Thin Junction Tree Filters for Simultaneous Localization and Mapping

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Simultaneous Localization and Mapping

A mobile robot navigating in an unknown environment must incrementally
1. build a map of its surroundings and
2. localize itself within that map.
The traditional approach: Kalman filters

- View SLAM as a state estimation problem in a linear-Gaussian dynamical system
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- System state at time $t$: $[R_t; L_1; \ldots; L_{N_t}]$
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- System state at time $t$: $[R_t; L_1; \ldots; L_{N_t}]$
- The belief state is a Gaussian $\mathcal{N}(\mu_t, \Sigma_t)$
The traditional approach: Kalman filters

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- System state at time $t$: $[R_t; L_1; \ldots; L_{N_t}]$
- The belief state is a Gaussian $\mathcal{N}(\mu_t, \Sigma_t)$
- Time & space complexity: $\Theta(N_t^2)$
Thin junction tree filters

- TJTF: a novel algorithm for approximate filtering in dynamic Bayesian networks
Thin junction tree filters

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- When applied to the SLAM problem, we obtain space complexity: $O(N_t)$
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- When applied to the SLAM problem, we obtain
  
  space complexity: $O(N_t)$
  
  time complexity: $O(N_t)$ or $O(1)$
Filtering in dynamic Bayesian networks

A dynamic Bayesian network (DBN) is a compact representation of a complex stochastic process.
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Belief state at time $t$: $b_t = p(x_t, y_t, z_t \mid \overline{w}_{1:t-1})$
Filtering in dynamic Bayesian networks

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Belief state at time $t$: $b_t = p(x_t, y_t, z_t \mid \overline{w}_{1:t-1})$

Filtering: iteratively update $b_t \xrightarrow{\overline{w}_t} b_{t+1}$
Complexity of filtering in DBNs

The DBN is compact, but the belief state is not:

\[ p(x_1)p(y_1)p(z_1) \]
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Exact filtering in DBNs is intractable.
The Boyen & Koller (1998) Algorithm

Choose a fixed, tractable form for the belief state and **project** to the closest density of that form:

\[ p(x_1)p(y_1)p(z_1) \quad p(x_2, y_2, z_2 | \overline{w}_1) \]
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p(x_1)p(y_1)p(z_1) \quad p(x_2, y_2, z_2 \mid \overline{w}_1)
\]

\[
p(x_2 \mid \overline{w}_1) \times p(y_2, z_2 \mid \overline{w}_1)
\]
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Choose a fixed, tractable form for the belief state and project to the closest density of that form:

\[ p(x_1)p(y_1)p(z_1) \quad p(x_2, y_2, z_2 \mid w_1) \quad p(x_2 \mid w_1) \times p(y_2, z_2 \mid w_1) \]

Problem: what is the best tractable form?
An example SLAM DBN

- **R₁**: robot pose and velocity
- **Z₁, Z₂, Z₃**: landmark measurements
- **L₁, L₂, L₃, L₄, L₅**: landmarks' positions
- **Y₂, Y₃**: odometry measurements
Filtering the SLAM DBN: estimation

\[ p(r_1) \]
Filtering the SLAM DBN: estimation

\[ p(r_1, l_1 \mid z_1) \propto p(r_1) \cdot p(l_1) \cdot p(z_1 \mid r_1, l_1) \]
Filtering the SLAM DBN: estimation

\[ p(r_1, l_1 | z_1) \propto p(r_1) \cdot p(l_1) \cdot p(z_1 | r_1, l_1) \]
Filtering the SLAM DBN: estimation

\[ p(r_1, l_1 \mid z_1) \propto p(r_1) \cdot p(l_1) \cdot p(z_1 \mid r_1, l_1) \]

Observing landmark \( i \) connects \( R_t \) and \( L_i \).
Filtering the SLAM DBN: prediction

\[ p(r_1, l_{1:3} \mid \bar{z}_{1:3}) \]
Filtering the SLAM DBN: prediction

\[ p(r_{1:2}, l_{1:3} \mid z_{1:3}) = p(r_1, l_{1:3} \mid z_{1:3}) \cdot p(r_2 \mid r_1) \]
Filtering the SLAM DBN: prediction

\[ p(r_{1:2}, l_{1:3} \mid \bar{z}_{1:3}) = p(r_1, l_{1:3} \mid \bar{z}_{1:3}) \cdot p(r_2 \mid r_1) \]

The prediction phase connects \( R_t \) and \( R_{t+1} \).
Filtering the SLAM DBN: roll-up

\[ p(r_{1:2}, l_{1:3} \mid \overline{z}_{1:3}) \]
Filtering the SLAM DBN: roll-up

\[
p(r_2, l_{1:3} \mid \bar{z}_{1:3}) = \sum_{r_1} p(r_{1:2}, l_{1:3} \mid \bar{z}_{1:3})
\]
Filtering the SLAM DBN: roll-up

After roll-up, the SLAM belief state has no conditional independencies.
Junction trees
An undirected tree whose nodes are sets of variables...
Junction trees

An undirected tree whose nodes are sets of variables...with the running intersection property.
Junction trees

An undirected tree whose nodes are sets of variables... with the **running intersection** property.
Junction trees

An undirected tree whose nodes are sets of variables... with the **running intersection** property.

A density $p$ **decomposes on** $T$ if we can write

$$p(x) = \frac{\prod_C \phi_C(x_C)}{\prod_S \phi_S(x_S)}$$
Junction tree inference

\[ p = \frac{\prod_c \phi_C}{\prod_s \phi_S} \]
Junction tree inference

initialize

\[ p = \frac{\prod_C \phi_C}{\prod_S \phi_S} \]

pass messages
Junction tree inference

initialize

$\phi(u, v, w)$

$\phi(v, w)$

$\phi(v, w, y)$

$\phi(v, x)$

$\phi(v, y)$

$\phi(v, y, z)$

$\mathbf{p} = \prod_C \phi_C \prod_S \phi_S$

pass messages

$\phi(v, w)$

$\phi(v, x)$

$\phi(v, y)$

$\phi(v, y, z)$

$\mathbf{p} = \prod_C \phi_C \prod_S \phi_S$

calibrated

$p(u, v, w)$

$p(v, w)$

$p(v, w, y)$

$p(v)$

$p(v, x)$

$p(v, y)$

$p(v, y, z)$

$p(v, y)$

$p(v, x)$

$p(v, y, z)$
Junction tree inference

initialize

\[ \phi(u, v, w) \]

\[ \phi(v, w) \]

\[ \phi(v, w, y) \]

\[ \phi(v, y) \]

\[ \phi(v, y, z) \]

\[ p = \frac{\prod C \phi_C}{\prod S \phi_S} \]

pass messages

\[ \phi(v, y) \]

\[ \phi(v, x) \]

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\[ \phi(v, w) \]

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Junction tree inference

**initialize**

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\phi(v, w) \\
\phi(v, w, y) \\
\phi(v, y) \\
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p = \frac{\prod_C \phi_C}{\prod_S \phi_S}
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**pass messages**

**calibrated**

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p(v, w) \\
p(v, w, y) \\
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p = \frac{\prod_C p_C}{\prod_S p_S}
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complexity scales with width
Junction tree filters

The belief state is a calibrated junction tree.

\[ p^{(t)} = \prod_C p_C^{(t)} \prod_S p_S^{(t)} \]
The belief state is a calibrated junction tree.

$$p^{(t)} = \frac{\prod_C p_C^{(t)}}{\prod_S p_S^{(t)}}$$

$$p^{(t+1)} = \frac{\prod_C p_C^{(t+1)}}{\prod_S p_S^{(t+1)}}$$
Estimation and prediction

To multiply $\psi(x_1, \ldots, x_k)$ into $p$ and recalibrate:

1. Find a cluster $C$ that contains $X_1, \ldots, X_k$.
2. Multiply $\psi$ into $\phi_C$.
3. Distribute evidence from $C$ (if needed).
Estimation and prediction

To multiply $\psi(x_1, \ldots, x_k)$ into $p$ and recalibrate:

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If there is no cover, we must make one.
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If there is no cover, we must make one.
Pushing variables to create covers

$V, X, Y$

$Y$

$Y, Z$
Pushing variables to create covers

\[ \text{push } X \]
Pushing variables to create covers

\[ V, X, Y \]

\[ Y \]

\[ Y, Z \]

\[ X, Y, Z \]

push \( X \)

pass message
To marginalize out a variable $X$ that is in only one cluster $C$ . . .

$\phi(v, x, y)$
Roll-up

To marginalize out a variable $X$ that is in only one cluster $C$ . . . marginalize $X$ out of $\phi_C$. 

\[ \sum_x \phi(v, x, y) \]
Roll-up

To marginalize out a variable $X$ that is in only one cluster $C$ . . . marginalize $X$ out of $\phi_C$.

If $X$ is in more than one cluster, we must first merge the clusters containing $X$ . . .
Merging adjacent clusters

\[ \phi(x, y) \]

\[ \phi(y) \]

\[ \phi(y, z) \]

\[ \phi(x, y) \]

\[ \phi(y, z) \]

\[ \phi(y) \]
Thin junction tree filters (TJTF)

Pushing and merging increase the width of the junction tree, and therefore the complexity.

estimation prediction roll-up
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Pushing and merging increase the width of the junction tree, and therefore the complexity.

TJTF chooses the projection adaptively to minimize the approximation error.
Variable contraction

\( \phi_C \) \( \phi_S \)

contract \( X \) from \( C \)

\[ \sum_x \phi_C \]
\[ \sum_x \phi_S \]

\( S \setminus X \) \( C \setminus X \)
Variable contraction

This cuts all edges between $X$ and $C - S$, the variables $X$ no longer resides with.
Variable contraction is an $I$-projection

Proposition. If $\tilde{\rho}$ is the density obtained by contracting $X$ from $C$, then

$$\tilde{\rho} = \arg \min_{\{q : X \perp (C-S) \mid (S\setminus X)\}} D(p \parallel q)$$
Proposition. If $\tilde{p}$ is the density obtained by contracting $X$ from $C$, then

$$D(p \| \tilde{p}) = I(X; C - S \mid S \setminus X)$$
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$$D(p \parallel \tilde{p}) = I(X; C - S \mid S \setminus X)$$

- This can be computed using $\phi_C \propto p_C$. 
Adaptive approximation

Proposition. If \( \tilde{p} \) is the density obtained by contracting \( X \) from \( C \), then

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D(p \parallel \tilde{p}) = I(X; C - S | S \setminus X)
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- For Gaussian \( p \), this is \( O(\dim(X)^3) \) time!
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• To thin \( C \), perform the contraction that minimizes this approximation error.
Thin junction tree filters for SLAM

- The junction tree has $O(N_t)$ clusters.
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Thin junction tree filters for SLAM

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• Use greedy-optimal variable contractions to keep the width bounded by $w$.

• Space complexity: $O(w^2 \cdot N_t)$

• Time complexity: $O(w^3 \cdot N_t)$

• This $O(N_t)$ time complexity is due (mainly) to message passing in the estimation step.
Adaptive message passing

Propagate messages only as long as they induce significant change in the belief state.
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Propagate messages only as long as they induce significant change in the belief state.

Significance is measured by $D(\phi_S^* \parallel \phi_S)$, which decreases with distance.
Simulation results

- landmark location
- robot path
- odometry
- landmark observations
Simulation results

- landmark location
- robot path
- odometry
- landmark observations

- floating point operations \( \times 10^6 \)
- localization error
- mapping error

- time step

- Kalman filter
- FastSLAM
- TJTF
Summary

Thin junction tree filtering:

- a novel algorithm for adaptive approximate filtering in dynamic Bayesian networks
- an elegant solution to the Simultaneous Localization and Mapping problem

More movies and the implementation:

http://www.cs.berkeley.edu/~paskin/slam

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