Plan

✓ Motivating Examples

11/28: Formalizing the muddy children puzzle, Basic Modal Logic I

11/30: Basic Modal Logic II

12/3: Basic Modal Logic III

12/5: Dynamics in Logic I

12/7: Dynamics in Logic II
Three children are outside playing. Two of them get mud on their forehead. They cannot see or feel the mud on their own foreheads, but can see who is dirty.

Their mother enters the room and says “At least one of you have mud on your forehead”.

Then the children are repeatedly asked “do you know if you have mud on your forehead?”

What happens?

Claim: After first question, the children answer “I don’t know”,...
Three children are outside playing. Two of them get mud on their forehead. They cannot see or feel the mud on their own foreheads, but can see who is dirty.

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**Claim:** After first question, the children answer “I don’t know”, after the second question the muddy children answer “I have mud on my forehead!” (but the clean child is still in the dark).
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What happens?

**Claim:** After first question, the children answer “I don’t know”, after the second question the muddy children answer “I have mud on my forehead!” (but the clean child is still in the dark). Then the clean child says, “Oh, I must be clean.”
Muddy Children

Assume:

- There are three children: Ann, Bob and Charles.
- (Only) Ann and Bob have mud on their forehead.
Muddy Children

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![Diagram showing state-of-affairs and children]

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Muddy Children

Assume:

- There are three children: Ann, Bob and Charles.
- (Only) Ann and Bob have mud on their forehead.
Muddy Children

All 8 possible situations
Muddy Children

The actual situation
Muddy Children

Ann’s uncertainty
Muddy Children

Bob’s uncertainty
Muddy Children

Charles’ uncertainty
Muddy Children
Muddy Children

None of the children know if they are muddy
Muddy Children

None of the children know if they are muddy
“At least one has mud on their forehead.”
Muddy Children

“At least one has mud on their forehead.”
Muddy Children

“Who has mud on their forehead?”
Muddy Children

“Who has mud on their forehead?”
No one steps forward.
Muddy Children

No one steps forward.
Muddy Children

“Who has mud on their forehead?”
Muddy Children

Charles does not know he is clean.
Muddy Children

Ann and Bob step forward.
Now, Charles knows he is clean.
Muddy Children

Now, Charles knows he is clean.
Recall:

A wff of **Propositional Logic** is defined *inductively*:
- Any atomic propositional variable is a wff
- If $P$ and $Q$ are wff, then so are $\neg P$, $P \land Q$, $P \lor Q$ and $P \rightarrow Q$

A wff of **Modal Logic** is defined *inductively*:
1. Any atomic propositional variable is a wff
2. If $P$ and $Q$ are wff, then so are $\neg P$, $P \land Q$, $P \lor Q$ and $P \rightarrow Q$
3. If $P$ is a wff, then so is $\Box P$ and $\Diamond P$
One formal language, many interpretations

**Alethic**

$\Box P$ is intended to mean $P$ is **necessary**
One formal language, many interpretations

*Alethic*

□$P$ is intended to mean $P$ is **necessary**

*Deontic*

□$P$ is intended to mean $P$ is **obligatory**
One formal language, many interpretations

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**Epistemic**

□$P$ is intended to mean $P$ is **known**
One formal language, many interpretations

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**Doxastic**

\[\square P\] is intended to mean \( P \) is **believed**
Basic Modal Logic

One formal language, many interpretations

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\( \Box P \) is intended to mean \( P \) is **necessary**

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\( \Box P \) is intended to mean \( P \) is **believed**

*Temporal*

\( \Box P \) is intended to mean \( P \) will **always** be true (at every point in the future)
Some Questions:

1. Is \( (A \rightarrow B) \lor (B \rightarrow A) \) true or false?
   - \[\text{true}\]

2. Is \( A \land \neg (B \lor A) \) true or false?
   - \[\text{false}\]

3. Is \( A \rightarrow (B \lor C) \) true or false?
   - \[\text{It depends!}\]

4. Is \( \Box A \rightarrow (B \rightarrow \Box A) \) true or false?
   - \[\text{true}\]

5. Is \( \neg \Box A \land \neg (\Diamond B \lor \neg \Box A) \) true or false?
   - \[\text{false}\]

6. Is \( \neg \Box A \land \neg (\Diamond B \lor \Diamond \neg A) \) true or false?
   - \[\text{false}\]
   - (tricky: \( \Box A \) is equivalent to \( \neg \Diamond \neg A \).)

7. Is \( \Box A \rightarrow A \) true or false?
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Some Questions:

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Basic Modal Logic

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Some Questions:

1. Is \((A \rightarrow B) \vee (B \rightarrow A)\) true or false? \text{true.}

2. Is \(A \wedge \neg (B \vee A)\) true or false? \text{false.}

3. Is \(A \rightarrow (B \vee C)\) true or false? \text{It depends!}

(Tricky: \(\Box A\) is equivalent to \(\neg \Diamond \neg A\).)
Basic Modal Logic

Some Questions:

1. Is \((A \rightarrow B) \lor (B \rightarrow A)\) true or false?  true.
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7. Is \(\Box A \rightarrow A\) true or false? It depends!
Can we give find a natural *semantics* for the basic modal language?
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What about truth tables?
Can we give a natural semantics for the basic modal language?

What about truth tables? Won’t work! (Why?)
Proof (from the board): There are four possible truth tables:

<table>
<thead>
<tr>
<th>P</th>
<th>□P</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Suppose we want □P → P to be valid (i.e., true regardless of the interpretation of P), but allow for the possibility that both ¬□P and P → □P are false. (This is natural on an epistemic reading: it is a principle that knowledge of P entails the truth of P. Further it is possible that P is known (¬□P is false), and it is false that if P is true then P is known (P → □P is false).)

Assuming □P → P is true under all interpretations means we have to rule out all truth tables that contain a row with □P assigned T but P assigned F. Hence, we throw out T_1 and T_3.

Now in order to make P → □P false, there must at least one row in which P is assigned T, but □P is assigned F. Hence we throw out T_4.

This leaves us with truth table T_2, but here ¬□P is always true (i.e., □P is always assigned F).

Q.E.D.
Can we give a natural *semantics* for the basic modal language?

What about truth tables? Won’t work! (Why?)

The solution was provided by the American philosopher Saul Kripke (see also the work of Hintikka, McKinsey, and Tarski, and others).
The main idea:

▶ ‘Currently, it is sunny outside.’ is true
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▶ ‘Currently, it is sunny outside.’ is true, but it is not necessary (for example, if we were in Amsterdam).
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- ‘Currently, it is sunny outside.’ is true, but it is not necessary (for example, if we were in Amsterdam).

- We say $P$ is necessary provided $P$ is true in all (relevant) situations (states, worlds, possibilities).
Basic Modal Logic

The main idea:

- ‘Currently, it is sunny outside.’ is true, but it is not necessary (for example, if we were in Amsterdam).

- We say $P$ is necessary provided $P$ is true in all (relevant) situations (states, worlds, possibilities).

- A Kripke structure is

  1. A set of states, or worlds (each world specifies the truth value of all propositional variables)
  2. A relation on the set of states (specifying the “relevant situations”)

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A Kripke Structure

1. Set of states

- $w_1$
- $w_2$
- $w_3$
- $w_4$
- $w_5$
A Kripke Structure

1. Set of states (propositional valuations)

- $w_1$: $A$
- $w_2$: $B$
- $w_3$: $B$
- $w_4$: $B, C$
- $w_5$: $A, B$
A Kripke Structure

1. Set of states (propositional valuations)
2. Accessibility relation
A Kripke Structure

1. Set of states (propositional valuations)
2. Accessibility relation

denoted $w_3 R w_5$
A More Concrete Example of a Kripke Structure
Truth of Modal Formulas

We interpret formulas at states in a Kripke structure: \( w \models P \) means \( P \) is true at state \( w \).
Truth of Modal Formulas

We interpret formulas at states in a Kripke structure: $w \models P$ means $P$ is true at state $w$.

We write $wRv$ is $v$ is accessible from state $w$. 
Truth of Modal Formulas

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We write \( wRv \) is \( v \) is accessible from state \( w \).

1. \( \Box P \) is true at state \( w \) iff \( P \) is true in all accessible worlds.
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1. \( \Box P \) is true at state \( w \) iff \( P \) is true in all accessible worlds.
   \[
   w \models \Box P \iff \text{for all } v, \text{ if } wRv \text{ then } v \models P
   \]
Truth of Modal Formulas

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1. \( \Box P \) is true at state \( w \) iff \( P \) is true in all accessible worlds. \( w \models \Box P \) iff for all \( v \), if \( wRv \) then \( v \models P \)

2. \( \Diamond P \) is true at state \( w \) iff \( P \) is true at some accessible world.
Basic Modal Logic

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1. \( \Box P \) is true at state \( w \) iff \( P \) is true in all accessible worlds.
   \[ w \models \Box P \text{ iff for all } v, \text{ if } wRv \text{ then } v \models P \]

2. \( \Diamond P \) is true at state \( w \) iff \( P \) is true at some accessible world.
   \[ w \models \Diamond P \text{ iff there exists } v \text{ such that } wRv \text{ and } v \models P. \]
Example
Example

$w_4 \models B \land C$
Example

\[ \models w_3 \models \Box B \]
Example

\[ w_3 \models \Diamond C \]
Example

\[ B \xrightarrow{w_2} B, C \xrightarrow{w_4} B \xrightarrow{w_3} A, B \]

\[ w_3 \not\models \Box C \]
Basic Modal Logic

Example

\[ w_1 \models \diamondsuit \Box B \]
Example

\[ w_1 \models \Diamond \Box B \]
Example

\[ w_1 \models \Diamond \Box B \]
Basic Modal Logic

Example

\[
\begin{align*}
&\models □B \land B? \\
&\models ♦♦B? \\
&\models ♦♦♦B? \\
&\models □□B? \\
&\models □♦C? \\
&\models ♦□A?
\end{align*}
\]
Example

\[ w_1 \not\models \Box B \land B \]
\[ w_1 \models \Diamond \Diamond B? \]
\[ w_1 \models \Diamond \Diamond \Diamond B? \]
\[ w_1 \models \Box \Box B? \]
\[ w_1 \models \Box \Diamond C? \]
\[ w_1 \models \Diamond \Box A? \]
Example

\[
\begin{align*}
\text{Example} & \\
A & \leftarrow w_1 \leftarrow B & w_1 \not\models \Box B \land B \\
& \rightarrow B, C \rightarrow w_2 \rightarrow B & w_1 \models \diamond 
\diamond B \\
& \rightarrow B, C \rightarrow w_4 \rightarrow A, B & w_1 \models \diamond \Box A \\
& \rightarrow B & w_1 \models \diamond \Box C \\
B & \rightarrow B & w_1 \models \diamond \Box B \\
A, B & \rightarrow w_5 & w_1 \models \diamond \Box A
\end{align*}
\]
Example

\[ w_1 \not\models \Box B \land B \]
\[ w_1 \models \Diamond \Diamond B \]
\[ w_1 \models \Diamond \Diamond \Diamond B \]
\[ w_1 \not\models \Box \Box \Box B \]
\[ w_1 \models \Box \Diamond C? \]
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Basic Modal Logic

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\[ w_1 \not\models \Box B \land B \]
\[ w_1 \models \Diamond \Diamond B \]
\[ w_1 \models \Diamond \Diamond \Diamond B \]
\[ w_1 \not\models \Box \Box B \]
\[ w_1 \models \Box \Diamond C ? \]
\[ w_1 \models \Diamond \Box A ? \]
Basic Modal Logic

Example

\[ w_1 \not\models \lozenge B \land B \]
\[ w_1 \models \lozenge \lozenge B \]
\[ w_1 \models \lozenge \lozenge \lozenge B \]
\[ w_1 \not\models \lozenge \lozenge \Box B \]
\[ w_1 \not\models \lozenge \lozenge C \]
\[ w_1 \models \lozenge \lozenge \lozenge A ? \]
Basic Modal Logic

Example

\[ \begin{align*}
A \& w_1 \\
B \& w_2 \\
B, C \& w_4 \\
A, B \& w_5
\end{align*} \]

\[ \begin{align*}
w_1 \not\models & \Box B \land B \\
w_1 \models & \Diamond \Diamond B \\
w_1 \models & \Diamond \Diamond \Diamond B \\
w_1 \not\models & \Box \Box B \\
w_1 \not\models & \Box \Diamond C \\
w_1 \models & \Diamond \Box A?
\end{align*} \]
Example

$$\begin{align*}
  w_1 & \not\models \Box B \land B \\
  w_1 & \models \lozenge \lozenge B \\
  w_1 & \models \lozenge \lozenge \lozenge B \\
  w_1 & \models \Box \Box B \\
  w_1 & \not\models \Box \lozenge C \\
  w_1 & \models \lozenge \Box A
\end{align*}$$
Basic Modal Logic

Example

$w_1 \not\models \Box B \land B$

$w_1 \models \Diamond \Diamond B$

$w_1 \models \Diamond \Diamond \Diamond B$

$w_1 \models \Box \Box B$

$w_1 \not\models \Box \Diamond C$

$w_1 \models \Diamond \Box A$
Some Facts

- □P ∨ ¬□P is always true (i.e., true at any state in any Kripke structure), but what about □P ∨ □¬P?

- □P ↔ ¬♦¬P is true at any state in any Kripke structure.
Some Facts

- □P ∨ ¬□P is always true (i.e., true at any state in any Kripke structure), but what about □P ∨ □¬P?

- □P ∧ □Q → □(P ∧ Q) is true at any state in any Kripke structure.
Some Facts

- □\(P \lor \neg\Box P\) is always true (i.e., true at any state in any Kripke structure), but what about □\(P \lor \Box \neg P\)?

- □\(P \land \Box Q \rightarrow \Box (P \land Q)\) is true at any state in any Kripke structure. What about □\((P \lor Q) \rightarrow \Box (P \lor \Box Q)\)?
Some Facts

- $\Box P \lor \neg \Box P$ is always true (i.e., true at any state in any Kripke structure), but what about $\Box P \lor \Box \neg P$?

- $\Box P \land \Box Q \rightarrow \Box (P \land Q)$ is true at any state in any Kripke structure. What about $\Box (P \lor Q) \rightarrow \Box (P \lor \Box Q)$?

- $\Box P \leftrightarrow \neg \Diamond \neg P$ is true at any state in any Kripke structure.
Next time: continue our discussion of modal logic.

Homework: available on the course website.

Questions?
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