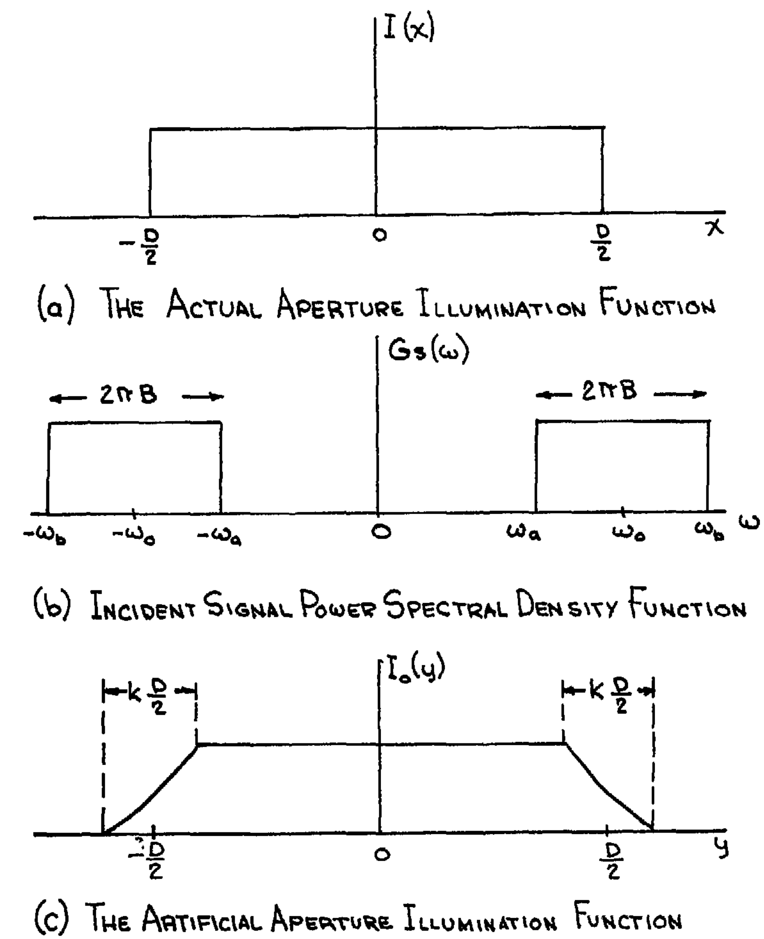

Fig. 5. Typical choices for I(x) and $G_s(\omega)$ and the corresponding $I_o(y)$.

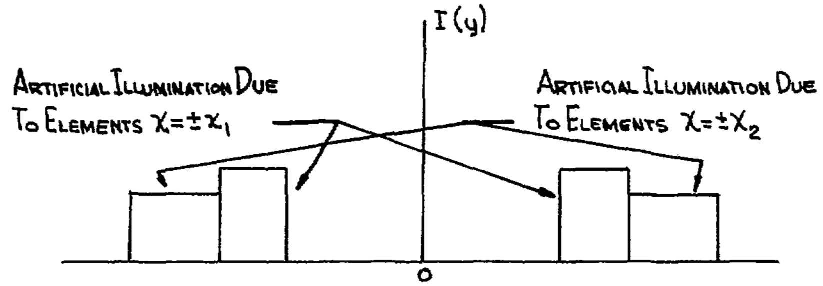

Fig. 6. The artificial illumination for a four-element array.

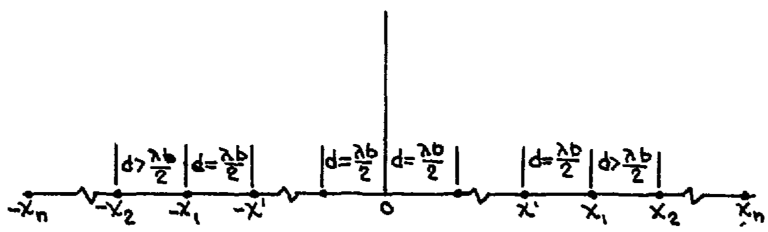

Fig. 7. Element spacing for a uniform artificial illumination.