Assignment #1 Solutions

1. (5 points)
   No, these two options are not the same. As a simple example, consider the case of rolling a sum of two. With two 6-sided dice, the probability of this outcome is $1/36$, since the roll $(1, 1)$ is one outcome out of 36 possibilities. With a 4-sided die and an 8-sided die, there are 32 ($= 4 \times 8$) possible outcomes, so the probability of rolling a sum of two (also denoted by the roll $(1, 1)$) is $1/32$. Since the probability mass function is not the same in the case of rolling a sum of two, it cannot be the same across the whole range of rolls.

2. (15 points)
   Let event $A = \text{missing aircraft is discovered}$.
   Let event $B = \text{missing aircraft has emergency locator}$.

   We are told in the problem that $P(A) = 0.7$, $P(B \mid A) = 0.6$, and $P(B^c \mid A^c) = 0.9$.

   We want to compute $P(A^c \mid B)$
   
   $= 1 - P(A \mid B)$ by complement
   
   $= 1 - \left[ P(B \mid A) P(A) / (P(B \mid A) P(A) + P(B \mid A^c) P(A^c)) \right]$ by Bayes Theorem

   Note that (by complement):
   
   $P(A^c) = 1 - P(A) = 1 - 0.7 = 0.3$.
   
   Also, $P(B \mid A^c) = 1 - P(B^c \mid A^c) = 1 - 0.9 = 0.1$

   $= 1 - [(0.6)(0.7) / ((0.6)(0.7) + (0.1)(0.3))]$ substitution of values
   
   $= 1 - (0.42 / (0.42 + 0.3)) = 1 - (0.42 / 0.45) = 0.0667$ algebra

3. (20 points)
   We have $P(\text{vehicle passing inspection}) = 0.7$, so $P(\text{vehicle failing inspection}) = 1 - 0.7 = 0.3$.

   a. $P(\text{all of the next three vehicles pass inspection}) = (0.7)^3 = 0.343$

   b. $P(\text{at least one of the next three vehicles fails inspection})$
   
   $= 1 - P(\text{all three vehicles pass inspection}) = 1 - (0.7)^3 = 0.657$

   c. $P(\text{exactly one of the next three vehicles pass inspection})$
   
   Let $X$ be the number of vehicle passing inspection. $X \sim \text{Bin}(3, 0.7)$. We want to compute $P(X = 1)$.

   $P(X = 1) = \binom{3}{1}(0.7)^1(0.3)^2 = 3(0.7)(0.09) = 0.189$
d. \( P(\text{at most one of the next three vehicles pass inspection}) \)

Let \( X \) be the number of vehicle passing inspection. \( X \sim \text{Bin}(3, 0.7) \). We want to compute \( P(X = 0 \text{ or } X = 1) = P(X = 0) + P(X = 1) \).

\[
P(X = 0) = \binom{3}{0} (0.7)^0 (0.3)^3 = 1(1)(0.027) = 0.027
\]

\[
P(X = 1) = 0.189 \quad \text{(computed in part (c) above)}
\]

\[
P(X = 0) + P(X = 1) = 0.027 + 0.189 = 0.216
\]

e. Given that at least one of the next three vehicles passes inspection, what is the conditional probability that all three pass?

Let \( X \) be the number of vehicle passing inspection. \( X \sim \text{Bin}(3, 0.7) \). We want to compute \( P(X = 3 \mid X \geq 1) = P(X = 3 \text{ and } X \geq 1) / P(X \geq 1) \).

Note that \( P(X = 3 \text{ and } X \geq 1) = P(X = 3) \). Also, note that \( P(X \geq 1) = 1 - P(X = 0) \).

So we want to compute: \( P(X = 3 \mid X \geq 1) = P(X = 3) / (1 - P(X = 0)) \)

\[
P(X = 3) = \binom{3}{3} (0.7)^3 (0.3)^0 = 1(0.343)(1) = 0.343
\]

\[
1 - P(X = 0) = 1 - 0.027 = 0.973 \quad \text{(using computation from part (d))}
\]

Finally, we get: \( P(X = 3 \mid X \geq 1) = 0.343/0.973 = 0.3525 \)

4. (15 points)

a. \( E[X] = (0.2)(16) + (0.5)(64) + (0.3)(128) = 73.6 \)

b. \( E[X^2] = (0.2)(16^2) + (0.5)(64^2) + (0.3)(128^2) = 7014.4 \)

c. \( \text{Var}(X) = E[X^2] - (E[X])^2 = 7014.4 - 5416.96 = 1597.44 \)

d. Expected price paid by a customer is \( 5 E[X] + 100 = 5(73.6) + 100 = 468 \)

5. (15 points) Let \( W \) be payoff of the game that has a maximum payoff of $128. The expectation for the game is given by: \( E[W] = \sum_{i=0}^{7} \left( \frac{1}{2} \right)^i \cdot 2^i + \left( 1 - \sum_{i=0}^{7} \left( \frac{1}{2} \right)^i \right) \cdot 128 \). The first term in the expectation represents the cases where the number of consecutive heads flipped (denoted by the index \( i \) leads to a payoff of \( 2^i \), and \( 2^i \) is less than or equal to $128, or equivalently \( i \leq \log_2 128 = 7 \). The second term represents the cases where the payoff of the game is the maximum value $128 (i.e., the player flipped more than 7 “heads” in a row). Note the probability in the second case is the complement of first case: \( 1 - P(\text{payoff is } 2^N) \).

We use the equation above to obtain the answer.

\[
E[W] = \sum_{i=0}^{7} \left( \frac{1}{2} \right)^i \cdot 2^i + \left( 1 - \sum_{i=0}^{7} \left( \frac{1}{2} \right)^i \right) \cdot 128 = $4.5
\]
6. (15 points)
Let \( X \) = the amount you win. \( P(X = \$2.00) = 4/9 \), since the first ball you draw can be of either color, and then there will be 4 balls left of that color in the remaining 9 balls to get a match on the second draw. So, \( P(X = -\$1.00) = 1 - P(X = \$2.00) = 5/9 \).

a. \( E[X] = (\$2.00)(4/9) + (-\$1.00)(5/9) = $3/9 \approx $0.33. \)

\[ \text{b. } \text{Var}(X) = E[X^2] - (E[X])^2 = \left[ (\$2.00)^2(4/9) + (-\$1.00)^2(5/9) \right] - \left( \frac{1}{3} \right)^2 = 20/9 \approx 2.2222 \]

7. (15 points)
Student One would assign utilities as follows:

Gamble A: Utility = \( U(\$5,000) \approx 0.0488 \)
Gamble B: Utility = \( (0.4)U(\$40,000) + (0.6)U(-\$10,000) \)
\[ \approx (0.4)(0.3297) + (0.6)(-0.1052) = 0.0688 \]
Gamble C: Utility = \( (1/3)U(\$21,000) + (1/3)U(9,000) + (1/3)U(-\$9,000) \)
\[ \approx (1/3)(0.1894) + (1/3)(0.0861) + (1/3)(-0.0942) = 0.0604 \]

Student One would prefer Gamble B as that provides the maximal expected utility.

Student Two would assign utilities as follows:

Gamble A: Utility = \( U(\$5,000) \approx 0.0952 \)
Gamble B: Utility = \( (0.4)U(\$40,000) + (0.6)U(-\$10,000) \)
\[ \approx (0.4)(0.5507) + (0.6)(-0.2214) = 0.0874 \]
Gamble C: Utility = \( (1/3)U(\$21,000) + (1/3)U(9,000) + (1/3)U(-\$9,000) \)
\[ \approx (1/3)(0.3430) + (1/3)(0.1647) + (1/3)(-0.1972) = 0.1035 \]

Student Two would prefer Gamble C as that provides the maximal expected utility.